Derived Algebraic Geometry XII: Proper Morphisms, Completions, and the Grothendieck Existence Theorem

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Introduction

Let R be a Noetherian ring which is complete with respect to an ideal I, let X be a proper R-scheme, let X_0 denote the closed subscheme $X \times_{\operatorname{Spec} R} \operatorname{Spec} R/I$ of X, and let $\mathfrak{X}$ denote the formal scheme obtained by completing X along X_0. A classical result of Grothendieck asserts that every coherent sheaf on $\mathfrak{X}$ extends uniquely to a coherent sheaf of X. More precisely, the *Grothendieck existence theorem* (Corollary 5.1.6 of [8]) implies that the restriction functor $\operatorname{Coh}(X) \to \operatorname{Coh}(\mathfrak{X})$ is an equivalence of categories.

Our primary objective in this paper is to prove a version of Grothendieck's existence theorem in the setting of spectral algebraic geometry. For this, we first need to develop a good theory of proper morphisms between spectral algebraic spaces. The main thing we need is the following version of the proper direct image theorem, which we prove in §3 (Theorem 3.2.2):

$(*)$ If $f : \mathfrak{X} \to \mathfrak{Y}$ is a proper morphism of spectral algebraic spaces which is locally almost of finite presentation and $\mathcal{F} \in \operatorname{QCoh}(\mathfrak{X})$ is almost perfect, then the pushforward $f_* \mathcal{F}$ again almost perfect.

As a first step towards proving $(*)$, we might ask when the pushforward functor f_* carries quasi-coherent sheaves on $\mathfrak{X}$ to quasi-coherent sheaves on $\mathfrak{Y}$. This requires only very mild hypotheses: namely, that f is quasi-compact and quasi-separated. This follows from very general considerations about spectral Deligne-Mumford stacks, and will be proven in §1.

One novel feature of assertion $(*)$ is that, unlike its classical counterpart, it does not require any Noetherian hypotheses on $\mathfrak{X}$ or $\mathfrak{Y}$. Nevertheless, our proof of $(*)$ will proceed by reduction to the Noetherian case, where it can be deduced from the classical coherence theorem for (higher) direct image sheaves. To carry out the reduction, we need to develop the technique of *Noetherian approximation* in the setting of spectral algebraic geometry. This is the subject of §2.

The second half of this paper is devoted to a study of formal completions in spectral algebraic geometry. To every spectral Deligne-Mumford stack $\mathfrak{X}$, we can associate an underlying topological space $|\mathfrak{X}|$, which we study in §1.4. In §5, we will associate to every closed subset $K \subseteq |\mathfrak{X}|$ a formal completion $\mathfrak{X}^{\wedge}_K$, which we regard as a functor from the ∞-category $\operatorname{CAlg}^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$-rings. In particular, we have an ∞-category of quasi-coherent sheaves $\operatorname{QCoh}(\mathfrak{X}^{\wedge}_K)$ and a restriction functor $\operatorname{QCoh}(\mathfrak{X}) \to \operatorname{QCoh}(\mathfrak{X}^{\wedge}_K)$. Our main result asserts that, if $\mathfrak{X}$ is proper and locally almost of finite presentation over a Noetherian $\mathbb{E}_\infty$-ring R which is complete with respect to an ideal $I \subseteq \pi_0 R$, and $K \subseteq |\mathfrak{X}|$ is the closed subset determined by I, then the restriction functor

$$\operatorname{QCoh}(\mathfrak{X}) \to \operatorname{QCoh}(\mathfrak{X}^{\wedge}_K)$$

restricts to an equivalence of ∞-categories on almost perfect objects (Theorem 5.3.2). The proof relies on $(*)$ together with some facts about completions of modules over Noetherian $\mathbb{E}_\infty$-rings, which we study in §4.

In the final section of this paper (§6), we study the relationship between the formal completions studied in this paper and the formal moduli problems of [46]. In particular, we show that if a local Noetherian $\mathbb{E}_\infty$-ring A is complete with respect to the maximal ideal $\mathfrak{m} \subseteq \pi_0 A$, then A is completely determined by an associated formal moduli problems (defined on local Artinian $\mathbb{E}_\infty$-rings with residue field $k = \pi_0 A/\mathfrak{m}$). Moreover, we characterize those formal moduli problems which arise via this construction (Theorem 6.2.2).

We have included in this paper an appendix, where we review some facts about Stone duality which we will use in this series of papers.

Remark 0.0.1. Many of the notions introduced in this paper are straightforward adaptations of the corresponding notions in classical algebraic geometry. For example, a map of spectral algebraic spaces $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is proper if and only if the underlying map of ordinary algebraic spaces $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathcal{O}_{\mathcal{Y}})$ is proper. Consequently, many of the results of this paper follow immediately from their classical counterparts. However, it seems worthwhile to give an independent exposition, since some of our definitions differ from those given in [31] and elsewhere in the literature (in particular, we do not require the diagonal of a spectral algebraic space to be schematic, though this follows from Theorem 1.2.1 in the quasi-separated case).

Notation and Terminology

We will use the language of ∞-categories freely throughout this paper. We refer the reader to [40] for a general introduction to the theory, and to [41] for a development of the theory of structured ring spectra from the ∞-categorical point of view. We will also assume that the reader is familiar with the formalism of spectral algebraic geometry developed in the earlier papers in this series. For convenience, we will adopt the following reference conventions:

(T) We will indicate references to [40] using the letter T.

(A) We will indicate references to [41] using the letter A.

(V) We will indicate references to [42] using the Roman numeral V.

(VII) We will indicate references to [43] using the Roman numeral VII.

($VIII$) We will indicate references to [44] using the Roman numeral VIII.

(IX) We will indicate references to [45] using the Roman numeral IX.

(X) We will indicate references to [46] using the Roman numeral X.

(XI) We will indicate references to [47] using the Roman numeral XI.

For example, Theorem T.6.1.0.6 refers to Theorem 6.1.0.6 of [40].

If $\mathcal{C}$ is an ∞-category, we let $\mathcal{C}^{\simeq}$ denote the largest Kan complex contained in $\mathcal{C}$: that is, the ∞-category obtained from $\mathcal{C}$ by discarding all non-invertible morphisms. We will say that a map of simplicial sets $f : S \to T$ is *left cofinal* if, for every right fibration $X \to T$, the induced map of simplicial sets $\mathrm{Fun}_T(T, X) \to \mathrm{Fun}_T(S, X)$ is a homotopy equivalence of Kan complexes (in [40], we referred to a map with this property as *cofinal*). We will say that f is *right cofinal* if the induced map $S^{op} \to T^{op}$ is left cofinal: that is, if f induces a homotopy equivalence $\mathrm{Fun}_T(T, X) \to \mathrm{Fun}_T(S, X)$ for every *left* fibration $X \to T$. If S and T are ∞-categories, then f is left cofinal if and only if for every object $t \in T$, the fiber product $S \times_T T_{t/}$ is weakly contractible (Theorem T.4.1.3.1).

Throughout this paper, we let CAlg denote the ∞-category of $\mathbb{E}_\infty$-rings and $\mathrm{CAlg}^{\mathrm{cn}}$ the full subcategory spanned by the connective $\mathbb{E}_\infty$-rings. We will need to study several different notions of the "spectrum" of a ring:

(i) Given a commutative ring R, we can consider its Zariski spectrum: that is, the topological space whose points are prime ideals of R. We will denote this topological space by $\mathrm{Spec}^{\mathrm{Z}} R$. More generally, if A is an $\mathbb{E}_\infty$-ring, we let $\mathrm{Spec}^{\mathrm{Z}} A$ denote the Zariski spectrum of the commutative ring $\pi_0 A$.

(ii) Given a commutative ring, we can consider the affine scheme given by the spectrum of R. We will denote this affine scheme simply by $\mathrm{Spec}\, R$ (so that the underlying topological space of $\mathrm{Spec}\, R$ is given by $\mathrm{Spec}^{\mathrm{Z}} R$).

(iii) Given a connective $\mathbb{E}_\infty$-ring A, we let $\mathrm{Spec}^{\text{ét}} A$ denote the affine spectral Deligne-Mumford stack associated to A. That is, we have $\mathrm{Spec}^{\text{ét}} A = (\mathcal{X}, \mathcal{O})$, where $\mathcal{X}$ is the ∞-topos of $\mathcal{S}\mathrm{hv}_A^{\text{ét}}$ of sheaves on the ∞-category $\mathrm{CAlg}_A^{\text{ét}}$ of étale A-algebras, and $\mathcal{O}$ is the sheaf of $\mathbb{E}_\infty$-rings given by the forgetful functor $\mathrm{CAlg}_A^{\text{ét}} \to \mathrm{CAlg}$.

(iv) If A is a connective $\mathbb{E}_\infty$-ring equipped with an ideal $I \subseteq \pi_0 A$, then we can consider the *formal spectrum* $\mathrm{Spf}\, A$ of A (with respect to I). We will regard this formal spectrum as a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, which assigns to a connective $\mathbb{E}_\infty$-ring R the summand of $\mathrm{Map}_{\mathrm{CAlg}}(A, R)$ consisting of maps from A into R which carry each element of I to a nilpotent element of $\pi_0 R$.

(v) If k is a field and A is an $\mathbb{E}_\infty$-ring equipped with a map $A \to k$, we let $\mathrm{Spec}_{\mathrm{f}}\, A$ denote the formal moduli problem corepresented by A (see §6.1).

(*vi*) If P is a distributive lattice (or, more generally, a distributive upper-semilattice), we let $\mathrm{Spt}(P)$ denote the set of prime ideals of P (see Construction A.2.8). In the special case where P is a Boolean algebra, we can identify $\mathrm{Spt}(P)$ with the Zariski spectrum $\mathrm{Spec}^{\mathrm{Z}} P$, where we regard P as commutative algebra over the finite field $\mathbf{F}_2$ (see Remark A.3.22).

1 Generalities on Spectral Algebraic Spaces

Our goal in this section is to prove some foundational results concerning spectral algebraic spaces. We begin in §1.1 with the following technical result: if $\mathfrak{X}$ is a quasi-compact separated spectral algebraic space, then $\mathfrak{X}$ admits a scallop decomposition (in the sense of Definition VIII.2.5.5). As a consequence, we will see that for any quasi-compact strongly separated morphism $f : \mathfrak{X} \to \mathfrak{Y}$, there is a well-behaved pushforward operation f_* on quasi-coherent sheaves (Corollary 1.1.3). We will apply this result in §1.2 to develop a theory of quasi-finite morphisms between spectral Deligne-Mumford stacks. In particular, we will prove the following version of Zariski's Main Theorem: if $f : \mathfrak{X} \to \mathfrak{Y}$ is quasi-compact, strongly separated, and locally quasi-finite, then f is quasi-affine (Theorem 1.2.1).

In §1.3, we introduce the notion of a *quasi-separated* spectral Deligne-Mumford stack (and, more generally, the notion of a *quasi-separated* morphism between spectral Deligne-Mumford stacks). Using Zariski's main theorem, we prove that a spectral Deligne-Mumford stack $\mathfrak{X}$ admits a scallop decomposition if and only if $\mathfrak{X}$ is a quasi-compact, quasi-separated spectral algebraic space (Theorem 1.3.8). Using this, we can generalize some of our earlier results (for example, the quasi-coherence of direct image sheaves) from the separated to the quasi-separated case.

In §1.4, we study the underlying topological space $|\mathfrak{X}|$ of a spectral Deligne-Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. We define $|\mathfrak{X}|$ to be the space of points of the locale underlying the ∞-topos $\mathcal{X}$. More or less by definition, we can identify open subsets of $|\mathfrak{X}|$ with open substacks of $\mathfrak{X}$. From this description, it is not immediately clear how to describe the *points* of $\mathfrak{X}$. In the special case where $\mathfrak{X}$ is a quasi-separated spectral algebraic space, we will use Theorem 1.3.8 to show that there is a bijection between points of $|\mathfrak{X}|$ and isomorphism classes of maps $i : \mathrm{Spec}^{\text{ét}} k \to \mathfrak{X}$, where k is a field and i induces a monomorphism between the underlying ordinary algebraic spaces (Proposition 1.4.10).

Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack and let $\mathrm{QStk}(\mathfrak{X})$ denote the ∞-category of quasi-coherent stacks on $\mathfrak{X}$ (see §XI.8). To each $\mathcal{C} \in \mathrm{QStk}(\mathfrak{X})$, we can associate an ∞-category $\mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ of quasi-coherent sheaves on $\mathfrak{X}$ with values in $\mathcal{C}$. This ∞-category is tensored over $\mathrm{QCoh}(\mathfrak{X})$ in a natural way. In §1.5, we will show that the construction $\mathcal{C} \mapsto \mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ induces an equivalence

$$\mathrm{QStk}(\mathfrak{X}) \simeq \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$$

whenever $\mathfrak{X}$ is a quasi-compact, quasi-separated spectral algebraic space (Theorem 1.5.3). Moreover, we show that this equivalence respects the property of (local) compact generation (Theorem 1.5.10).

1.1 Scallop Decompositions in the Separated Case

Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack. In §VIII.2.4, we introduced the notion of a *scallop decomposition* of $\mathfrak{X}$: that is, a sequence of open substacks

$$\emptyset = \mathfrak{U}_0 \subseteq \mathfrak{U}_1 \subseteq \cdots \subseteq \mathfrak{U}_n = \mathfrak{X},$$

where each $\mathfrak{U}_i$ is obtained as a pushout

$$\mathfrak{U}_{i-1} \coprod_{\mathfrak{V}} \mathrm{Spec}^{\text{ét}} A$$

for some étale map $\mathfrak{V} \to \mathfrak{U}_{i-1}$ and some quasi-compact open immersion $\mathfrak{V} \hookrightarrow \mathrm{Spec}^{\text{ét}} A$. Our goal in this section is to prove the following:

Proposition 1.1.1. *Let $\mathfrak{Y}$ be a quasi-compact separated spectral algebraic space. Then $\mathfrak{Y}$ admits a scallop decomposition.*

Remark 1.1.2. In §1.3, we will prove that the hypothesis of separatedness can be replaced by quasi-separatedness; see Theorem 1.3.8.

Corollary 1.1.3. *Let $f : \mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a quasi-compact strongly separated morphism of spectral Deligne-Mumford stacks. Then:*

(1) *The pushforward functor $f_* : \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}}$ carries quasi-coherent sheaves to quasi-coherent sheaves.*

(2) *The induced functor $\mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{Y})$ commutes with small colimits.*

(3) *For every pullback diagram*

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{g'} & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}, \end{array}$$

the associated diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathfrak{Y}) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathfrak{X}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathfrak{Y}') & \xrightarrow{f'^*} & \mathrm{QCoh}(\mathfrak{X}') \end{array}$$

is right adjointable. In other words, for every object $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, the canonical map $\lambda : g^* f_* \mathcal{F} \to f'_* g'^* \mathcal{F}$ is an equivalence in $\mathrm{QCoh}(\mathfrak{Y}')$.

Proof. Combine Propositions VIII.2.5.12, VIII.2.5.14, and 1.1.1. □

The proof of Proposition 1.1.1 will require some preliminaries.

Construction 1.1.4. Suppose we are given a strongly separated morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} = (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$, so that the diagonal map induces a closed immersion of ∞-topoi $\delta_* : \mathcal{X} \to \mathcal{Z}$. Let $\emptyset$ denote an initial object of $\mathcal{X}$ and let $U = \delta_*(\emptyset)$, so that U is a (-1)-truncated object of $\mathcal{Z}$ and δ_* induces an equivalence of ∞-topoi $\mathcal{X} \to \mathcal{Z}/U$.

For every finite set I, let $\overline{\mathrm{Conf}}^I_{\mathfrak{Y}}(\mathfrak{X})$ denote the I-fold product of $\mathfrak{X}$ with itself in the ∞-category $\mathrm{Stk}_{/\mathfrak{Y}}$. For every pair of distinct elements $i, j \in I$, we obtain an evaluation map

$$p_{i,j} : \overline{\mathrm{Conf}}^I_{\mathfrak{Y}}(\mathfrak{X}) \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}\,.$$

Let V denote the product $\prod_{i \neq j} p^*_{i,j}(U)$ in the underlying ∞-topos of $\overline{\mathrm{Conf}}^I_{\mathfrak{Y}}(\mathfrak{X})$. We let $\mathrm{Conf}^I_{\mathfrak{Y}}(\mathfrak{X})$ denote the open substack of $\overline{\mathrm{Conf}}^I_{\mathfrak{Y}}(\mathfrak{X})$ corresponding to the (-1)-truncated object V. We will refer to $\mathrm{Conf}^I_{\mathfrak{Y}}(\mathfrak{X})$ as the *spectral Deligne-Mumford stack of I-configurations in* $\mathfrak{X}$ (relative to $\mathfrak{Y}$).

Note that $\mathrm{Conf}^I_{\mathfrak{Y}}(\mathfrak{X})$ depends functorially on I. In particular, it is acted on by the group of all permutations of I, and (up to equivalence) depends only on the cardinality of the set I. When $I = \{1, 2, \ldots, n\}$, we will denote $\mathrm{Conf}^I_{\mathfrak{Y}}(\mathfrak{X})$ by $\mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X})$, so that $\mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X})$ carries an action of the symmetric group Σ_n.

Remark 1.1.5. In the situation of Construction 1.1.4, the projection map $\overline{\mathrm{Conf}}^I_{\mathfrak{Y}}(\mathfrak{X}) \to \mathfrak{Y}$ is strongly separated by Remark IX.4.19. The open immersion $\mathrm{Conf}^I_{\mathfrak{Y}}(\mathfrak{X}) \to \overline{\mathrm{Conf}}^I_{\mathfrak{Y}}(\mathfrak{X})$ is strongly separated by Remark IX.4.15, so the projection $\mathrm{Conf}^I_{\mathfrak{Y}}(\mathfrak{X}) \to \mathfrak{Y}$ is also strongly separated (Remark IX.4.18).

Remark 1.1.6. Recall that a map of spectral Deligne-Mumford stacks $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a *clopen immersion* if it is étale and the underlying map of ∞-topoi $f_* : \mathcal{X} \to \mathcal{Y}$ is a closed immersion (see Definition VIII.1.2.9).

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a strongly separated étale morphism of spectral Deligne-Mumford stacks. Then the diagonal map $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is a clopen immersion. If we write $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} = (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ and define $U \in \mathcal{Z}$ as in Construction 1.1.4, then it follows that U has a complement (in the underlying locale of $\mathcal{Z}$). It follows that for any finite set I, the object $V = \prod_{i \neq j} p_{i,j}^*(U)$ appearing in Construction 1.1.4 has a complement in the underlying locale of $\overline{\mathrm{Conf}}^I_{\mathfrak{Y}}(\mathfrak{X})$, so that the open immersion $\mathrm{Conf}^I_{\mathfrak{Y}}(\mathfrak{X}) \to \overline{\mathrm{Conf}}^I_{\mathfrak{Y}}(\mathfrak{X})$ is also a clopen immersion.

Remark 1.1.7. Every clopen immersion of spectral Deligne-Mumford stacks is also a closed immersion; in particular, it is an affine map. Let R be a connective $\mathbb{E}_\infty$-ring. A map of spectral Deligne-Mumford stacks $\mathfrak{X} \to \mathrm{Spec}^{\text{ét}} R$ is a clopen immersion if and only if $\mathfrak{X}$ has the form $\mathrm{Spec}\, R[\frac{1}{e}]$, where e is an idempotent element in the commutative ring $\pi_0 R$.

Notation 1.1.8. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack. If G is a discrete group, an *action* of G on $\mathfrak{X}$ is a diagram $\chi : BG \to \mathrm{Stk}$ carrying the base point of BG to $\mathfrak{X}$. Since every morphism in BG is an equivalence, χ is automatically a diagram consisting of étale morphisms in Stk, so there exists a colimit $\varinjlim(\chi)$ of the diagram χ (Proposition V.2.3.10). We will denote this colimit by $\mathfrak{X}/G$, and refer to it as the *quotient of* $\mathfrak{X}$ *by the action of* G. There is an evident étale surjection $\mathfrak{X} \to \mathfrak{X}/G$. Moreover, there is a canonical equivalence

$$\mathfrak{X} \times_{\mathfrak{X}/G} \mathfrak{X} \simeq \coprod_{g \in G} \mathfrak{X}.$$

Example 1.1.9. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a finite étale map of spectral Deligne-Mumford stacks of degree n. Then f is strongly separated, so the configuration stack $\mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X})$ is defined and carries an action of the symmetric group Σ_n. We claim that the canonical map $\mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X})/\Sigma_n \to \mathfrak{Y}$ is an equivalence. To prove this, we may work locally on $\mathfrak{Y}$ and thereby reduce to the case where $\mathfrak{Y} = \mathrm{Spec}\, R$ and $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} R^n$. In this case, the result follows from a simple calculation (note that $\mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X}) \simeq \coprod_{\sigma \in \Sigma_n} \mathrm{Spec}\, R$).

Proposition 1.1.10. *Let R be an $\mathbb{E}_\infty$-ring equipped with an action of a finite group G, and let R^G denote the $\mathbb{E}_\infty$-ring of invariants. Suppose that the action of G on the commutative ring $\pi_0 R$ is free (see Definition XI.4.2). Then the canonical map $(\mathrm{Spec}^{ét} R)/G \to \mathrm{Spec}^{ét} R^G$ is an equivalence of spectral Deligne-Mumford stacks. In particular, the quotient $(\mathrm{Spec}^{ét} R)/G$ is affine.*

Proof. Let $\mathfrak{X}_\bullet$ be the Čech nerve of the map $\mathrm{Spec}\, R \to (\mathrm{Spec}\, R)/G$, and let $\mathfrak{Y}_\bullet$ be the Čech nerve of the map $\mathrm{Spec}\, R \to \mathrm{Spec}\, R^G$. It follows from Corollary XI.4.15 that the map $R^G \to R$ is faithfully flat and étale, so that the vertical maps in the diagram

$$\begin{array}{ccc} |\mathfrak{X}_\bullet| & \longrightarrow & |\mathfrak{Y}_\bullet| \\ \downarrow & & \downarrow \\ (\mathrm{Spec}\, R)/G & \longrightarrow & \mathrm{Spec}\, R^G \end{array}$$

are equivalences. It will therefore suffice to show that the canonical map $\mathfrak{X}_n \to \mathfrak{Y}_n$ is an equivalence for every integer n. Since $\mathfrak{X}_\bullet$ and $\mathfrak{Y}_\bullet$ are groupoid objects of Stk, we only need to consider the cases $n = 0$ and $n = 1$. When $n = 0$, the result is obvious. When $n = 1$, we must show that the canonical map

$$\coprod_{g \in G} \mathfrak{X} \to \mathrm{Spec}(R \otimes_{R^G} R)$$

is an equivalence. Equivalently, we must show that the canonical map

$$R \otimes_{R^G} R \to \prod_{g \in G} R$$

is an equivalence of $\mathbb{E}_\infty$-rings, which follows from Corollary XI.4.15. $\square$

Lemma 1.1.11. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral algebraic spaces. If f is strongly separated, then the mapping space $\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X}))$ is discrete for every commutative ring R and every integer $n \geq 0$. If R is nonzero, then the symmetric group Σ_n acts freely on the set $\pi_0 \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X}))$.*

Proof. For any commutative ring R, the map $\theta : \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{X}) \to \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{Y})$ has discrete homotopy fibers (Remark IX.4.20). Since the codomain of θ is discrete, we conclude that the domain of θ is also discrete. Let $S = \pi_0 \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{X})$ and $T = \pi_0 \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{Y})$. There is an evident injective map from $\pi_0 \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, A, \mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X}))$ to the set $K = S \times_T \cdots \times_T S$ given by the n-fold fiber power of S over T. If $\sigma \in \Sigma_n$ is a nontrivial permutation which fixes an element $(s_1, \ldots, s_n)$ of K, then we must have $s_i = s_j$ for some $i \neq j$, in which case the corresponding map

$$\mathrm{Spec}\, R \to \overline{\mathrm{Conf}}^n_{\mathfrak{Y}}(\mathfrak{X}) \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$$

factors through the diagonal. By construction, the fiber product

$$\mathfrak{X} \times_{\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}} \mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X})$$

is empty, which is impossible unless $R \simeq 0$. $\square$

Proposition 1.1.12. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a finite étale map of rank $n > 0$ between spectral Deligne-Mumford stacks. Assume that $\mathfrak{Y}$ is a spectral algebraic space and that $\mathfrak{X}$ is affine. Then $\mathfrak{Y}$ is affine.*

Proof. Example 1.1.9 implies that $\mathfrak{Y}$ can be described as the quotient $\mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X})$ by the action of the symmetric group Σ_n. Note that $\mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X})$ admits a clopen immersion into the iterated fiber product $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} \times \cdots \times_{\mathfrak{Y}} \mathfrak{X}$. Since $n > 0$, we have a finite étale projection map $\mathfrak{X} \times_{\mathfrak{Y}} \cdots \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{X}$. Since $\mathfrak{X}$ is affine, it follows that $\mathfrak{X} \times_{\mathfrak{Y}} \cdots \times_{\mathfrak{Y}} \mathfrak{X}$ is affine and therefore $\mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X}) \simeq \mathrm{Spec}\, A$ is affine. To complete the proof, it will suffice to show that the action of Σ_n on $\mathrm{Spec}\, A$ is free, which follows from Lemma 1.1.11. $\square$

Proposition 1.1.13. *Let $\mathfrak{Y}$ be a separated spectral algebraic space. Suppose we are given an étale map $\mathfrak{X} \to \mathfrak{Y}$. If $\mathfrak{X}$ is affine, then $\mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X})$ is affine for every $n > 0$.*

Proof. Since the diagonal of $\mathfrak{Y}$ is affine, the fiber product

$$\overline{\mathrm{Conf}}^n_{\mathfrak{Y}}(\mathfrak{X}) \simeq \mathfrak{X} \times_{\mathfrak{Y}} \cdots \times_{\mathfrak{Y}} \mathfrak{X}$$

is affine. The desired result now follows from Remarks 1.1.6 and 1.1.7. $\square$

Corollary 1.1.14. *Let $\mathfrak{Y}$ be a separated spectral algebraic space and let $f : \mathfrak{X} \to \mathfrak{Y}$ be an étale map, where $\mathfrak{X}$ is affine. For every $n > 0$, the quotient $\mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X})/\Sigma_n$ is affine.*

Proof. Proposition 1.1.13 implies that $\mathrm{Conf}^n_{\mathfrak{Y}}(\mathfrak{X})$ is affine, hence of the form $\mathrm{Spec}\, R$ for some connective $\mathbb{E}_\infty$-ring R. According to Proposition 1.1.10, it will suffice to show that the action of the symmetric group Σ_n on R is free, which follows from Lemma 1.1.11. $\square$

Lemma 1.1.15. *Let $f : R \to R'$ be an étale morphism of commutative rings. For every prime ideal $\mathfrak{p} \subseteq R$, let $r(\mathfrak{p})$ denote the dimension of the $\kappa(\mathfrak{p})$-vector space $R' \otimes_R \kappa(\mathfrak{p})$. Then:*

(1) *For every integer n, the set $\{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}} R : r(\mathfrak{p}) > n\}$ is quasi-compact and open in $\mathrm{Spec}^{\mathrm{Z}} R$.*

(2) *The function r is constant with value $n \in \mathbf{Z}$ if and only if f exhibits R' as a finite flat R-module of rank n.*

(3) *The function r is bounded above.*

Proof. We will prove (1) and (2) using induction on n. We begin with the case $n = 0$. In this case, assertion (2) is obvious, and assertion (1) follows from the fact that the map $\mathrm{Spec}^{\mathrm{Z}} R' \to \mathrm{Spec}^{\mathrm{Z}} R$ has quasi-compact open image.

Now suppose $n > 0$. We first prove (1). Let $U = \{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}} R : r(\mathfrak{p}) > 0\}$. and let $V = \{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}} R : r(\mathfrak{p}) > n\}$. The inductive hypothesis implies that U is open, so it will suffice to show that V is a quasi-compact open subset of U. Using Proposition VII.5.9, we deduce that the map $\phi : \mathrm{Spec}^{\mathrm{Z}} R' \to U$ is a quotient map; it will therefore suffice to show that $\phi^{-1}U$ is a quasi-compact open subset of $\mathrm{Spec}^{\mathrm{Z}} R'$. Since f is étale, the tensor product $R' \otimes_R R'$ factors as a product $R' \times R''$. Then the set

$$\phi^{-1}U = \{\mathfrak{q} \in \mathrm{Spec}^{\mathrm{Z}} R' : \dim_{\kappa(\mathfrak{q})}(R'' \otimes_{R'} \kappa(\mathfrak{q})) > n-1\}$$

is open by the inductive hypothesis.

It remains to prove (2). The "only if" direction is obvious. For the converse, assume that r is a constant function with value $n > 0$. Then $U = \mathrm{Spec}^{\mathrm{Z}} R$, so f is faithfully flat. It therefore suffices to show that $R' \otimes_R R'$ is a finite flat R'-module of rank n. This is equivalent to the requirement that R'' be a finite flat R'-module of rank $(n-1)$, which follows from the inductive hypothesis.

We now prove (3). Using (1), we see that each of the sets $\{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}} R : r(\mathfrak{p}) > n\}$ is closed with respect to the constructible topology on $\mathrm{Spec}^{\mathrm{Z}} R$ (see Example A.3.33). Since

$$\bigcap_n \{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}} R : r(\mathfrak{p}) > n\} = \emptyset$$

and $\mathrm{Spec}^{\mathrm{Z}} R$ is compact with respect to the constructible topology, we conclude that there exists an integer n such that $\{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}} R : r(\mathfrak{p}) > n\} = \emptyset$. □

Proof of Proposition 1.1.1. We may assume without loss of generality that $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is connective. Since $\mathfrak{Y}$ is quasi-compact, we can choose an étale surjection $u : \mathfrak{X} \to \mathfrak{Y}$, where $\mathfrak{X}$ is affine. For every map $\eta : \mathrm{Spec}\, A \to \mathfrak{Y}$, the pullback $\mathfrak{X} \times_{\mathfrak{Y}} \mathrm{Spec}\, A$ has the form $\mathrm{Spec}\, A'$, for some étale A-algebra A'. Let $r_\eta : \mathrm{Spec}^{\mathrm{Z}} \pi_0 A \to \mathbf{Z}_{\geq 0}$ be defined by the formula

$$r_\eta(\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})}(A' \otimes_A \kappa(\mathfrak{p})).$$

Using Lemma 1.1.15, we can define open immersions $\mathfrak{V}_i \hookrightarrow \mathfrak{Y}$ so that the following universal property is satisfied: a map $\eta : \mathrm{Spec}\, A \to \mathfrak{Y}$ factors through $\mathfrak{V}_i$ if and only if $r_\eta(\mathfrak{p}) \geq i$ for every prime ideal $\mathfrak{p} \subseteq \pi_0 A$. Lemma 1.1.15 implies that the fiber product $\mathrm{Spec}\, A \times_{\mathfrak{Y}} \mathfrak{V}_i$ is empty for $i \gg 0$. Using the quasi-compactness of $\mathfrak{Y}$, we conclude that there exists an integer n such that $\mathfrak{V}_{n+1}$ is empty. The surjectivity of u guarantees that $\mathfrak{V}_1 \simeq \mathfrak{Y}$. For $0 \leq i \leq n$, let $U_i \in \mathcal{Y}$ be the (-1)-truncated object corresponding to the open substack $\mathfrak{V}_{n+1-i}$. We claim that the sequence of morphisms

$$U_0 \to U_1 \to \cdots \to U_n$$

gives a scallop decomposition of $\mathfrak{Y}$.

Note that each $0 \leq i < n$, the étale map $\mathrm{Conf}^{n-i}_{\mathfrak{Y}}(\mathfrak{X})/\Sigma_{n-i} \to \mathfrak{Y}$ determines an object $X_i \in \mathcal{Y}$. It follows from Corollary 1.1.14 that X_i is affine. Choose an equivalence $\mathrm{Conf}^{n-i}_{\mathfrak{Y}}(\mathfrak{X})/\Sigma_{n-i} \simeq \mathrm{Spec}\, R_i$, so that we have an étale map $v_i : \mathrm{Spec}\, R_i \to \mathfrak{Y}$. For every map $\eta : \mathrm{Spec}\, A \to \mathfrak{Y}$, choose an equivalence $\mathrm{Spec}\, A \times_{\mathfrak{Y}} (\mathrm{Conf}^{n-i}_{\mathfrak{Y}}(\mathfrak{X})/\Sigma_i) \simeq \mathrm{Spec}\, A^{(i)}$, and define a function $r^{(i)}_\eta : \mathrm{Spec}^{\mathrm{Z}}(\pi_0 A) \to \mathbf{Z}_{\geq 0}$ by the formula

$$r^{(i)}_\eta(\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})}(A^{(i)} \otimes_A \kappa(\mathfrak{p})).$$

An easy calculation shows that $r^{(i)}_\eta(\mathfrak{p})$ is equal to the binomial coefficient $\binom{r_\eta(\mathfrak{p})}{n-i}$. In particular, $r^{(i)}_\eta(\mathfrak{p})$ takes positive values if and only if $r^{(1)}_\eta(\mathfrak{p}) \geq n-i$ for every $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}}(\pi_0 A)$. It follows that the map v_i factors through $\mathfrak{V}_i$. Form a pullback diagram σ_i:

$$\begin{array}{ccc} X_i \times_{U_{i+1}} U_i & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & U_{i+1}. \end{array}$$

Note that there is an effective epimorphism

$$\coprod_{0 \leq j < i} X_i \times X_j \to X_i \times_{U_{i+1}} U_i.$$

Since $\mathfrak{Y}$ is separated, each product $X_i \times X_j$ is affine and therefore quasi-compact, so that $X_i \times_{U_{i+1}} U_i$ is quasi-compact.

To complete the proof, it will suffice to show that each σ_i is an excision square. For this, we may replace $\mathfrak{Y}$ by the reduced closed substack $\mathfrak{K}_i$ of $\mathfrak{U}_{i+1}$ which is complementary to $\mathfrak{U}_i$, and thereby reduce to the case where the function r_η takes the constant value i, for every $\eta : \operatorname{Spec} A \to \mathfrak{Y}$. In this case, the function $r_\eta^{(i)}$ is constant with value 1, so that the map $A \to A^{(i)}$ is finite étale of rank 1 and therefore an equivalence (Lemma 1.1.15). It follows that the map $\operatorname{Spec} R_i \to \mathfrak{Y}$ is an equivalence, as desired. □

1.2 Quasi-Finite Morphisms

Recall that a map $\phi : A \to B$ of commutative rings is said to be *quasi-finite* if the following conditions are satisfied:

(i) The commutative ring B is finitely generated as an A algebra.

(ii) For each residue field κ of A, the fiber $\operatorname{Tor}_0^A(B, k)$ is a finite-dimensional vector space over κ. Assuming (i), this is equivalent to the requirement that the induced map of topological spaces $\operatorname{Spec}^Z B \to \operatorname{Spec}^Z A$ has finite fibers.

A morphism of schemes $f : X \to Y$ is said to be *locally quasi-finite* if, for every point $x \in X$, there exist affine open neighborhoods $\operatorname{Spec} B \simeq U \subseteq X$ of x and $\operatorname{Spec} A \simeq V \subseteq Y$ such that $f(U) \subseteq V$ and the induced map of commutative rings $A \to B$ is quasi-finite. Our goal in this section is generalize the notion of locally quasi-finite morphism to the setting of spectral algebraic geometry. Our main result is the following version of Zariski's Main Theorem:

Theorem 1.2.1. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of spectral Deligne-Mumford stacks. Assume that f is quasi-compact, strongly separated, and locally quasi-finite. Then f is quasi-affine.*

We begin by defining the class of locally quasi-finite morphisms.

Definition 1.2.2. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. We will say that f is *locally quasi-finite* if the following condition is satisfied: for every commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}^{\text{ét}} B & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow f \\ \operatorname{Spec}^{\text{ét}} A & \longrightarrow & \mathfrak{Y} \end{array}$$

in which the horizontal maps are étale, the induced map of commutative rings $\pi_0 A \to \pi_0 B$ is quasi-finite.

Remark 1.2.3. Every locally quasi-finite morphism of spectral Deligne-Mumford stacks is locally of finite presentation to order 0 (in the sense of Definition IX.8.16).

Example 1.2.4. Every étale map of spectral Deligne-Mumford stacks is locally quasi-finite.

Example 1.2.5. Every closed immersion of spectral Deligne-Mumford stacks is locally quasi-finite.

Proposition 1.2.6. *The condition that a map of spectral Deligne-Mumford stacks $f : \mathfrak{X} \to \mathfrak{Y}$ be locally quasi-finite is local on the source with respect to the étale topology (see Definition VIII.1.5.7).*

Proof. It is clear that if $f : \mathfrak{X} \to \mathfrak{Y}$ is locally quasi-finite and $g : \mathfrak{U} \to \mathfrak{X}$ is étale, then the composite map $f \circ g$ is locally quasi-finite. To complete the proof, let us suppose that $f : \mathfrak{X} \to \mathfrak{Y}$ is arbitrary and that we are given a jointly surjective collection of étale morphisms $\{g_\alpha : \mathfrak{U}_\alpha \to \mathfrak{X}\}$ such that each composition $f \circ g_\alpha$ is a locally quasi-finite morphism from $\mathfrak{U}_\alpha$ to $\mathfrak{X}$. We wish to show that f has the same property. Choose a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}^{\text{ét}} B & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}^{\text{ét}} A & \longrightarrow & \mathfrak{Y} \end{array}$$

where the horizontal maps are étale. We wish to show that $\pi_0 B$ is quasi-finite over $\pi_0 A$. It follows from Proposition IX.8.18 that $\pi_0 B$ is finitely generated over $\pi_0 A$. It will therefore suffice to show that the map $\operatorname{Spec}^{\mathrm{Z}} B \to \operatorname{Spec}^{\mathrm{Z}} A$ has finite fibers. Since the maps g_α are jointly surjective, we can choose an étale covering $\{B \to B_i\}_{1 \leq i \leq n}$ such that each of the composite maps $\operatorname{Spec}^{\text{ét}} B_i \to \operatorname{Spec}^{\text{ét}} B \to \mathfrak{X}$ factors through some $\mathfrak{U}_\alpha$. Using our assumption on $f \circ g_\alpha$, we deduce that each of the commutative rings $\pi_0 B_i$ is quasi-finite over $\pi_0 A$. It follows that the composite map

$$\coprod_{1 \leq i \leq n} \operatorname{Spec}^{\mathrm{Z}} B_i \xrightarrow{\theta} \operatorname{Spec}^{\mathrm{Z}} B \to \operatorname{Spec}^{\mathrm{Z}} A$$

has finite fibers. Since the map θ is surjective, we conclude that $\operatorname{Spec}^{\mathrm{Z}} B \to \operatorname{Spec}^{\mathrm{Z}} A$ has finite fibers as desired. □

Corollary 1.2.7. *Suppose we are given a morphisms of spectral Deligne-Mumford stacks*

$$\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}.$$

If f and g are locally quasi-finite, then so is $g \circ f$.

Proof. Suppose we are given a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}^{\text{ét}} C & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec}^{\text{ét}} A & \longrightarrow & \mathfrak{Z} \end{array}$$

where the horizontal maps are étale. We wish to show that $\pi_0 C$ is quasi-finite over $\pi_0 A$. The proof of Proposition 1.2.6 shows that this condition is étale local on C; we may therefore assume that the map $\operatorname{Spec}^{\text{ét}} C \to \operatorname{Spec}^{\text{ét}} A \times_{\mathfrak{Z}} \mathfrak{Y}$ factors through some étale map $\operatorname{Spec}^{\text{ét}} B \to \operatorname{Spec}^{\text{ét}} A \times_{\mathfrak{Z}} \mathfrak{Y}$. Since f and g are locally quasi-finite, we see that $\pi_0 B$ is quasi-finite over $\pi_0 A$ and that $\pi_0 C$ is quasi-finite over $\pi_0 B$, so that $\pi_0 C$ is quasi-finite over $\pi_0 A$ as desired. □

Proposition 1.2.8. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. Then:*

(1) *The map f is locally quasi-finite if and only if, for every étale map $\operatorname{Spec}^{\text{ét}} A \to \mathfrak{Y}$, the induced map $\operatorname{Spec}^{\text{ét}} A \times_{\mathfrak{Y}} \mathfrak{X} \to \operatorname{Spec}^{\text{ét}} A$ is locally quasi-finite.*

(2) *Assume that* $\mathfrak{Y} \simeq \mathrm{Spec}^{ét} A$ *is affine. Then* f *is locally quasi-finite if and only if, for every étale map* $\mathrm{Spec}^{ét} B \to \mathfrak{X}$*, the induced map of commutative rings* $\pi_0 A \to \pi_0 B$ *is quasi-finite.*

(3) *Assume that* $\mathfrak{Y} \simeq \mathrm{Spec}^{ét} A$ *and* $\mathfrak{X} \simeq \mathrm{Spec}^{ét} B$ *are both affine. Then* f *is locally quasi-finite if and only if the underlying map of commutative rings* $\pi_0 A \to \pi_0 B$ *is quasi-finite.*

Proof. Assertion (1) follows immediately from the definition, and the "only if" directions of (2) and (3) are obvious. To complete the proof of (2), assume that $\mathfrak{Y} \simeq \mathrm{Spec}^{\text{ét}} A$ and consider an arbitrary commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}^{\text{ét}} B & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}^{\text{ét}} A' & \longrightarrow & \mathrm{Spec}^{\text{ét}} A \end{array}$$

where the horizontal maps are étale. If $\pi_0 B$ is quasi-finite over $\pi_0 A$, then it is also quasi-finite over $\pi_0 A'$. The proof of (3) is similar. □

Proposition 1.2.9. *The condition that a map of spectral Deligne-Mumford stacks* $f : \mathfrak{X} \to \mathfrak{Y}$ *be locally quasi-finite is local on the target with respect to the étale topology (see Definition VIII.3.1.1).*

Proof. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. From Proposition 1.2.8, we see immediately that if $\mathfrak{U} \to \mathfrak{Y}$ is an étale map, then the induced map $\mathfrak{U} \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{U}$ is locally quasi-finite. Conversely, suppose we are given a jointly surjective collection of étale maps $\mathfrak{U}_\alpha \to \mathfrak{Y}$ such that each of the induced maps $\mathfrak{U}_\alpha \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{U}_\alpha$ are locally quasi-finite. Using Example 1.2.4 and Corollary 1.2.7, we see that each of the induced maps $\mathfrak{U}_\alpha \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{Y}$ is locally quasi-finite. Applying Proposition 1.2.6, we deduce that f is locally quasi-finite. □

Proposition 1.2.10. *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}. \end{array}$$

If f *is locally quasi-finite, then so is* f'*. The converse holds if* g *is faithfully flat and quasi-compact.*

Proof. Assume first that f is locally quasi-finite; we wish to show that f' has the same property. Using Proposition 1.2.9 we can reduce to the case where $\mathfrak{Y} = \mathrm{Spec}^{\text{ét}} A$ is affine, and the map g factors as a composition

$$\mathfrak{Y}' \to \mathrm{Spec}^{\text{ét}} A_0 \xrightarrow{g'} \mathfrak{Y},$$

where g' is étale. Replacing $\mathfrak{Y}$ by $\mathrm{Spec}^{\text{ét}} A_0$, we may assume that $\mathfrak{Y}$ is affine. Using Proposition 1.2.6, we may further suppose that $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} B_0$ is affine, so that $\mathfrak{X}' \simeq \mathrm{Spec}^{\text{ét}}(A \otimes_{A_0} B_0)$ is also affine. We wish to show that $R = \pi_0(A \otimes_{A_0} B_0) = \mathrm{Tor}_0^{\pi_0 A_0}(\pi_0 A, \pi_0 B_0)$ is quasi-finite over $\pi_0 A$. It is clear that R is finitely generated over $\pi_0 A$ (since $\pi_0 B_0$ is finitely generated over $\pi_0 A_0$). For each residue field κ of $\pi_0 A$, if we let κ_0 denote the corresponding residue field of $\pi_0 A_0$, then we have canonical isomorphisms

$$\mathrm{Tor}_0^{\pi_0 A}(\kappa, R) \simeq \mathrm{Tor}_0^{\pi_0 A_0}(\kappa, \pi_0 B_0) \simeq \mathrm{Tor}_0^{\pi_0 A_0}(\kappa_0, \pi_0 B) \otimes_{\kappa_0} \kappa,$$

which proves that $\mathrm{Tor}_0^{\pi_0 A}(\kappa, R)$ is finite dimensional as a vector space over κ.

Now suppose that g is faithfully flat and quasi-compact and that f' is locally quasi-finite; we wish to show that f is locally quasi-finite. Using Propositions 1.2.6 and 1.2.9, we may assume that $\mathfrak{Y} = \mathrm{Spec}^{\text{ét}} A_0$ and $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} B_0$ are affine. Replacing $\mathfrak{Y}'$ by an étale cover if necessary, we may suppose htat $\mathfrak{Y}' = \mathrm{Spec}^{\text{ét}} A$

for some flat A_0-algebra A. Let R be defined as above, so that R is quasi-finite over $\pi_0 A$. Using Proposition IX.8.24, we see that $\pi_0 B_0$ is finitely generated over $\pi_0 A_0$. Moreover, for every residue field κ_0 of $\pi_0 A_0$, the surjectivity of the map $\mathrm{Spec}^{\mathrm{Z}} A \to \mathrm{Spec}^{\mathrm{Z}} A_0$ implies that we can lift κ_0 to a residue field κ of $\pi_0 A$. The finite-dimensionality of

$$\mathrm{Tor}_0^{\pi_0 A_0}(\kappa_0, \pi_0 B) \otimes_{\kappa_0} \kappa \simeq \mathrm{Tor}_0^{\pi_0 A}(\kappa, R)$$

over κ then implies the finite dimensionality of $\mathrm{Tor}_0^{\pi_0 A_0}(\kappa_0, \pi_0 B)$ over κ_0. □

We also have the following converse to Corollary 1.2.7:

Proposition 1.2.11. *Suppose we are given a morphisms of spectral Deligne-Mumford stacks*

$$\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}.$$

If $g \circ f$ is locally quasi-finite, then so is f.

Proof. Using Propositions 1.2.6 and 1.2.9, we can reduce to the case where $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} C$, $\mathfrak{Y} = \mathrm{Spec}^{\text{ét}} B$, and $\mathfrak{Z} = \mathrm{Spec}^{\text{ét}} A$ are affine. Then $\pi_0 C$ is quasi-finite over $\pi_0 A$. It follows immediately that $\pi_0 C$ is finitely generated over $\pi_0 B$. Since the composite map

$$\mathrm{Spec}^{\mathrm{Z}} C \xrightarrow{\theta} \mathrm{Spec}^{\mathrm{Z}} B \to \mathrm{Spec}^{\mathrm{Z}} A$$

has finite fibers, we conclude that θ has finite fibers. □

The essential step in the proof of Theorem 1.2.1 is the following:

Proposition 1.2.12. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of spectral Deligne-Mumford stacks. Suppose that:*

(*a*) *The map f is quasi-compact, strongly separated and locally quasi-finite.*

(*b*) *Set $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. Then the unit map $\mathcal{O}_{\mathcal{Y}} \to f_* \mathcal{O}_{\mathcal{X}}$ exhibits $\mathcal{O}_{\mathcal{Y}}$ as a connective cover of $f_* \mathcal{O}_{\mathcal{X}}$.*

Then f is an open immersion.

Remark 1.2.13. It follows from Corollary 1.1.3 that condition (*b*) of Proposition 1.2.12 is stable under flat base change.

Before giving the proof, we make a few simple observations.

Definition 1.2.14. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. We will say that f is *finite étale of rank n* if, for every morphism $\mathrm{Spec}^{\text{ét}} R \to \mathfrak{Y}$, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} \mathrm{Spec}^{\text{ét}} R$ has the form $\mathrm{Spec}^{\text{ét}} R'$, where R' is a finite étale R-algebra of rank n.

Remark 1.2.15. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a finite étale map of spectral Deligne-Mumford stacks. Then f determines a decomposition $\mathfrak{Y} \simeq \coprod_{n \geq 0} \mathfrak{Y}_n$, where each of the induced maps $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}_n \to \mathfrak{Y}_n$ is finite étale of rank n.

Proof of Proposition 1.2.12. The assertion is local on $\mathfrak{Y}$; we may therefore reduce to the case where $\mathfrak{Y} = \mathrm{Spec}^{\text{ét}} R$ is affine (so that $\mathfrak{X}$ is a separated spectral algebraic space). Then $\mathfrak{X}$ is quasi-compact, so we can choose an étale surjection $u : \mathrm{Spec}^{\text{ét}} A \to \mathfrak{X}$. Let $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}}(\pi_0 A)$ and let $\mathfrak{q}$ be its image in $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)$. We will show that there exists an open set $U_{\mathfrak{q}} \subseteq \mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)$ such that, if $\mathfrak{U}_{\mathfrak{q}}$ denotes the corresponding open substack of $\mathfrak{Y}$, then the projection map $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{U}_{\mathfrak{q}} \to \mathfrak{U}_{\mathfrak{q}}$ is an equivalence. Let $U = \bigcup_{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}}(\pi_0 A)} U_{\mathfrak{q}}$ and let $\mathfrak{U}$ be the corresponding open substack of $\mathfrak{Y}$. Then the projection $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{U} \to \mathfrak{U}$ is an equivalence. Moreover, the open substack $\mathfrak{U} \times_{\mathfrak{Y}} \mathrm{Spec}^{\text{ét}} A$ is equivalent to $\mathrm{Spec}^{\text{ét}} A$. Since u is surjective, it follows that $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{U} \simeq \mathfrak{X}$, so that we can identify f with the open immersion $\mathfrak{U} \to \mathfrak{Y}$.

It remains to construct the open set $U_{\mathfrak{q}}$. Let κ denote the residue field of $\pi_0 R$ at the prime ideal $\mathfrak{q}$. Since f is locally quasi-finite and u is étale, the map of commutative rings $\pi_0 R \to \pi_0 A$ is quasi-finite. It follows from Proposition VII.7.14 that the map $\pi_0 R \to \kappa$ factors as a composition $\pi_0 R \to R'_0 \to \kappa$, where R'_0 is an étale $(\pi_0 R)$-algebra and $(\pi_0 A) \otimes_{\pi_0 R} R'_0$ decomposes as a product $B'_0 \times B''_0$, where B' is a finite R'_0-module and $\mathrm{Tor}_0^{R'_0}(B'', \kappa) \simeq 0$. Using Theorem A.7.5.0.6, we can choose an étale R-algebra R' with $\pi_0 R' \simeq R'_0$, so that $A \otimes_R R'$ decomposes as a product $B' \times B''$. with $\pi_0 B' \simeq B'_0$ and $\pi_0 B'' \simeq B''_0$. Since the map $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R') \to \mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)$ is open and its image contains $\mathfrak{q}$, we can replace R by R' and thereby assume that $A \simeq B' \times B''$, where $\pi_0 B'$ is a finitely generated module over $\pi_0 R$ and $B'' \otimes_R \kappa \simeq 0$. Since $A \otimes_R \kappa \neq 0$, it follows that $B' \otimes_R \kappa \neq 0$. The composite map

$$u' : \mathrm{Spec}^{\text{ét}} B' \to \mathrm{Spec}^{\text{ét}} A \to \mathfrak{X}$$

is étale. Since $\mathfrak{X}$ is strongly separated, the map u' is affine. Using the fact that $\pi_0 B'$ is finitely generated as a $\pi_0 R$-module, we deduce that u' is finite étale. Using Remark 1.2.15, we deduce that $\mathfrak{X}$ admits a decomposition $\mathfrak{X} \simeq \coprod_{n \geq 0} \mathfrak{X}_n$, where each of the induced maps

$$\mathrm{Spec}^{\text{ét}} B' \times_{\mathfrak{X}} \mathfrak{X}_n \to \mathfrak{X}_n$$

is finite étale of rank n. Each fiber product $\mathrm{Spec}^{\text{ét}} B' \times_{\mathfrak{X}} \mathfrak{X}_n$ is a summand of $\mathrm{Spec}^{\text{ét}} B'$, and therefore affine. Since $\mathfrak{X}$ is quasi-compact, the stacks $\mathfrak{X}_n$ are empty for $n \gg 0$. It follows that $\mathfrak{X}' = \coprod_{n>0} \mathfrak{X}_n$ is an affine open substack of $\mathfrak{X}$. Note that since $\mathrm{Spec}^{\text{ét}} \kappa \times_{\mathrm{Spec}^{\text{ét}} R} \mathrm{Spec}^{\text{ét}} B'$ is nonempty, the fiber product $\mathrm{Spec}^{\text{ét}} \kappa \times_{\mathrm{Spec}^{\text{ét}} R} \mathfrak{X}'$ is also nonempty.

Using (b), we can choose an idempotent element $e \in \pi_0 R$ which vanishes on $\mathfrak{X}_0$ but not on $\mathfrak{X}'$. Since $\mathrm{Spec}^{\text{ét}} \kappa \times_{\mathrm{Spec}^{\text{ét}} R} \mathfrak{X}' \neq \emptyset$, we must have $e \notin \mathfrak{q}$. We may therefore replace R by $R[\frac{1}{e}]$ and thereby reduce to the case where $\mathfrak{X}_0$ is empty. In this case, $\mathfrak{X} \simeq \mathfrak{X}'$ is affine. Using (b) again, we deduce that f is an equivalence. □

Proof of Theorem 1.2.1. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be strongly separated, quasi-compact, and locally quasi-finite; we wish to show that f is quasi-affine. The assertion is local on $\mathfrak{Y}$; we may therefore assume that $\mathfrak{Y} \simeq \mathrm{Spec}^{\text{ét}} R$ is affine. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, and let A be the connective cover $f_* \mathcal{O}_{\mathcal{X}} \in \mathrm{CAlg}_R$. Then f factors as a composition $\mathfrak{X} \xrightarrow{f'} \mathrm{Spec}\, A \xrightarrow{f''} \mathrm{Spec}\, R$. Since f is locally quasi-finite, f' is also locally quasi-finite (Proposition 1.2.11). Using Proposition 1.2.12, we deduce that f' is an open immersion, so that $\mathfrak{X}$ can be identified with a quasi-compact open substack of $\mathrm{Spec}\, A$ and is therefore quasi-affine. □

1.3 Quasi-Separatedness

In this section, we will introduce the notion of a *quasi-separated* spectral Deligne-Mumford stack. Our main result (Theorem 1.3.8) asserts that a spectral Deligne-Mumford stack $\mathfrak{X}$ admits a scallop decomposition (in the sense of Definition VIII.2.5.5) if and only it is is a quasi-compact, quasi-separated spectral algebraic space. From this we will deduce a number of consequences concerning the global sections functor $\Gamma(\mathfrak{X}; \bullet)$ on the ∞-category $\mathrm{QCoh}(\mathfrak{X})$ of quasi-coherent sheaves on $\mathfrak{X}$.

Definition 1.3.1. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. We will say that f is *quasi-separated* if the diagonal map $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is quasi-compact. We say that a spectral Deligne-Mumford stack $\mathfrak{X}$ is *quasi-separated* if the map $\mathfrak{X} \to \mathrm{Spec}\, S$ is quasi-separated, where S denotes the sphere spectrum. In other words, $\mathfrak{X}$ is quasi-separated if the absolute diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is quasi-compact.

Example 1.3.2. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. If f is quasi-geometric (that is, if the diagonal map $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is quasi-affine), then f is quasi-separated. In particular, every strongly separated morphism is quasi-separated (see Definition IX.4.11).

Proposition 1.3.3. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. The following conditions are equivalent:*

(1) *The spectral Deligne-Mumford stack $\mathfrak{X}$ is quasi-separated.*

(2) *For every pair of maps $f, g : \operatorname{Spec} R \to \mathfrak{X}$, the fiber product $\operatorname{Spec} R \times_{\mathfrak{X}} \operatorname{Spec} R$ is quasi-compact.*

(3) *For every pair of maps $f : \operatorname{Spec} R \to \mathfrak{X}$, $g : \operatorname{Spec} R' \to \mathfrak{X}$, the fiber product $\operatorname{Spec} R \times_{\mathfrak{X}} \operatorname{Spec} R'$ is quasi-compact.*

(4) *For every pair of étale maps maps $f : \operatorname{Spec} R \to \mathfrak{X}$, $g : \operatorname{Spec} R' \to \mathfrak{X}$, the fiber product $\operatorname{Spec} R \times_{\mathfrak{X}} \operatorname{Spec} R'$ is quasi-compact.*

(5) *For every pair of affine objects $U, V \in \mathcal{X}$, the product $U \times V \in \mathcal{X}$ is quasi-compact.*

(6) *For every pair of quasi-compact objects $U, V \in \mathcal{X}$, the product $U \times V \in \mathcal{X}$ is quasi-compact.*

Proof. The implications (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Leftarrow$ (6) are obvious. We next prove that (2) $\Rightarrow$ (3). Suppose we are given a pair of maps $f : \operatorname{Spec} R \to \mathfrak{X}$, $g : \operatorname{Spec} R' \to \mathfrak{X}$. Let $A = R \otimes R'$, so that f and g define maps $f', g' : \operatorname{Spec} A \to \mathfrak{X}$. Note that

$$\operatorname{Spec} R \times_{\mathfrak{X}} \operatorname{Spec} R' \simeq (\operatorname{Spec} A \times_{\mathfrak{X}} \operatorname{Spec} A) \times_{\operatorname{Spec}(A \otimes A)} \operatorname{Spec} A.$$

If (2) is satisfied, then there exists an étale surjectiion $\operatorname{Spec} B \to \operatorname{Spec} A \times_{\mathfrak{X}} \operatorname{Spec} A$. It follows that there is an étale surjection

$$\operatorname{Spec}(B \otimes_{A \otimes A} A) \to \operatorname{Spec} R \times_{\mathfrak{X}} \operatorname{Spec} R',$$

so that $\operatorname{Spec} R \times_{\mathfrak{X}} \operatorname{Spec} R'$ is quasi-compact.

We next show that (4) $\Rightarrow$ (3). Assume we are given arbitrary maps $f : \operatorname{Spec} R \to \mathfrak{X}$ and $g : \operatorname{Spec} R' \to \mathfrak{X}$. Choose a faithfully flat étale map $R \to A$ such that the composite map $\operatorname{Spec} A \to \operatorname{Spec} R \xrightarrow{f} \mathfrak{X}$ factors through some étale map $\operatorname{Spec} B \to \mathfrak{X}$, and a faithfully flat étale map $R' \to A'$ such that the composite map $\operatorname{Spec} A' \to \operatorname{Spec} R' \xrightarrow{g} \mathfrak{X}$ factors through an étale map $\operatorname{Spec} B' \to \mathfrak{X}$. Condition (4) implies that $\operatorname{Spec} B \times_{\mathfrak{X}} \operatorname{Spec} B'$ is quasi-compact, so there is an étale surjection $\operatorname{Spec} T \to \operatorname{Spec} B \times_{\mathfrak{X}} \operatorname{Spec} B'$. It follows that the composite map

$$\operatorname{Spec}(T \otimes_{B \otimes B'} (A \otimes A'')) \to \operatorname{Spec} A \times_{\mathfrak{X}} \operatorname{Spec} A' \to \operatorname{Spec} R \times_{\mathfrak{X}} \operatorname{Spec} R'$$

is an étale surjection, so that $\operatorname{Spec} R \times_{\mathfrak{X}} \operatorname{Spec} R'$ is also quasi-compact.

We complete the proof by showing that (5) $\Rightarrow$ (6). Assume $U, V \in \mathcal{X}$ are quasi-compact. Then there exist effective epimorphisms $U' \to U$ and $V' \to V$, where U' and V' are affine. Condition (5) implies that $U' \times V'$ is quasi-compact. Since we have an effective epimorphism $U' \times V' \to U \times V$, it follows that $U \times V$ is quasi-compact. □

Proposition 1.3.4. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-compact, quasi-separated spectral algebraic space. Then the ∞-topos $\mathcal{X}$ is coherent.*

Proof. We first suppose that $\mathfrak{X}$ is strongly separated. Using Corollary VII.3.10, it suffices to show that if we are given affine objects $U, V \in \mathcal{X}$, then the product $U \times V \in \mathcal{X}$ is coherent. Let $\mathfrak{U}$ and $\mathfrak{V}$ be the spectral Deligne-Mumford stacks determined by U and V. In fact, we claim that $U \times V$ is affine. This follows from Theorem IX.4.4, since $\mathfrak{Y} \simeq \mathfrak{U} \times_{\mathfrak{X}} \mathfrak{V}$ admits a closed immersion into the affine spectral Deligne-Mumford stack $\mathfrak{U} \times \mathfrak{V}$.

We now treat the general case. Once again, it suffices to show that if $U, V \in \mathcal{X}$ are affine, then $U \times V$ is coherent. By the first part of the proof, we are reduced to proving that $\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{V}$ is separated. For this, it suffices to show that the map $\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{V} \to \mathfrak{U} \times \mathfrak{V}$ is strongly separated, which follows from Example IX.4.24 (since $\mathfrak{X}$ is a spectral algebraic space). □

Remark 1.3.5. From Proposition 1.3.4 we deduce the following stronger assertion:

(∗) Let $\mathfrak{X}$ be a spectral Deligne-Mumford m-stack. If $\mathfrak{X}$ is $(m+1)$-quasi-compact, then it is ∞-quasi-compact.

Proposition 1.3.6. *Let $\mathfrak{X}$ be a quasi-separated spectral algebraic space. Suppose we are given étale maps $\operatorname{Spec}^{ét} R \to \mathfrak{X} \leftarrow \operatorname{Spec}^{ét} R'$. Then the fiber product $\mathfrak{Y} \simeq \operatorname{Spec}^{ét} R \times_{\mathfrak{X}} \operatorname{Spec}^{ét} R'$ is quasi-affine.*

Proof. Since $\mathfrak{X}$ is quasi-separated, $\mathfrak{Y}$ is quasi-compact. Since $\mathfrak{X}$ is a spectral algebraic space, the canonical map

$$\operatorname{Map}_{\mathrm{Stk}}(\operatorname{Spec} A, \mathfrak{Y}) \to \operatorname{Map}_{\mathrm{Stk}}(\operatorname{Spec} A, \operatorname{Spec}(R \otimes R'))$$

is (-1)-truncated for any commutative ring A. In particular, $\operatorname{Map}_{\mathrm{Stk}}(\operatorname{Spec} A, \mathfrak{Y})$ is discrete, so that $\mathfrak{Y}$ is a spectral algebraic space. It follows from Example IX.4.24 that the map $\mathfrak{Y}$ is separated. The projection map $\mathfrak{Y} \to \operatorname{Spec} R$ is étale and therefore locally quasi-finite. It follows from Theorem 1.2.1 that $\mathfrak{Y}$ is quasi-affine. □

Remark 1.3.7. In the situation of Proposition 1.3.6, Remark VIII.2.4.2 implies that the fiber product $\operatorname{Spec} R \times_{\mathfrak{X}} \operatorname{Spec} R'$ is schematic. If we let $\mathcal{C}$ denote the full subcategory of Stk spanned by the quasi-separated, 0-truncated spectral algebraic spaces, the $\mathcal{C}$ is equivalent to (the nerve of) the category of algebraic spaces defined in [31].

We can now state the main result of this section.

Theorem 1.3.8. *Let $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a spectral Deligne-Mumford stack. Then $\mathfrak{Y}$ admits a scallop decomposition if and only if it is a quasi-compact, quasi-separated spectral algebraic space.*

We will give the proof of Theorem 1.3.8 at the end of this section. First, let us collect some consequences.

Corollary 1.3.9. *Let $f : \mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a map of spectral Deligne-Mumford stacks. Assume that f is quasi-compact, quasi-separated, and that every fiber product $\operatorname{Spec}^{ét} R \times_{\mathfrak{Y}} \mathfrak{X}$ is a spectral algebraic space. Then:*

(1) *The pushforward functor $f_* : \operatorname{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \operatorname{Mod}_{\mathcal{O}_{\mathcal{Y}}}$ carries quasi-coherent sheaves to quasi-coherent sheaves.*

(2) *The induced functor $\operatorname{QCoh}(\mathfrak{X}) \to \operatorname{QCoh}(\mathfrak{Y})$ commutes with small colimits.*

(3) *For every pullback diagram*

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{g'} & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}, \end{array}$$

the associated diagram of ∞-categories

$$\begin{array}{ccc} \operatorname{QCoh}(\mathfrak{Y}) & \xrightarrow{f^*} & \operatorname{QCoh}(\mathfrak{X}) \\ \downarrow & & \downarrow \\ \operatorname{QCoh}(\mathfrak{Y}') & \xrightarrow{f'^*} & \operatorname{QCoh}(\mathfrak{X}') \end{array}$$

is right adjointable. In other words, for every object $\mathcal{F} \in \operatorname{QCoh}(\mathfrak{X})$, the canonical map $\lambda : g^* f_* \mathcal{F} \to f'_* g'^* \mathcal{F}$ is an equivalence in $\operatorname{QCoh}(\mathfrak{Y}')$.

Proof. Combine Theorem 1.1.1 with Propositions VIII.2.5.12 and VIII.2.5.14. □

Corollary 1.3.10. *Let $\mathfrak{X}$ be a quasi-compact quasi-separated spectral algebraic space. Then there exists an integer n such that the global sections functor $\Gamma : \operatorname{QCoh}(\mathfrak{X}) \to \mathrm{Sp}$ carries $\operatorname{QCoh}(\mathfrak{X})_{\geq 0}$ into $\mathrm{Sp}_{\geq -n}$.*

Proof. Combine Proposition VIII.2.5.13 with Theorem 1.3.8. □

Definition 1.3.11. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. We will say that $\mathfrak{X}$ is *empty* if $\mathcal{X}$ is equivalent to the ∞-topos $\mathcal{S}\mathrm{hv}(\emptyset)$: that is, if $\mathcal{X}$ is a contractible Kan complex. Otherwise, we will say that $\mathfrak{X}$ is *nonempty.*

Corollary 1.3.12. *Let $\mathfrak{X}$ be a quasi-separated spectral algebraic space. If $\mathfrak{X}$ is nonempty, then there exists an open immersion $j : \mathrm{Spec}^{\acute{e}t} R \to \mathfrak{X}$ for some nonzero connective $\mathbb{E}_\infty$-ring R.*

Proof. Replacing $\mathfrak{X}$ by an open substack if necessary, we may suppose that $\mathfrak{X}$ is quasi-compact. Choose a scallop decomposition

$$\mathfrak{U}_0 \to \mathfrak{U}_1 \to \cdots \to \mathfrak{U}_n$$

of $\mathfrak{X}$. Let i be the smallest integer such that $\mathfrak{U}_i$ is nonempty. Then $\mathfrak{U}_i$ is an affine open substack of $\mathfrak{X}$. □

Another consequence of Theorem 1.3.8 is that it is possible to choose a "Nisnevich neighborhood" around any point of quasi-separated spectral algebraic space.

Corollary 1.3.13. *Let $\mathfrak{Y}$ be a quasi-separated spectral algebraic space. Let k be a field, and suppose we are given a map $\eta : \mathrm{Spec}^{\acute{e}t} k \to \mathfrak{Y}$. Then η admits a factorization*

$$\mathrm{Spec}^{\acute{e}t} k \xrightarrow{\eta'} \mathrm{Spec}^{\acute{e}t} R \xrightarrow{\eta''} \mathfrak{Y},$$

where η' is étale.

Remark 1.3.14 (Projection Formula)**.** Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a quasi-compact, quasi-separated relative spectral algebraic space, and suppose we are given quasi-coherent sheaves $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ and $\mathcal{G} \in \mathrm{QCoh}(\mathfrak{Y})$. The counit map $f^* f_* \mathcal{F} \to \mathcal{F}$ induces a morphism $f^*(f_* \mathcal{F} \otimes \mathcal{G}) \simeq f^* f_* \mathcal{F} \otimes f^* \mathcal{G} \to \mathcal{F} \otimes f^* \mathcal{G}$, which is adjoint to a map

$$\theta : f_* \mathcal{F} \otimes \mathcal{G} \to f_*(\mathcal{F} \otimes f^* \mathcal{G}).$$

We claim that θ is an equivalence. To prove this, we may work locally on $\mathfrak{Y}$ and thereby reduce to the case where $\mathfrak{Y} = \mathrm{Spec}^{\text{ét}} R$ is affine. The collection of those objects $\mathcal{G} \in \mathrm{QCoh}(\mathfrak{Y}) \simeq \mathrm{Mod}_R$ for which θ is an equivalence is stable under shifts and colimits in $\mathrm{QCoh}(\mathfrak{Y})$. It will therefore suffice to show that θ is an equivalence in the special case where $\mathcal{G}$ corresponds to the unit object $R \in \mathrm{QCoh}(\mathfrak{Y})$, which is obvious.

We now turn to the proof of Theorem 1.3.8.

Lemma 1.3.15. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-affine spectral Deligne-Mumford stack acted on by a finite group G. Assume that the action of G is free in the following sense: for every nonzero commutative ring R, G acts freely on the set $\pi_0 \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{X})$. Then there exist a finite collection of G-equivariant (-1)-truncated objects $\{U_i \in \mathcal{X}\}_{1 \le i \le n}$ with the following properties:*

1. *For $1 \le i \le n$, let $\mathfrak{U}_i$ denote the open substack $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}} | U_i)$ of $\mathfrak{X}$. The the quotients $\mathfrak{U}_i / G$ are affine.*
2. *The objects U_i cover $\mathcal{X}$. That is, if $\mathbf{1}$ denotes a final object of $\mathcal{X}$, then the canonical map $\coprod_{1 \le i \le n} U_i \to \mathbf{1}$ is an effective epimorphism.*

Proof. Let R denote the connective cover of the $\mathbb{E}_\infty$-ring $\Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}}) \simeq \mathcal{O}_{\mathcal{X}}(\mathbf{1})$. Since $\mathfrak{X}$ is quasi-affine, the canonical map $j : \mathfrak{X} \to \mathrm{Spec}^{\text{ét}} R$ is an open immersion (Proposition VIII.2.4.3), classified by some quasi-compact open subset $U \subseteq \mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)$. Then $U = \{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}}(\pi_0 R) : I \nsubseteq \mathfrak{p}\}$ for some radical ideal $I \subseteq \pi_0 R$. Note that the finite group G acts on the commutative ring $\pi_0 R$ and the ideal I is G-invariant. For every point $\mathfrak{p} \in U$, none of the prime ideals $\{\sigma(\mathfrak{p})\}_{\sigma \in G}$ contains I. Consequently, there exists an element $x \in I$ such that $\sigma(x) \notin \mathfrak{p}$ for each $\sigma \in G$. Replacing x by $\prod_{\sigma \in G} \sigma(x)$ if necessary, we may suppose that x is G-invariant. Let $U_{\mathfrak{p}} = \{\mathfrak{q} \in \mathrm{Spec}^{\mathrm{Z}}(\pi_0 R) : x \notin \mathfrak{q}\}$. Then $U_{\mathfrak{p}}$ is an open subset of U containing the point $\mathfrak{p}$. The collection of open sets $\{U_{\mathfrak{p}}\}_{\mathfrak{p} \in U}$ is an open covering of U. Since U is quasi-compact, there exists a finite subcovering by open sets $U_{\mathfrak{p}_1}, \ldots, U_{\mathfrak{p}_n}$, which we can identify with (-1)-truncated objects $U_1, \ldots, U_n \in \mathcal{X}$. It is clear that these objects satisfy condition (2). To verify (1), we note that each of the open substacks $\mathfrak{U}_i = (\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}} | U_i)$ of $\mathfrak{X}$ has the form $\operatorname{Spec}^{\mathbf{Z}} R[\frac{1}{x}]$ for some G-invariant element $x \in \pi_0 R$. Since G acts freely on $\mathfrak{X}$, it also acts freely on the open substack $\mathfrak{U}_i$, so that $\mathfrak{U}_i / G$ is affine by virtue of Lemma 1.3.15. □

Lemma 1.3.16. *Let $u : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral algebraic spaces. If $\mathfrak{X}$ is separated, then u is strongly separated.*

Proof. The map u factors as a composition

$$\mathfrak{X} \xrightarrow{u'} \mathfrak{X} \times \mathfrak{Y} \xrightarrow{u''} \mathfrak{Y}.$$

Since $\mathfrak{X}$ is separted, u'' is strongly separated. The map u' is a pullback of the diagonal map $\delta : \mathfrak{Y} \to \mathfrak{Y} \times \mathfrak{Y}$. Since $\mathfrak{Y}$ is a spectral algebraic space, Example IX.4.24 implies that δ is strongly separated. It follows that u' is strongly separated, so that $u = u'' \circ u'$ is also strongly separated. □

Lemma 1.3.17. *Let $j : \mathfrak{U} \to \mathfrak{X}$ be a map of spectral Deligne-Mumford stacks. Assume that j is strongly separated, quasi-compact, and that for every map* $\operatorname{Spec} k \to \mathfrak{X}$ *where k is a field, the fiber product* $\mathfrak{U} \times_{\mathfrak{X}} \operatorname{Spec} k$ *is either empty or equivalent to* $\operatorname{Spec} k$*. Then j is an open immersion.*

Proof. The assertion is local on $\mathfrak{X}$, so we may assume that $\mathfrak{X}$ is affine. In this case, Theorem 1.2.1 implies that $\mathfrak{U}$ is quasi-affine. Choose a covering of $\mathfrak{U}$ by affine open substacks $\mathfrak{U}_i$, and for each index i let $\mathfrak{V}_i$ be the open substack of $\mathfrak{X}$ given by the image of $\mathfrak{U}_i$. Then each $\mathfrak{V}_i$ is quasi-affine and therefore a separated spectral algebraic space. Since $\mathfrak{U}_i$ is affine, the maps $u_i : \mathfrak{U}_i \to \mathfrak{V}_i$ are affine and étale. Our condition on the fibers of j guarantee that each u_i is finite étale of rank 1 and therefore an equivalence. It follows that j induces an equivalence from $\mathfrak{U}$ to the open substack of $\mathfrak{X}$ given by the union of the open substacks $\mathfrak{V}_i$. □

We will say that a diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \widetilde{\mathfrak{U}} & \xrightarrow{j} & \widetilde{\mathfrak{Y}} \\ \downarrow{\scriptstyle g} & & \downarrow \\ \mathfrak{U} & \longrightarrow & \mathfrak{Y} \end{array}$$

is an *excision square* if it is a psuhout square where j is an open immersion and g is étale (see Definition VIII.2.5.4).

Lemma 1.3.18. *Let $\mathfrak{Y}$ be a spectral Deligne-Mumford stack. Suppose that there exists an excision square of spectral Deligne-Mumford stacks σ :*

$$\begin{array}{ccc} \widetilde{\mathfrak{U}} & \longrightarrow & \widetilde{\mathfrak{Y}} \\ \downarrow & & \downarrow \\ \mathfrak{U} & \longrightarrow & \mathfrak{Y} \end{array}$$

where $\mathfrak{U}$ is a quasi-compact quasi-separated spectral algebraic space, $\widetilde{\mathfrak{Y}}$ is affine, and $\widetilde{\mathfrak{U}}$ is quasi-compact. Then $\mathfrak{Y}$ is a quasi-compact quasi-separated spectral algebraic space.

Proof. The map $\mathfrak{U} \coprod \widetilde{\mathfrak{Y}} \to \mathfrak{Y}$ is an étale surjection. Since $\widetilde{\mathfrak{Y}}$ and $\mathfrak{U}$ are quasi-compact, it follows immediately that $\mathfrak{Y}$ is quasi-compact. We next prove that $\mathfrak{Y}$ is quasi-separated. Choose maps $\mathfrak{V}_0, \mathfrak{V}_1 \to \mathfrak{Y}$, where $\mathfrak{V}_0$ and $\mathfrak{V}_1$ are affine. We wish to prove that the fiber product $\mathfrak{V}_0 \times_{\mathfrak{Y}} \mathfrak{V}_1$ is quasi-compact. Passing to an étale covering of $\mathfrak{V}_0$ and $\mathfrak{V}_1$ if necessary we may suppose that the maps $\mathfrak{V}_i \to \mathfrak{Y}$ factor through either $\mathfrak{U}$ or $\widetilde{\mathfrak{Y}}$. There are three cases to consider:

(a) Suppose that both of the maps $\mathfrak{V}_i \to \mathfrak{Y}$ factor through $\mathfrak{U}$. Then $\mathfrak{V}_0 \times_{\mathfrak{Y}} \mathfrak{V}_1 \simeq \mathfrak{V}_0 \times_{\mathfrak{U}} \mathfrak{V}_1$ is quasi-compact by virtue of our assumption that $\mathfrak{U}$ is quasi-separated.

(b) Suppose that the map $\mathfrak{V}_0 \to \mathfrak{Y}$ factors through $\mathfrak{U}$ and the map $\mathfrak{V}_1 \to \mathfrak{Y}$ factors through $\widetilde{\mathfrak{Y}}$.

$$\mathfrak{V}_0 \times_{\mathfrak{Y}} \mathfrak{V}_1 \simeq \mathfrak{V}_0 \times_{\mathfrak{U}} (\mathfrak{U} \times_{\mathfrak{Y}} \mathfrak{V}_1) \simeq \mathfrak{V}_0 \times_{\mathfrak{U}} (\widetilde{\mathfrak{U}} \times_{\widetilde{\mathfrak{Y}}} \mathfrak{V}_1).$$

Since $\widetilde{\mathfrak{U}}$ is quasi-compact and $\widetilde{\mathfrak{Y}}$ is quasi-separated, the fiber product $\widetilde{\mathfrak{U}} \times_{\widetilde{\mathfrak{Y}}} \mathfrak{V}_1$ is quasi-compact. Using the quasi-separateness of $\mathfrak{U}$ we deduce that $\mathfrak{V}_0 \times_{\mathfrak{Y}} \mathfrak{V}_1$ is quasi-compact.

(c) Suppose that both of the maps $\mathfrak{V}_i \to \mathfrak{Y}$ factor through $\widetilde{\mathfrak{Y}}$. Since σ is an excision square, the map

$$\widetilde{\mathfrak{Y}} \coprod (\widetilde{\mathfrak{U}} \times_{\mathfrak{Y}} \widetilde{\mathfrak{U}}) \to \widetilde{\mathfrak{Y}} \times_{\mathfrak{Y}} \widetilde{\mathfrak{Y}}$$

is an étale surjection. We therefore obtain an étale surjection

$$(\mathfrak{V}_0 \times_{\widetilde{\mathfrak{Y}}} \mathfrak{V}_1) \coprod ((\mathfrak{V}_0 \times_{\widetilde{\mathfrak{Y}}} \widetilde{\mathfrak{U}}) \times_{\mathfrak{U}} (\mathfrak{V}_1 \times_{\widetilde{\mathfrak{Y}}} \widetilde{\mathfrak{U}})) \to \mathfrak{V}_0 \times_{\mathfrak{Y}} \mathfrak{V}_1 .$$

The fiber product $\mathfrak{V}_0 \times_{\widetilde{\mathfrak{Y}}} \mathfrak{V}_1$ is affine and therefore quasi-compact. Since $\widetilde{\mathfrak{U}}$ is quasi-compact, the fiber products $\mathfrak{V}_i \times_{\widetilde{\mathfrak{Y}}} \widetilde{\mathfrak{U}}$ are quasi-compact. Using the quasi-separateness of $\mathfrak{U}$, we deduce that

$$(\mathfrak{V}_0 \times_{\widetilde{\mathfrak{Y}}} \widetilde{\mathfrak{U}}) \times_{\mathfrak{U}} (\mathfrak{V}_1 \times_{\widetilde{\mathfrak{Y}}} \widetilde{\mathfrak{U}})$$

is quasi-compact, so that $\mathfrak{V}_0 \times_{\mathfrak{Y}} \mathfrak{V}_1$ is quasi-compact.

It remains to prove that $\mathfrak{Y}$ is a spectral algebraic space. We may assume without loss of generality that $\mathfrak{Y}$ is connective; we wish to show that the mapping space $\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{Y})$ is discrete for every commutative ring R. For every map $f : \mathrm{Spec}\, R \to \mathfrak{Y}$, the fiber product $\mathfrak{U} \times_{\mathfrak{Y}} \mathrm{Spec}\, R$ is an open substack of $\mathrm{Spec}\, R$ corresponding to an open subset $V_f \subseteq \mathrm{Spec}^{\mathrm{Z}} R$. Fix an open set $V \subseteq \mathrm{Spec}^{\mathrm{Z}} R$, and let $\mathrm{Map}^V_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{Y})$ denote the summand of $\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{Y})$ spanned by those maps f with $V_f = V$. Then $\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{Y}) \simeq \coprod_V \mathrm{Map}^V_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{Y})$, so it will suffice to show that each $\mathrm{Map}^V_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{Y})$ is discrete.

Let $\mathfrak{V}$ denote the open substack of $\mathrm{Spec}\, R$ corresponding to V. Write $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ and $\mathrm{Spec}\, R = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, so we can identify V with a (-1)-truncated object of $\mathcal{X}$. The étale map $\widetilde{\mathfrak{Y}} \to \mathfrak{Y}$ determines an object $\widetilde{Y} \in \mathcal{Y}$. Every map $f : \mathrm{Spec}\, R \to \mathfrak{Y}$ determines an object $f^* \widetilde{Y} \in \mathcal{X}$. This construction determines a functor θ fitting into a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}^V_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{Y}) & \longrightarrow & \mathrm{Map}_{\mathrm{Stk}}(\mathfrak{V}, \mathfrak{U}) \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta_0} \\ \mathcal{X} & \longrightarrow & \mathcal{X}_{/V} . \end{array}$$

Since $\widetilde{\mathfrak{U}} \to \mathfrak{U}$ is a map between spectral algebraic spaces, the homotopy fibers of the induced map

$$\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R', \widetilde{\mathfrak{U}}) \to \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R', \mathfrak{U})$$

are discrete for every étale R-algebra R'. It follows that θ_0 factors through the full subcategory $\tau_{\leq 0} \mathcal{X}_{/V} \subseteq \mathcal{X}_{/V}$ spanned by the discrete objects. Let $\mathcal{X}_0$ denote the full subcategory of $\mathcal{X}$ spanned by those objects X such that the image of X in $\mathcal{X}/V$ is a final object, so that θ factors through $\mathcal{X}_0$. Using Proposition A.A.8.15, we deduce that the homotopy fiber of the forgetful functor $\mathcal{X}_0 \to \mathcal{X}_{/V}$ over an object $\widetilde{V} \in \mathcal{X}_{/V}$ can be identified with the space $\mathrm{Map}_{\mathcal{X}_{/V}}(V, \widetilde{V})$; in particular, it is discrete if $\widetilde{V}$ is discrete. It follows that the map

$$\mathrm{Map}_{\mathrm{Stk}}(\mathfrak{V}, \mathfrak{U}) \times_{\mathcal{X}_{/V}} \mathcal{X} \to \mathrm{Map}_{\mathrm{Stk}}(\mathfrak{V}, \mathfrak{U})$$

has discrete homotopy fibers. Since the structure sheaf of $\mathfrak{V}$ is discrete and $\mathfrak{U}$ is a spectral algebraic space, the mapping space $\mathrm{Map}_{\mathrm{Stk}}(\mathfrak{V}, \mathfrak{U})$ is discrete. We conclude that the Kan complex $\mathrm{Map}_{\mathrm{Stk}}(\mathfrak{V}, \mathfrak{U}) \times_{\mathcal{X}_{/V}} \mathcal{X}$ is discrete. To complete the proof, it will suffice to show that the canonical map

$$\phi : \mathrm{Map}^{V}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{Y}) \to \mathrm{Map}_{\mathrm{Stk}}(\mathfrak{V}, \mathfrak{U}) \times_{\mathcal{X}_{/V}} \mathcal{X}$$

has discrete homotopy fibers. To this end, we fix an object $\widetilde{X} \in \mathcal{X}$ having image $\widetilde{V} \in \mathcal{X}_{/V}$; we will show that the map

$$\phi_{\widetilde{X}} : \mathrm{Map}^{V}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{Y}) \times_{\mathcal{X}} \{\widetilde{X}\} \to \mathrm{Map}_{\mathrm{Stk}}(\mathfrak{V}, \mathfrak{U}) \times_{\mathcal{X}_{/V}} \{\widetilde{V}\}$$

has discrete homotopy fibers. To prove this, we observe that $\phi_{\widetilde{X}}$ is a pullback of the map

$$\mathrm{Map}^{V}_{\mathrm{Stk}}(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{Y}}) \to \mathrm{Map}_{\mathrm{Stk}}(\widetilde{\mathfrak{V}}, \widetilde{\mathfrak{U}}),$$

where $\mathrm{Map}^{V}_{\mathrm{Stk}}(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{Y}})$ is the summand of $\mathrm{Map}_{\mathrm{Stk}}(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{Y}})$ corresponding to those maps satisfying $\widetilde{\mathfrak{V}} \simeq \widetilde{\mathfrak{U}} \times_{\widetilde{\mathfrak{Y}}} \widetilde{\mathfrak{X}}$. It now suffices to observe that $\mathrm{Map}^{V}_{\mathrm{Stk}}(\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{Y}})$ and $\mathrm{Map}_{\mathrm{Stk}}(\widetilde{\mathfrak{V}}, \widetilde{\mathfrak{U}})$ are both discrete, since both $\widetilde{\mathfrak{X}}$ and $\widetilde{\mathfrak{V}}$ have discrete structure sheaves and both $\widetilde{\mathfrak{Y}}$ and $\widetilde{\mathfrak{U}}$ are spectral algebraic spaces. □

Proof of Theorem 1.3.8. If $\mathfrak{Y}$ is a spectral Deligne-Mumford stack which admits a scallop decomposition, then Lemma 1.3.16 immediately implies that $\mathfrak{Y}$ is a quasi-compact, quasi-separated spectral algebraic space (using induction on the length of the scallop decomposition). To prove the converse, we use a slightly more complicated version of the proof of Proposition 1.1.1. Assume that $\mathfrak{Y}$ is a quasi-compact, quasi-separated spectral algebraic space. Since $\mathfrak{Y}$ is quasi-compact, we can choose an étale surjection $u : \mathfrak{X} \to \mathfrak{Y}$ where $\mathfrak{X}$ is affine. Lemma 1.3.16 implies that u is strongly separated. For $i \geq 1$, each of the evaluation maps $\mathrm{Conf}^{i}_{\mathfrak{Y}}(\mathfrak{X}) \to \mathfrak{X}$ is étale, strongly separated, and quasi-compact (since $\mathfrak{Y}$ is assumed to be quasi-separated). Since $\mathfrak{X}$ is affine, we conclude that $\mathrm{Conf}^{i}_{\mathfrak{Y}}(\mathfrak{X})$ is quasi-affine (Theorem 1.2.1). Using the quasi-compactness of $\mathfrak{Y}$, we deduce the existence of an integer n such that $\mathrm{Conf}^{n+1}_{\mathfrak{Y}}(\mathfrak{X})$ is empty. For $0 \leq i < n$, we can use Lemma 1.3.15 to obtain a finite covering of $\mathrm{Conf}^{n-i}_{\mathfrak{Y}}(\mathfrak{X})$ by Σ_i-invariant open substacks $\{\mathfrak{U}_{i,j}\}_{1 \leq j \leq m_i}$ such that each quotient $\mathfrak{U}_{i,j}/\Sigma_{n-i}$ is affine. Let $m = \sum_{0 \leq i < n} m_i$. If $1 \leq k \leq m$, then we can write $k = m_0 + \cdots + m_{i-1} + j$ where $1 \leq j \leq m_i$, and we let $\mathfrak{U}_k$ denote the spectral Deligne-Mumford stack $\mathfrak{U}_{i,j}$. For $0 \leq k \leq m$, we let $\mathfrak{V}_k$ denote the open substack of $\mathfrak{Y}$ given by the image of the étale map $\coprod_{1 \leq k' \leq k} \mathfrak{U}_{k'} \to \mathfrak{Y}$. We claim that the sequence of open immersions

$$\mathfrak{V}_0 \to \mathfrak{V}_1 \to \cdots \to \mathfrak{V}_m$$

is a scallop decomposition of $\mathfrak{Y}$. Since u is surjective, it is clear that $\mathfrak{V}_m \simeq \mathfrak{Y}$, and $\mathfrak{V}_0$ is empty by construction. Let $0 < k \leq m$, and write $k = m_0 + \cdots + m_{i-1} + j$ for $1 \leq j \leq m_i$. Form a pullback square

$$\begin{array}{ccc} \mathfrak{W} & \longrightarrow & \mathfrak{U}_{i,j}/\Sigma_{n-i} \\ \downarrow & & \downarrow{\scriptstyle q} \\ \mathfrak{V}_{k-1} & \longrightarrow & \mathfrak{V}_k . \end{array}$$

We claim that this diagram is an excision square. To prove this, can replace $\mathfrak{Y}$ by the reduced closed substack complementary to $\mathfrak{V}_{k-1}$, and thereby reduce to the case where $\mathrm{Conf}^{n-i+1}_{\mathfrak{Y}}(\mathfrak{X})$ is empty. In this case, we wish to show that q is an equivalence. Since q is an étale surjection by construction, it suffices to show that the map $\mathfrak{U}_{i,j}/\Sigma_{n-i} \to \mathfrak{Y}$ is an open immersion. In fact, we claim that the map $j : \mathrm{Conf}^{n-i}_{\mathfrak{Y}}(\mathfrak{X})/\Sigma_{n-i} \to \mathfrak{Y}$ is an open immersion: this follows from Lemma 1.3.17 (since $\mathrm{Conf}^{n-i+1}_{\mathfrak{Y}}(\mathfrak{X})$ is empty). □

1.4 Points of Spectral Algebraic Spaces

Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. The collection of equivalence classes of open substacks of $\mathfrak{X}$ forms a locale (namely, the locale consisting of (-1)-truncated objects of $\mathcal{X}$; see §T.6.4.2). We let $|\mathfrak{X}|$ denote the collection of all points of this locale. More concretely, an element $x \in |\mathfrak{X}|$ is given by a collection F_x of open substacks of $\mathfrak{X}$ having the following properties:

(a) If $\mathfrak{U} \subseteq \mathfrak{X}$ belongs to F_x, then so does any larger open substack of $\mathfrak{X}$.

(b) Given a finite collection of open substacks $\mathfrak{U}_i \subseteq \mathfrak{X}$ belonging to F_x, the intersection $\bigcap \mathfrak{U}_i$ belongs to F_x.

(c) Given an arbitrary collection of open substacks $\mathfrak{U}_\alpha \subseteq \mathfrak{X}$, if the union $\bigcup_\alpha \mathfrak{U}_\alpha$ belongs to F_x, then $\mathfrak{U}_\alpha$ belongs to F_x for some α.

Here we should think of F_x as the collection of those open substacks of $\mathfrak{X}$ which contain the point x. We will regard $|\mathfrak{X}|$ as a topological space, with open sets given by $\{x \in |\mathfrak{X}| : \mathfrak{U} \in F_x\}$, where $\mathfrak{U}$ ranges over open substacks of $\mathfrak{X}$.

Remark 1.4.1. All of the results in this section follow immediately from their counterparts in the classical theory of algebraic spaces (see, for example, [31]). We include proofs here for the sake of completeness.

Remark 1.4.2. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a spectral Deligne-Mumford stack. The ∞-topos $\mathcal{X}$ is locally coherent, so that the hypercompletion $\mathcal{X}^\wedge$ has enough points (Theorem VII.4.1). It follows immediately that the locale of open substacks of $\mathfrak{X}$ has enough points: that is, there is a one-to-one correspondence between open subsets of $|\mathfrak{X}|$ and equivalence classes of open substacks of $\mathfrak{X}$.

Example 1.4.3. Let R be an $\mathbb{E}_\infty$-ring. Then there is a bijective correspondence between equivalence classes of open substacks of $\operatorname{Spec}^{\text{ét}} R$ and open subsets of the topological space $\operatorname{Spec}^{\mathrm{Z}} \pi_0 R$ (see Lemma VII.9.7). Since every irreducible closed subset of $\operatorname{Spec}^{\mathrm{Z}} \pi_0 R$ has a unique generic point, we obtain a canonical homeomorphism $|\operatorname{Spec}^{\text{ét}} R| \simeq \operatorname{Spec}^{\mathrm{Z}} \pi_0 R$. In particular, if R is a field, then the topological space $|\operatorname{Spec}^{\text{ét}} R|$ consists of a single point.

Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a spectral Deligne-Mumford stack. Every point of the ∞-topos $\mathcal{X}$ determines a point of the topological space $|\mathfrak{X}|$. This observation determines a map $\theta : \pi_0 \operatorname{GPt}(\mathfrak{X}) \to |\mathfrak{X}|$, where $\operatorname{GPt}(\mathfrak{X})$ denotes the space of geometric points of $\mathfrak{X}$ (Proposition VIII.1.1.15). In good cases, one can show that the map θ is bijective. One of our goals in this section is prove this in the case where $\mathfrak{X}$ is a quasi-separated spectral algebraic space. To do so, it will be convenient to describe the elements of $|\mathfrak{X}|$ in a different way.

Definition 1.4.4. Let $\mathfrak{X}$ be a spectral algebraic space. A *point* of $\mathfrak{X}$ is a map $\eta : \mathfrak{X}_0 \to \mathfrak{X}$ with the following properties:

(1) The object $\mathfrak{X}_0 \in \mathrm{Stk}$ is equivalent to the spectrum of a field k.

(2) For every commutative ring R, the map

$$\operatorname{Hom}_{\mathrm{Ring}}(k, R) \simeq \operatorname{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X}_0) \to \operatorname{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, X)$$

is injective.

Example 1.4.5. Let R be a connective $\mathbb{E}_\infty$-ring. A map $\eta : \operatorname{Spec} k \to \operatorname{Spec} R$ is a point of R if and only if k is a field and η induces a map $R \to k$ which exhibits k as the residue field of the commutative ring $\pi_0 R$ at some prime ideal $\mathfrak{p} \subseteq \pi_0 R$.

Remark 1.4.6. Let $\mathfrak{X}$ be a quasi-separated spectral algebraic space, and suppose we are given a point $\eta : \operatorname{Spec} k \to \mathfrak{X}$. Choose any étale map $u : \operatorname{Spec} R \to \mathfrak{X}$, and let $\mathfrak{Y}$ denote the fiber product $\operatorname{Spec} k \times_\mathfrak{X} \operatorname{Spec} R$. Since the diagonal of $\mathfrak{X}$ is quasi-affine, $\mathfrak{Y}$ has the form $\operatorname{Spec} k'$ for some étale k-algebra k'. We may therefore write k' as a finite product $\prod_\alpha k'_\alpha$, where each k'_α is a finite separable extension of the field k. Each of the induced maps $\operatorname{Spec} k'_\alpha \to \operatorname{Spec} k' \to \operatorname{Spec} R$ is a point of $\operatorname{Spec} R$, so that each k'_α can be identified with a residue field of the commutative ring $\pi_0 R$ (Example 1.4.5) at some prime ideal $\mathfrak{p}_\alpha$. Moreover, the prime ideals $\mathfrak{p}_\alpha$ are distinct from one another.

Remark 1.4.7. Let $\mathfrak{X}$ be a quasi-separated spectral algebraic space, and suppose we are given a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} k & \xrightarrow{\theta} & \operatorname{Spec} k' \\ & {\scriptstyle \eta} \searrow \quad \swarrow {\scriptstyle \eta'} & \\ & \mathfrak{X} & \end{array}$$

where η is a point of $\mathfrak{X}$ and k' is a field. Choose an étale map $u : \operatorname{Spec} R \to \mathfrak{X}$ such that $\mathfrak{Y} = \operatorname{Spec} k' \times_{\mathfrak{X}} \operatorname{Spec} R$ is nonempty, so that $\mathfrak{Y}$ has the form $\operatorname{Spec} k''$ for some nonzero étale k'-algebra k''. Then $\operatorname{Spec} k \times_{\mathfrak{X}} \operatorname{Spec} R$ is the spectrum of the commutative ring $k \otimes_{k'} k''$. It follows from Remark 1.4.6 that the composite map $\pi_0 R \to k'' \to k \otimes_{k'} k''$ is surjective. In particular, the map $k'' \to k \otimes_{k'} k''$ is surjective, so we must have $k' \simeq k$: that is, the map θ is an equivalence.

Notation 1.4.8. Let $\mathfrak{X}$ be a quasi-separated spectral algebraic space, and let $\operatorname{Pt}(\mathfrak{X})$ be the full subcategory of $\operatorname{Stk}_{/\mathfrak{X}}$ spanned by the points of $\mathfrak{X}$. Remark 1.4.7 implies that $\operatorname{Pt}(\mathfrak{X})$ is a Kan complex, and it follows immediately from the definition that all mapping spaces in $\operatorname{Pt}(\mathfrak{X})$ are either empty or contractible. It follows that $\operatorname{Pt}(\mathfrak{X})$ is homotopy equivalent to the discrete space $\pi_0 \operatorname{Pt}(\mathfrak{X})$. We will generally abuse notation by identifying $\operatorname{Pt}(\mathfrak{X})$ with $\pi_0 \operatorname{Pt}(\mathfrak{X})$. If $\eta \in \operatorname{Pt}(\mathfrak{X})$ corresponds to a morphism $\operatorname{Spec}^{\text{ét}} k \to \mathfrak{X}$, we will refer to k as the *residue field* of $\mathfrak{X}$ at the point η, and denote it by $\kappa(\eta)$.

Remark 1.4.9. Let $i : \mathfrak{X}_0 \to \mathfrak{X}$ be a closed immersion of quasi-separated spectral algebraic spaces, and let $j : \mathfrak{U} \to \mathfrak{X}$ be the complementary open immersion. Then the induced maps $\pi_0 \operatorname{Pt}(\mathfrak{X}_0) \to \pi_0 \operatorname{Pt}(\mathfrak{X})$ and $\pi_0 \operatorname{Pt}(\mathfrak{U}) \to \pi_0 \operatorname{Pt}(\mathfrak{X})$ induce a bijection $\pi_0 \operatorname{Pt}(\mathfrak{X}_0) \amalg \pi_0 \operatorname{Pt}(\mathfrak{U}) \to \pi_0 \operatorname{Pt}(\mathfrak{X})$.

Let $\mathfrak{X}$ be a spectral algebraic space. For every point $\eta : \operatorname{Spec}^{\text{ét}} k \to \mathfrak{X}$, the induced map $|\operatorname{Spec}^{\text{ét}} k| \to |\mathfrak{X}|$ determines an element of $|\mathfrak{X}|$ (see Example 1.4.3). This construction determines a map of sets $\pi_0 \operatorname{Pt}(\mathfrak{X}) \to |\mathfrak{X}|$. Under mild hypotheses, this map is bijective:

Proposition 1.4.10. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-separated spectral algebraic space. Then construction above determines a bijection $\theta : \pi_0 \operatorname{Pt}(\mathfrak{X}) \to |\mathfrak{X}|$.*

Proof. The topological space $|\mathfrak{X}|$ is sober: that is, every irreducible closed subset of $|\mathfrak{X}|$ has a unique generic point. It will therefore suffice to show that for every irreducible closed subset $K \subseteq |\mathfrak{X}|$, there is a unique equivalence class of points $\eta : \operatorname{Spec}^{\text{ét}} k \to \mathfrak{X}$ which determine a generic point of K. Using Proposition IX.4.29, we can assume that K is the image of $|\mathfrak{X}_0|$ for some closed immersion $\mathfrak{X}_0 \to \mathfrak{X}$. Replacing $\mathfrak{X}$ by $\mathfrak{X}_0$, we may suppose $\mathfrak{X}$ is reduced and that $|\mathfrak{X}|$ is itself irreducible. In particular, $\mathfrak{X}$ is nonempty; we may therefore choose an open immersion $j : \operatorname{Spec}^{\text{ét}} R \to \mathfrak{X}$ for some nonzero $\mathbb{E}_\infty$-ring R (Corollary 1.3.12). Since $\mathfrak{X}$ is reduced, R is an ordinary commutative ring. Then the Zariski spectrum $\operatorname{Spec}^{\mathrm{Z}} R$ is homeomorphic to nonempty open subset of $|\mathfrak{X}|$ and is therefore irreducible. It follows that R is an integral domain. Let $\eta \in |\mathfrak{X}|$ be the image of the zero ideal $(0) \in \operatorname{Spec}^{\mathrm{Z}} R$, so that η corresponds to the point given by the composition

$$\operatorname{Spec} k \to \operatorname{Spec} R \to \mathfrak{X}$$

where k is the fraction field of R. We claim that η is a generic point of $|\mathfrak{X}|$: that is, that η belongs to every nonempty open subset $V \subseteq |\mathfrak{X}|$. To see this, we note that because $|\mathfrak{X}|$ is irreducible, the inverse image of V in $\operatorname{Spec}^{\mathrm{Z}} R$ is nonempty and therefore contains the ideal (0). This proves the surjectivity of θ. To prove injectivity, let us suppose we are given any other point $\eta' : \operatorname{Spec}^{\text{ét}} k' \to \mathfrak{X}$ which determines a generic point of $|\mathfrak{X}'|$. Since η' determines a generic point of $|\mathfrak{X}|$, it must factor through the nonempty open substack $\operatorname{Spec}^{\text{ét}} R$ of $\mathfrak{X}$. We may therefore identify k' with the residue field of R at some prime ideal $\mathfrak{p} \subseteq R$. which belongs to every nonempty open subset of $\operatorname{Spec}^{\mathrm{Z}} R$. It follows that for every nonzero element $x \in R$, $\mathfrak{p} \in \operatorname{Spec}^{\mathrm{Z}} R[x^{-1}]$ and therefore $x \notin \mathfrak{p}$. This proves that $\mathfrak{p}$ coincides with the zero ideal (0), so that $\eta' \simeq \eta$. ∎

Corollary 1.4.11. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \mathfrak{Y}' & \longrightarrow & \mathfrak{Y} \end{array}$$

of quasi-separated spectral algebraic spaces. Then the induced map $|\mathfrak{X}'| \rightarrow |\mathfrak{X}| \times_{|\mathfrak{Y}|} |\mathfrak{Y}'|$ *is a surjection of topological spaces.*

Proof. Every point $\eta : |\mathfrak{X}| \times_{|\mathfrak{Y}|} |\mathfrak{Y}'|$ can be lifted to a commutative diagram

$$\begin{array}{ccccc} \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k' & \longleftarrow & \operatorname{Spec} k'' \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{Y} & \longleftarrow & \mathfrak{Y}' \end{array}$$

where k, k', and k'' are fields. To prove that η can be lifted to a point of $|\mathfrak{X}|$, it suffices to observe that $|\operatorname{Spec} k \times_{\operatorname{Spec} k'} \operatorname{Spec} k''|$ is nonempty: that is, that commutative ring $k \otimes_{k'} k''$ is nonzero. □

We now summarize some of the formal properties enjoyed by the underlying space of a spectral algebraic space.

Proposition 1.4.12. *Let $\mathfrak{X}$ be a quasi-separated spectral algebraic space. Then:*

(1) *The topological space $|\mathfrak{X}|$ is sober, and is quasi-compact if $\mathfrak{X}$ is quasi-compact.*

(2) *The topological space $|\mathfrak{X}|$ has a basis consisting of quasi-compact open sets.*

(3) *The topological space $|\mathfrak{X}|$ is quasi-separated.*

Moreover, if $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a quasi-compact morphism of quasi-separated spectral algebraic spaces, and $U \subseteq |\mathfrak{Y}|$ is quasi-compact, then $f^{-1}U \subseteq |\mathfrak{X}|$ is quasi-compact.

Proof. Assertion (1) follows immediately from the definitions. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, and identify the collection of open sets in $|\mathfrak{X}|$ with the collection of (equivalence classes of) (-1)-truncated objects of $\mathcal{X}$. For every object $U \in \mathcal{X}$, if we write $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} |U)$, then the open subset of $|\mathfrak{X}|$ corresponding to $\tau_{\leq -1}U$ can be described as the image of the map $|\mathfrak{X}_U| \rightarrow |\mathfrak{X}|$. Since $\mathcal{X}$ is generated under small colimits by affine objects, we see that $|\mathfrak{X}|$ has a basis of open sets given by the images of maps $|\mathfrak{U}| \rightarrow |\mathfrak{X}|$, where $\mathfrak{U}$ is an affine spectral algebraic space which is étale over $\mathfrak{X}$. In this case, $|\mathfrak{U}|$ is quasi-compact by (1), so that $|\mathfrak{X}|$ has a basis of quasi-compact open sets.

We now prove that $|\mathfrak{X}|$ is quasi-separated. Suppose we are given quasi-compact open sets $U, V \subseteq |\mathfrak{X}|$; we wish to show that $U \cap V$ is quasi-compact. Without loss of generality, we may assume that U and V are the images of maps $|\mathfrak{U}| \rightarrow |\mathfrak{X}|$ and $|\mathfrak{V}| \rightarrow |\mathfrak{X}|$, where $\mathfrak{U}$ and $\mathfrak{V}$ are affine spectral algebraic spaces which are étale over $\mathfrak{X}$. Using Corollary 1.4.11, we see that $U \cap V$ is the image of the map $\theta : |\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{V}| \rightarrow |\mathfrak{X}|$. Since $\mathfrak{X}$ is quasi-separated, the fiber product $\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{V}$ is quasi-compact, so that the underlying topological space $|\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{V}|$ is also quasi-compact by (1). It follows that the image of θ is quasi-compact, as desired.

Now suppose that $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a quasi-compact morphism between quasi-separated spectral algebraic spaces and let $U \subseteq |\mathfrak{Y}|$ be a quasi-compact open set; we wish to show that its inverse image is a quasi-compact open subset of $|\mathfrak{X}|$. Without loss of generality, we may suppose that U is the image of a map $|\mathfrak{U}| \rightarrow |\mathfrak{Y}|$, where $\mathfrak{U}$ is affine and étale over $\mathfrak{Y}$. Using Corollary 1.4.11 we see that the inverse image of U is the image of the map $\theta : |\mathfrak{U} \times_{\mathfrak{Y}} \mathfrak{X}| \rightarrow |\mathfrak{X}|$. Since f is quasi-compact, $\mathfrak{U} \times_{\mathfrak{Y}} \mathfrak{X}$ is quasi-compact. It follows from (1) that the topological space $|\mathfrak{U} \times_{\mathfrak{Y}} \mathfrak{X}|$ is quasi-compact, from which it follows that the image of θ is quasi-compact. □

Remark 1.4.13. Let $\mathfrak{X}$ be a quasi-compact, quasi-separated spectral algebraic space, and let $\mathcal{U}(\mathfrak{X})$ denote the collection of quasi-compact open subsets of $|\,\mathfrak{X}\,|$. Combining Propositions 1.4.12 and A.3.14, we see $\mathcal{U}(\mathfrak{X})$ is a distributive lattice and that we can recover the topological space $|\,\mathfrak{X}\,|$ as the specturm of $\mathcal{U}(\mathcal{X})$.

Proposition 1.4.14. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a faithfully flat, quasi-compact morphism between quasi-separated spectral algebraic spaces. Then the induced map $|\,\mathfrak{X}\,| \to |\,\mathfrak{Y}\,|$ is a quotient map of topological spaces.*

Proof. Writing $\mathfrak{Y}$ as a union of its quasi-compact open substacks, we can reduce to the case where $\mathfrak{Y}$ (and therefore also $\mathfrak{X}$) is quasi-compact. Choose an étale surjection $\operatorname{Spec}^{\text{ét}} R \to \mathfrak{Y}$ and an étale surjection $\operatorname{Spec}^{\text{ét}} R' \to \operatorname{Spec}^{\text{ét}} R \times_{\mathfrak{Y}} \mathfrak{X}$. Then R' is faithfully flat over R, so that $\operatorname{Spec}^{\mathrm{Z}} R' \to \operatorname{Spec}^{\mathrm{Z}} R$ is a quotient map (Proposition VII.5.9). It will therefore suffice to show that the vertical maps appearing in the diagram

$$\begin{array}{ccc} \operatorname{Spec}^{\mathrm{Z}} R' & \longrightarrow & \operatorname{Spec}^{\mathrm{Z}} R \\ \Big\downarrow{\scriptstyle \phi'} & & \Big\downarrow{\scriptstyle \phi} \\ |\,\mathfrak{X}\,| & \longrightarrow & |\,\mathfrak{Y}\,| \end{array}$$

are quotient maps. We will prove that ϕ is a quotient map; the proof for ϕ' is similar. Fix a subset $U \subseteq \mathfrak{Y}$, and suppose that $\phi^{-1}U$ is an open subset of $\operatorname{Spec}^{\mathrm{Z}} R$. Then the inverse images of $\phi^{-1}U$ under the two projection maps

$$|\operatorname{Spec}^{\text{ét}} R \times_{\mathfrak{Y}} \operatorname{Spec}^{\text{ét}} R| \to |\operatorname{Spec}^{\text{ét}} R|$$

coincide, so that $\phi^{-1}U = \phi^{-1}V$ for some open set $V \subseteq |\,\mathfrak{Y}\,|$. Since ϕ is surjective, we obtain

$$U = \phi(\phi^{-1}U) = \phi(\phi^{-1}V) = V,$$

so that U is open. □

We next show that the construction $\mathfrak{X} \mapsto |\,\mathfrak{X}\,|$ behaves well with respect to certain filtered inverse limits.

Proposition 1.4.15. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathfrak{X}$ be a quasi-compact quasi-separated spectral algebraic space over R. For every map of connective $\mathbb{E}_\infty$-rings $R \to R'$, let $\mathfrak{X}_{R'} = \operatorname{Spec}^{\text{ét}} R' \times_{\operatorname{Spec}^{\text{ét}} R} \mathfrak{X}$, and let $\mathcal{U}_{\mathfrak{X}}(R')$ denote the distributive lattice of quasi-compact open subsets of $|\,\mathfrak{X}_{R'}\,|$. Then:*

(1) *The functor $R' \mapsto \mathcal{U}_{\mathfrak{X}}(R')$ commutes with filtered colimits.*

(2) *The functor $R' \mapsto |\,\mathfrak{X}_{R'}\,|$ carries filtered colimits of R-algebras to filtered limits of topological spaces.*

Proof. By virtue of Remarks A.3.12 and 1.4.13, assertion (2) follows from (1). We now prove (1). Since $\mathfrak{X}$ is quasi-compact, we can choose an étale surjection $\operatorname{Spec}^{\text{ét}} A^0 \to \mathfrak{X}$. Since $\mathfrak{X}$ is quasi-separated, we can choose an étale surjection $\operatorname{Spec}^{\text{ét}} A^1 \to \operatorname{Spec}^{\text{ét}} A^0 \times_{\mathfrak{X}} \operatorname{Spec}^{\text{ét}} A^0$. For every commutative ring B, let $\mathcal{U}(B)$ be defined as in Proposition A.3.34. Then for $R' \in \mathrm{CAlg}^{\mathrm{cn}}_R$, we have an equalizer diagram of sets

$$\mathcal{U}_{\mathfrak{X}}(R') \longrightarrow \mathcal{U}(\pi_0(R' \otimes_R A^0)) \rightrightarrows \mathcal{U}(\pi_0(R' \otimes_R A^1)).$$

Since $\mathcal{U}$ commutes with filtered colimits, we conclude that $\mathcal{U}_{\mathfrak{X}}$ commutes with filtered colimits. □

We close this section with a discussion of the relationship between points of a spectral algebraic space $\mathfrak{X}$ and geometric points of $\mathfrak{X}$.

Notation 1.4.16. Let $\mathfrak{X}$ be a quasi-separated spectral algebraic space. Recall that $\mathrm{GPt}(\mathfrak{X})$ denotes the full subcategory of $\mathrm{Stk}_{/\,\mathfrak{X}}$ spanned by the minimal geometric points of $\mathfrak{X}$ (see Definition VIII.1.1.10). We let $\mathrm{GPt}'(\mathfrak{X})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Stk}_{/\mathfrak{X}})$ whose objects are equivalent to commutative diagrams

$$\begin{array}{ccc} \operatorname{Spec} \overline{k} & \longrightarrow & \operatorname{Spec} k \\ & {\scriptstyle \eta'} \searrow \quad \swarrow {\scriptstyle \eta} & \\ & \mathfrak{X} & \end{array}$$

where η is a point of $\mathfrak{X}$ and η' is a minimal geometric point of $\mathfrak{X}$.

Proposition 1.4.17. *Let $\mathfrak{X}$ be a quasi-separated spectral algebraic space. Then the forgetful functor*

$$\mathrm{GPt}'(\mathfrak{X}) \to \mathrm{GPt}(\mathfrak{X})$$

is an equivalence of ∞-categories.

More informally: every geometric point of a separated spectral algebraic space $\mathfrak{X}$ determines a point of $\mathfrak{X}$.

Proof. It is clear that the forgetful functor $\theta : \mathrm{GPt}'(\mathfrak{X}) \to \mathrm{GPt}(\mathfrak{X})$ is fully faithful. We must prove that θ is essentially surjective. Fix a geometric point $\eta : \operatorname{Spec} \overline{k} \to \mathfrak{X}$. Replacing $\mathfrak{X}$ by an open substack if necessary, we may suppose that $\mathfrak{X}$ is quasi-compact. Using Theorem 1.3.8, we can choose a scallop decomposition

$$\emptyset = \mathfrak{U}_0 \to \mathfrak{U}_1 \to \cdots \to \mathfrak{U}_n \simeq \mathfrak{X}\,.$$

Let i be the smallest integer such that η factors through $\mathfrak{U}_i$. Let $\mathfrak{K}$ be the reduced closed substack of $\mathfrak{U}_i$ complementary to $\mathfrak{U}_{i-1}$. Since $\overline{k}$ is a field and η does not factor through $\mathfrak{U}_{i-1}$, it must factor through $\mathfrak{K}$. Note that $\mathfrak{K} \simeq \operatorname{Spec} R$ is affine. It follows that η factors as a composition

$$\operatorname{Spec} \overline{k} \to \operatorname{Spec} k \to \operatorname{Spec} R \to \mathfrak{U}_i \to \mathfrak{X}$$

where k is the residue field of $\pi_0 R$ at some prime ideal $\mathfrak{p} \subseteq \pi_0 R$. We now observe that the map $\operatorname{Spec} k \to \mathfrak{X}$ is a point of $\mathfrak{X}$. □

Proposition 1.4.18. *Let $\mathfrak{X}$ be a quasi-separated spectral algebraic space, let $\eta : \operatorname{Spec} k \to \mathfrak{X}$ be a point of $\mathfrak{X}$, and let $\overline{k}$ be a field extension of k. The following conditions are equivalent:*

(1) *The field $\overline{k}$ is a separable closure of k.*

(2) *The composite map*
$$\eta' : \operatorname{Spec}^{\acute{e}t} \overline{k} \to \operatorname{Spec}^{\acute{e}t} k \to \mathfrak{X}$$
is a geometric point of $\mathfrak{X}$.

Proof. We may assume without loss of generality that $\overline{k}$ is separably closed (since this follows from both (1) and (2)). Choose an étale map $u : \operatorname{Spec}^{\text{ét}} R \to \mathfrak{X}$ such that the fiber product $\operatorname{Spec}^{\text{ét}} R \times_{\mathfrak{X}} \operatorname{Spec}^{\text{ét}} k$ is nonempty. Using Remark 1.4.6, we deduce that there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} k' & \xrightarrow{u'} & \operatorname{Spec} R \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \mathfrak{X} \end{array}$$

where k' is a finite separable extension of k, and u' exhibits k' as a residue field of the commutative ring $\pi_0 R$. Since $\overline{k}$ is separably closed, we can choose a map of k-algebras $k' \to \overline{k}$, so that η' factors as a composition

$$\operatorname{Spec} \overline{k} \xrightarrow{v} \operatorname{Spec} R \xrightarrow{v'} \mathfrak{X}\,.$$

Then η' is a geometric point of $\mathfrak{X}$ if and only if v exhibits $\overline{k}$ as a separable closure of the residue field of $\pi_0 R$: that is, if and only if $\overline{k}$ is a separable closure of k'. Since k' is a separably algebraic extension of k, this is equivalent to the requirement that $\overline{k}$ be a separable closure of k. □

Combining Proposition 1.4.18, Proposition 1.4.17, and the discussion of Notation 1.4.8, we deduce:

Corollary 1.4.19. *Let $\mathfrak{X}$ be a quasi-separated spectral algebraic space. Then the ∞-category $\mathrm{GPt}(\mathfrak{X})$ is canonically equivalent to the nerve of the groupoid whose objects are pairs $(\eta, \overline{k})$, where $\eta \in \pi_0 \mathrm{Pt}(\mathfrak{X})$ is a point of $\mathfrak{X}$ and $\overline{k}$ is a separable closure of the residue field $\kappa(\eta)$.*

Corollary 1.4.20. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a surjective map between quasi-separated spectral algebraic spaces. Then the induced map $|\,\mathfrak{X}\,| \to |\,\mathfrak{Y}\,|$ is a surjection of topological spaces.*

1.5 Quasi-Coherent Stacks and Local Compact Generation

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Recall that a *quasi-coherent stack* $\mathcal{C}$ on X is a rule which assigns to each point $\eta \in X(R)$ an R-linear ∞-category $\eta^* \mathcal{C}$, depending functorially on the pair (R, η) (see §XI.8 for more details). The collection of all quasi-coherent stacks on X forms an ∞-category which we denote by $\mathrm{QStk}(X)$. If $\mathfrak{X}$ is a spectral Deligne-Mumford stack and X is the functor represented by $\mathfrak{X}$ (so that $X(R) = \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}^{\text{ét}} R, \mathfrak{X})$), then we define a *quasi-coherent stack* on $\mathfrak{X}$ to be a quasi-coherent stack on the functor X, and set $\mathrm{QStk}(\mathfrak{X}) = \mathrm{QStk}(X)$.

Our first objective in this section is to study the functorial aspects of the construction $\mathfrak{X} \mapsto \mathrm{QStk}(\mathfrak{X})$. If $f : \mathfrak{X} \to \mathfrak{Y}$ is a map of spectral Deligne-Mumford stacks, then f induces a pullback functor $f^* : \mathrm{QStk}(\mathfrak{Y}) \to \mathrm{QStk}(\mathfrak{X})$. Note that the functor f^* preserves small limits. To prove this, we can reduce to the case where $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} B$ and $\mathfrak{Y} = \mathrm{Spec}^{\text{ét}} A$ are affine, in which case f^* is given by the construction

$$\mathcal{C} \mapsto \mathrm{Mod}_B \otimes_{\mathrm{Mod}_A} \mathcal{C} \simeq \mathrm{LMod}_B(\mathcal{C})$$

(see Theorem A.6.3.4.6).

Proposition 1.5.1. (1) *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. Then the pullback functor $f^* : \mathrm{QStk}(\mathfrak{Y}) \to \mathrm{QStk}(\mathfrak{X})$ admits a right adjoint, which we will denote by f_*.*

(2) *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{g'} & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y} \end{array}$$

of spectral Deligne-Mumford stacks. Then the associated diagram

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{f^*} & \mathfrak{X} \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\ \mathfrak{Y}' & \xrightarrow{f'^*} & \mathfrak{X}' \end{array}$$

is right adjointable: that is, the canonical natural transformation

$$g^* f_* \to f'_* g'^*$$

is an equivalence.

Proof. We first prove (1) in the special case where $\mathfrak{Y} = \operatorname{Spec}^{\text{ét}} A$ is affine. Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. For every object $U \in \mathcal{X}$, let $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} | U)$. We prove more generally that each of the pullback functors $f_U^* : \mathrm{LinCat}_A \to \mathrm{QStk}(\mathfrak{X}_U)$ admits a right adjoint $\Gamma(U; \bullet)$. Note that the collection of those objects $U \in \mathcal{X}$ for which $\Gamma(U; \bullet)$ exists is closed under colimits in $\mathcal{X}$. Using Lemma V.2.3.11, we can reduce to the case where $\mathcal{X}_U$ is affine, hence of the form $\operatorname{Spec}^{\text{ét}} B$ for some $\mathbb{E}_\infty$-ring B. In this case, $\Gamma(U; \bullet)$ can be identified with the forgetful functor $\mathrm{LinCat}_B \to \mathrm{LinCat}_A$.

We now prove (2) under the assumption that $\mathfrak{Y} = \operatorname{Spec}^{\text{ét}} A$ and $\mathfrak{Y}' = \operatorname{Spec}^{\text{ét}} A'$ are both affine. Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ as above. For each $U \in \mathcal{X}$, let U' denote the inverse image of U in the underlying ∞-topos of $\mathfrak{X}'$, and let $\Gamma(U'; \bullet) : \mathrm{QStk}(\mathfrak{X}') \to \mathrm{LinCat}_{A'}$ be defined as above. We will prove that for each $U \in \mathcal{X}$, the canonical map

$$\alpha_U : \mathrm{Mod}_{A'} \otimes_{\mathrm{Mod}_A} \Gamma(U; \mathcal{C}) \to \Gamma(U'; g'^* \mathcal{C}).$$

is an equivalence. When regarded as a functors of U, both the domain and codomain of α_U carry colimits in $\mathcal{X}$ to limits of ∞-categories. It will therefore suffice to prove that α_U is an equivalence when U is affine. We may therefore reduce to the case where $\mathfrak{X} = \operatorname{Spec}^{\text{ét}} B$, so that $\mathfrak{X}' = \operatorname{Spec}^{\text{ét}} B'$ for $B' = A' \otimes_A B$. The desired result now follows from Lemma VII.6.15, since the canonical map

$$\mathrm{Mod}_{A'} \otimes_{\mathrm{Mod}_A} \mathrm{Mod}_B \to \mathrm{Mod}_{B'}$$

is an equivalence of ∞-categories.

We now treat the general case of (1). Write $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. For each $V \in \mathcal{Y}$, write $\mathfrak{Y}_V = (\mathcal{Y}_{/V}, \mathcal{O}_{\mathcal{Y}} | V)$ and $\mathfrak{X}_V = \mathfrak{Y}_V \times_{\mathfrak{Y}} \mathfrak{X}$, and let $f_V : \mathfrak{X}_V \to \mathfrak{Y}_V$ denote the projection map. Let us say that an object $V \in \mathcal{Y}$ is *good* if the following conditions are satisfied:

(a) The functor f_V^* admits a right adjoint.

(b) For every pullback diagram

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{g'} & \mathfrak{X}_V \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f_V} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}_V \end{array}$$

where $\mathfrak{Y}'$ is affine, the associated diagram

$$\begin{array}{ccc} \mathfrak{Y}_V & \xrightarrow{f_V^*} & \mathfrak{X}_V \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\ \mathfrak{Y}' & \xrightarrow{f'^*} & \mathfrak{X}' \end{array}$$

is right adjointable.

It follows from the first part of the proof that every affine $V \in \mathcal{Y}$ is good. We next prove the following:

($*$) Let $V \to V'$ be a morphism between good objects of $\mathcal{Y}$. Then the diagram

$$\begin{array}{ccc} \mathfrak{Y}_{V'} & \xrightarrow{f_{V'}^*} & \mathfrak{X}_{V'} \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\ \mathfrak{Y}_V & \xrightarrow{f_V^*} & \mathfrak{X}_V \end{array}$$

is right adjointable.

To prove $(*)$, we must show that the canonical natural transformation $g^* f_{V'*} \to f_{V*} g'^*$. Equivalently, we must show that for any map $\eta : \mathfrak{Y}' \to \mathfrak{Y}$ where $\mathfrak{Y}'$ is affine, the induced map $\eta^* g^* f_{V'*} \to \eta^* f_{V*} g'^*$ is an equivalence. Using the assumption that V is good, we can rewrite the target of this map as $p_{1*} p_2^* g'^*$, where $p_1, p_2 : \mathfrak{Y}' \times_{\mathfrak{Y}_V} \mathfrak{X}_V \to \mathfrak{Y}'$ denote the projections onto the first and second factors, respectively. The desired result now follows from the assumption that V' is good. This completes the proof of $(*)$.

We next show that every object $Y \in \mathcal{Y}$ is good. By virtue of Lemma V.2.3.11, it will suffice to show that the collection of good objects of $\mathcal{Y}$ is stable under small colimits. Suppose that $V \in \mathcal{Y}$ is given by the colimit of a diagram of good objects $\{V_\alpha\}$; we wish to show that V is good. Let us regard the morphism f_V^* as an object of the ∞-category $\mathrm{Fun}(\Delta^1, \widehat{\mathcal{C}\mathrm{at}}_\infty)$, so that f_V^* is a limit of the morphisms $f_{V_\alpha}^*$. Let $\mathcal{Z} \subseteq \mathrm{Fun}(\Delta^1, \widehat{\mathcal{C}\mathrm{at}}_\infty)$ be the subcategory whose objects are morphisms which admit right adjoint, and whose morphisms are right adjointable squares. Using $(*)$, we see that the diagram $\alpha \mapsto f_{V_\alpha}^*$ takes values in $\mathcal{Z}$. Since the inclusion $\mathcal{Z} \hookrightarrow \mathrm{Fun}(\Delta^1, \widehat{\mathcal{C}\mathrm{at}}_\infty)$ preserves limits (Corollary A.6.2.3.18), we deduce that $f_V^* \in \mathcal{Z}$. To complete the proof that V is good, let us suppose we are given a diagram

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{g'} & \mathfrak{X}_V \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f_V} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}_V, \end{array}$$

where $\mathfrak{Y}'$ is affine. We wish to show that the natural transformation

$$g^* f_{V*} \to f'_* g'^*$$

is an equivalence. Using the first part of the proof, we see that this assertion is local on $\mathfrak{Y}'$ (with respect to the étale topology); we may therefore assume that the map g factors through $\mathfrak{Y}_{V_\alpha}$ for some index α. In this case, the desired result follows from the fact that V_α is good and that the morphism $f_{V_\alpha}^* \to f_V^*$ belongs to $\mathcal{Z}$. This completes the proof of (1). Moreover, we have proven the following version of (2):

(2′) Suppose we are given a pullback diagram

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{g'} & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y} \end{array}$$

of spectral Deligne-Mumford stacks. If $\mathfrak{Y}'$ is affine, then the canonical natural transformation

$$g^* f_* \to f'_* g'^*$$

is an equivalence.

To prove (2), suppose we are given an arbitrary pullback diagram

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{g'} & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}; \end{array}$$

we wish to show that the induced map $\theta : g^* f_* \to f'_* g'^*$ is an equivalence. To prove this, choose an arbitrary map $\eta : \mathfrak{Y}'' \to \mathfrak{Y}'$, where $\mathfrak{Y}''$ is affine, and set $\mathfrak{X}'' = \mathfrak{Y}'' \times_{\mathfrak{Y}} \mathfrak{X}$. We will show that θ induces an equivalence

$$\eta^* g^* f_* \to \eta^* f'_* g'^*.$$

This follows by applying $(2')$ to the left square and the outer rectangle of the diagram

$$\begin{array}{ccccc} \mathfrak{X}'' & \longrightarrow & \mathfrak{X}' & \xrightarrow{g'} & \mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}'' & \xrightarrow{\eta} & \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}\,. \end{array}$$

□

Notation 1.5.2. Let $\mathfrak{X}$ be an arbitrary spectral Deligne-Mumford stack, let S denote the sphere spectrum, and let $q : \mathfrak{X} \to \operatorname{Spec}^{\text{ét}} S$ be the canonical map. We will identify $\operatorname{QStk}(\operatorname{Spec}^{\text{ét}} S) \simeq \operatorname{LinCat}_S$ with the full subcategory of $\mathcal{P}r^{\mathrm{L}}$ spanned by the presentable stable ∞-categories. If $\mathcal{C} \in \operatorname{QStk}(\mathfrak{X})$ is a quasi-coherent stack on $\mathfrak{X}$, we let $\operatorname{QCoh}(\mathfrak{X}; \mathcal{C})$ denote the image of $q_* \mathcal{C}$ under this identification. We will refer to $\operatorname{QCoh} \mathfrak{X}; \mathcal{C})$ as the *∞-category of global sections of* $\mathcal{C}$.

More generally, suppose we are given a map $f : \mathfrak{X} \to \operatorname{Spec}^{\text{ét}} A$, for some connective $\mathbb{E}_\infty$-ring A. We can then identify $f_* \mathcal{C}$ with an A-linear ∞-category, whose underlying stable ∞-category is given by $q_* \mathcal{C} = \operatorname{QCoh}(\mathfrak{X}; \mathcal{C})$. We will generally abuse notation by identifying $f_* \mathcal{C}$ with $\operatorname{QCoh}(\mathfrak{X}; \mathcal{C})$. We can summarize the situation informally as follows: if $\mathfrak{X}$ is a spectral Deligne-Mumford stack over A, then $\operatorname{QCoh}(\mathfrak{X}; \mathcal{C})$ is an A-linear ∞-category for each $\mathcal{C} \in \operatorname{QStk}(\mathfrak{X})$.

If $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of spectral Deligne-Mumford stacks, then the pullback functor $f^* : \operatorname{QStk}(\mathfrak{Y}) \to \operatorname{QStk}(\mathfrak{X})$ is symmetric monoidal. It follows that the pushforward functor $f_* : \operatorname{QStk}(\mathfrak{X}) \to \operatorname{QStk}(\mathfrak{Y})$ is lax symmetric monoidal. In particular, if we let $\mathcal{Q}_{\mathfrak{X}}$ denote the unit object of $\operatorname{QStk}(\mathfrak{X})$, then $f_* \mathcal{Q}_{\mathfrak{X}}$ has the structure of a commutative algebra object of $\operatorname{QStk}(\mathfrak{Y})$, and the functor f_* induces a map

$$\operatorname{QStk}(\mathfrak{X}) \to \operatorname{Mod}_{f_* \mathcal{Q}_{\mathfrak{X}}}(\operatorname{QStk}(\mathfrak{Y})).$$

Theorem 1.5.3. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of spectral Deligne-Mumford stacks. Suppose that f is quasi-compact, quasi-separated, and a relative spectral algebraic space. Then the pushforward f_* indues an equivalence of ∞-categories*

$$G : \operatorname{QStk}(\mathfrak{X}) \to \operatorname{Mod}_{f_* \mathcal{Q}_{\mathfrak{X}}}(\operatorname{QStk}(\mathfrak{Y})).$$

Using Proposition 1.5.1, we see that the assertion of Theorem 1.5.3 is local on $\mathfrak{Y}$ (with respect to the étale topology). It therefore suffices to treat the case where $\mathfrak{Y} = \operatorname{Spec}^{\text{ét}} A$ is affine, which reduces to the following assertion:

Proposition 1.5.4. *Let $\mathfrak{X}$ be a quasi-compact, quasi-separated spectral algebraic space. Then the global sections functor* $\operatorname{QCoh}(\mathfrak{X}; \bullet)$ *induces an equivalence of ∞-categories*

$$\operatorname{QStk}(\mathfrak{X}) \to \operatorname{Mod}_{\operatorname{QCoh}(\mathfrak{X})}(\mathcal{P}r^{\mathrm{L}}).$$

Proof. Combine Corollary XI.8.9 with Theorem 1.3.8. □

We now investigate how the equivalences provided by Theorem 1.5.3 and Proposition 1.5.4 interact with finiteness conditions on quasi-coherent stacks.

Definition 1.5.5. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack and let $\mathcal{C} \in \operatorname{QStk}(\mathfrak{X})$ be a quasi-coherent stack on $\mathfrak{X}$. We will say that $\mathcal{C}$ is *locally compactly generated* if, for every map $\eta : \operatorname{Spec}^{\text{ét}} A \to \mathfrak{X}$, the pullback $\eta^* \mathcal{C} \in \operatorname{LinCat}_A$ is a compactly generated A-linear ∞-category.

Example 1.5.6. Let $\mathfrak{X} = \operatorname{Spec} A$ be an affine spectral Deligne-Mumford stack. Then a quasi-coherent stack on $\mathfrak{X}$ is locally compactly generated if and only if the corresponding A-linear ∞-category is compactly generated.

Lemma 1.5.7. *Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack and let $\mathcal{C} \in \mathrm{QStk}(\mathfrak{X})$. Suppose that there exists a surjective étale map $f : \mathfrak{X}' \to \mathfrak{X}$ such that $f^* \mathcal{C}$ is locally compactly generated. Then $\mathcal{C}$ is locally compactly generated.*

Proof. Choose a map $\eta : \mathrm{Spec}^{\text{ét}} A \to \mathfrak{X}$. Since f is surjective, we can choose a faithfully flat étale map $A \to B$ and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}^{\text{ét}} B & \xrightarrow{\eta'} & \mathfrak{X}' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}^{\text{ét}} A & \xrightarrow{\eta} & \mathfrak{X}. \end{array}$$

Since $f^* \mathcal{C}$ is locally compactly generated, the ∞-category

$$\eta'^* f^* \mathcal{C} \simeq f'^* \eta^* \mathcal{C} \simeq \mathrm{LMod}_B(\eta^* \mathcal{C})$$

is compactly generated. Using Theorem XI.6.1, we conclude that $\eta^* \mathcal{C}$ is compactly generated. $\square$

Lemma 1.5.8. *Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack, let $\mathcal{C} \in \mathrm{QStk}(\mathfrak{X})$ be locally compactly generated, and let $M \in \mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ be an object. The following conditions are equivalent:*

(1) *For every map $\eta : \mathrm{Spec}^{ét} A \to \mathfrak{X}$, the image of M is a compact object of $\eta^* \mathcal{C}$.*

(2) *For every étale map $\eta : \mathrm{Spec}^{ét} A \to \mathfrak{X}$, the image of M is a compact object of $\eta^* \mathcal{C}$.*

(3) *There exists a collection of étale maps $\eta_\alpha : \mathrm{Spec}^{ét} A_\alpha \to \mathfrak{X}$ which are jointly surjective, such that the image of M in each $\eta_\alpha^* \mathcal{C}$ is compact.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. We prove that (3) $\Rightarrow$ (1). Assume that (3) is satisfied for some jointly surjective $\eta_\alpha : \mathrm{Spec}\, A_\alpha \to \mathfrak{X}$, and choose any map $\xi : \mathrm{Spec}\, R \to \mathfrak{X}$. Then there exists a finite collection of étale maps $\{R \to R_\beta\}$ for which the induced map $R \to \prod_\beta R_\beta$ is faithfully flat, such that each of the induced maps $\mathrm{Spec}\, R_\beta \to \mathrm{Spec}\, R$ fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\, R_\beta & \longrightarrow & \mathrm{Spec}\, A_\alpha \\ \downarrow & & \downarrow{\scriptstyle \eta_\alpha} \\ \mathrm{Spec}\, R & \xrightarrow{\xi} & \mathfrak{X} \end{array}$$

for some index α. It follows that the image of M in $\mathrm{Mod}_{R_\beta}(\xi^* \mathcal{C})$ is compact for every index β. Using Proposition XI.6.21, we deduce that the image of M in $\xi^* \mathcal{C}$ is compact. $\square$

Definition 1.5.9. Let $\mathfrak{X}$ be a spectral Deligne-Mumfrod stack and let $\mathcal{C} \in \mathrm{QStk}(\mathfrak{X})$ be locally compactly generated. We will say that an object of $\mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ is *locally compact* if it satisfies the equivalent conditions of Lemma 1.5.8.

The main result of this section is the following:

Theorem 1.5.10. *Let $\mathfrak{X}$ be a quasi-compact quasi-separated spectral algebraic space, and let $\mathcal{C} \in \mathrm{QCoh}(\mathfrak{X})$ be a quasi-coherent stack on $\mathfrak{X}$. If $\mathcal{C}$ is locally compactly generated, then the ∞-category $\mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ is compactly generated. Moreover, an object $M \in \mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ is compact if and only if it is locally compact.*

Corollary 1.5.11. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. Suppose that f is quasi-compact, quasi-separated, and a relative spectral algebraic space. If $\mathcal{C} \in \mathrm{QStk}(\mathfrak{X})$ is locally compactly generated, then $f_* \mathcal{C} \in \mathrm{QStk}(\mathfrak{Y})$ is locally compactly generated.*

Corollary 1.5.12. *Let $\mathfrak{X}$ be a quasi-compact quasi-separated spectral algebraic space. Then the ∞-category* $\mathrm{QCoh}(\mathfrak{X})$ *is compactly generated. Moreover, an object of* $\mathrm{QCoh}(\mathfrak{X})$ *is compact if and only if it is perfect. In other words, $\mathfrak{X}$ is a perfect stack, in the sense of Definition XI.8.14.*

Proof of Theorem 1.5.10. Fix an object $M \in \mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$. Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, and for each $U \in \mathcal{X}$ let $\mathfrak{X}_U$ denote the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} | U$ and M_U the image of M in $\mathrm{QCoh}(\mathfrak{X}_U; \mathcal{C})$. Let $\mathcal{X}_0 \subseteq \mathcal{X}$ denote the full subcategory spanned by those objects U for which M_U is a compact object of $\mathrm{QCoh}(\mathfrak{X}_U; \mathcal{C})$. Note that every pushout diagram

$$\begin{array}{ccc} U & \longrightarrow & U' \\ \downarrow & & \downarrow \\ V & \longrightarrow & V' \end{array}$$

in $\mathcal{X}$ induces a pullback diagram of presentable ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathfrak{X}_U; \mathcal{C}) & \longleftarrow & \mathrm{QCoh}(\mathfrak{X}_{U'}; \mathcal{C}) \\ \uparrow & & \uparrow \\ \mathrm{QCoh}(\mathfrak{X}_V; \mathcal{C}) & \longleftarrow & \mathrm{QCoh}(\mathfrak{X}_{V'}; \mathcal{C}) \end{array}$$

It follows that if U, U', and V belong to $\mathcal{X}_0$, then V' also belongs to $\mathcal{X}_0$. Using Theorem 1.3.8 and Corollary VIII.2.5.9, we deduce that if M is locally compact, then M is compact.

Let $\mathcal{D}$ denote the full subcategory of $\mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ spanned by the locally compact objects. Using Proposition T.5.3.5.11, we deduce that the inclusion $\mathcal{D} \hookrightarrow \mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ extends to a fully faithful functor $F : \mathrm{Ind}(\mathcal{D}) \to \mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$. We will prove that F is an equivalence of ∞-categories. This implies that $\mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ is compactly generated, and that every compact object of $\mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ is a retract of an object of $\mathcal{D}$. Since $\mathcal{D}$ is idempotent complete, it will follow that every compact object of $\mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ is locally compact, thereby completing the proof.

Using Proposition T.5.5.1.9, we deduce that F preserves small colimits. Since $\mathcal{D}$ is essentially small and admits finite colimits, the ∞-category $\mathrm{Ind}(\mathcal{D})$ is presentable, so that F admits a right adjoint G (Corollary T.5.5.2.9). To prove that F is an equivalence, it will suffice to show that G is conservative. Since $\mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ is stable, we must show that if $N \in \mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ is an object such that $G(N) \simeq 0$, then $N \simeq 0$.

Choose a scallop decomposition

$$\emptyset = \mathfrak{U}_0 \to \mathfrak{U}_1 \to \cdots \to \mathfrak{U}_n \simeq \mathfrak{X}$$

and excision squares

$$\begin{array}{ccc} \mathrm{Spec}\, R_i \times_{\mathfrak{U}_i} \mathfrak{U}_{i-1} & \longrightarrow & \mathrm{Spec}\, R_i \\ \downarrow & & \downarrow \eta_i \\ \mathfrak{U}_{i-1} & \longrightarrow & \mathfrak{U}_i . \end{array}$$

For $0 \leq i \leq n$, let N_i denote the image of N in $\mathrm{QCoh}(\mathfrak{U}_i; \mathcal{C})$. We will prove that $N_i \simeq 0$ by induction on i. The case $i = 0$ is trivial, and when $i = n$ we will deduce that $N \simeq 0$ as desired. To carry out the inductive step, let us assume that $N_{i-1} \simeq 0$. To prove that $N_i \simeq 0$, it will suffice to show that the image of N is a zero object of the R_i-linear ∞-category $\eta_i^* \mathcal{C}$. Let us denote this image by N'. Note that the fiber product $\mathfrak{V}_i = \mathfrak{U}_{i-1} \times_{\mathfrak{X}} \mathrm{Spec}\, R_i$ is a quasi-compact open substack of $\mathrm{Spec}\, R_i$, and that the image of N' in $\mathrm{QCoh}(\mathfrak{V}_i; \mathcal{C})$ is zero. The proof of Lemma XI.6.17 shows that we can write N' as a filtered colimit of compact objects of $\eta_i^* \mathcal{C}$ whose restriction to $\mathfrak{V}$ is trivial. Consequently, if N' is nonzero, then there exists a compact object $M' \in \mathrm{QCoh}(\mathrm{Spec}\, R_i; \mathcal{C})$ and a nonzero map $f : M' \to N'$, such that the restriction of M' to $\mathfrak{V}$ is trivial.

Using the pullback diagram of ∞-categories

$$\begin{CD} \mathrm{QCoh}(\mathfrak{U}_i; \mathcal{C}) @>>> \mathrm{QCoh}(\mathfrak{U}_{i-1}; \mathcal{C}) \\ @VVV @VVV \\ \mathrm{QCoh}(\operatorname{Spec} R_i; \mathcal{C}) @>>> \mathrm{QCoh}(\mathfrak{V}_i; \mathcal{C}), \end{CD}$$

we can lift M' (in an essentially unique way) to a locally compact object $M_i \in \mathrm{QCoh}(\mathfrak{U}_i; \mathcal{C})$ whose image in $\mathrm{QCoh}(\mathfrak{U}_{i-1}; \mathcal{C})$ is zero. The same argument shows that we can lift f to a nonzero map $f_i : M_i \to N_i$.

We next prove the following assertion:

$(*)$ For $i \leq j \leq n$, there exists a nonzero morphism $f_j : M_j \to N_j$ in $\mathrm{QCoh}(\mathfrak{U}_j; \mathcal{C})$, where M_j is locally compact.

The proof proceeds by induction on j, the case $j = i$ having been handled above. When $j = n$, we will obtain a nonzero morphism from a locally compact object of $\mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ into N, contradicting our assumption that $G(N) \simeq 0$ and completing the proof.

Let us assume that $i < j$ and that $f_{j-1} : M_{j-1} \to N_{j-1}$ has been constructed. We let u denote the composite map

$$M_{j-1} \oplus M_{j-1}[1] \to M_{j-1} \xrightarrow{f_{j-1}} N_{j-1},$$

and let u_0 be the image of u in $\mathrm{QStk}(\mathfrak{V}_j; \mathcal{C})$. Let N'' denote the image of N in $\mathrm{QCoh}(\operatorname{Spec} R_j; \mathcal{C})$. Using the pullback diagram

$$\begin{CD} \mathrm{QCoh}(\mathfrak{U}_j; \mathcal{C}) @>>> \mathrm{QCoh}(\mathfrak{U}_{j-1}; \mathcal{C}) \\ @VVV @VV{g^*}V \\ \mathrm{QCoh}(\operatorname{Spec} R_j; \mathcal{C}) @>{h^*}>> \mathrm{QCoh}(\mathfrak{V}_j; \mathcal{C}), \end{CD}$$

we are reduced to proving that u_0 can be lifted to a morphism $v : M'' \to N''$ in $\mathrm{QCoh}(\operatorname{Spec} R_j; \mathcal{C})$ for some compact object $M'' \in \mathrm{QCoh}(\operatorname{Spec} R_j; \mathcal{C})$.

Lemma XI.6.19 implies that we can lift $g^*(M_{j-1} \oplus M_{j-1}[1])$ to a compact object $M'' \in \mathrm{QCoh}(\operatorname{Spec} R_j; \mathcal{C})$, so that u_0 can be regarded as a morphism from $h^* M''$ to $h^* N''$. Let h_* denote a right adjoint to h^*, so that u_0 determines a map $v_0 : M'' \to h_* h^* N''$. Let K denote the cofiber of the unit map $N'' \to h_* h^* N''$. Then $h^* K \simeq 0$. Using Lemma XI.6.17, we can write K as a filtered colimit of compact objects $K_\alpha \in \mathrm{QCoh}(\operatorname{Spec} R_j; \mathcal{C})$ satisfying $h^* K_\alpha \simeq 0$. Since M'' is compact, the composite map $M'' \to h_* h^* N'' \to K$ factors through some K_α. Replacing M'' by the fiber of the map $M'' \to K_\alpha$, we can assume that the composition $M'' \to h_* h^* N'' \to K$ is nullhomotopic, so that v_0 factors as a composition $M'' \xrightarrow{v} N'' \to h_* h^* N''$, where v is a morphism in $\mathrm{QCoh}(\operatorname{Spec} R_j; \mathcal{C})$ having the desired properties. □

We close this section with a few remarks about the formation of global sections of quasi-coherent stacks.

Proposition 1.5.13. *Let $\mathfrak{X}$ be a quasi-compact quasi-separated spectral algebraic space and suppose we are given a map $f : \mathfrak{X} \to \operatorname{Spec}^{\acute{e}t} A$. Let $\mathcal{C}$ be an A-linear ∞-category, regarded as an object of* $\mathrm{QStk}(\operatorname{Spec}^{\acute{e}t} A)$. *Then the unit map $\lambda : \mathcal{C} \to \mathrm{QCoh}(\mathfrak{X}; f^* \mathcal{C})$ admits a right adjoint, which we will denote by* $\Gamma(\mathfrak{X}; \bullet)$.

Proof. Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. For each object $U \in \mathcal{X}$, we let $\mathfrak{X}_U$ denote $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} | U)$, and $f_U : \mathfrak{X}_U \to \operatorname{Spec} A$ the induced map, and $\lambda_U : \mathcal{C} \to \mathrm{QCoh}\, \mathfrak{X}_U; f_U^*(\mathcal{C}))$ the canonical. Let us say that an object $U \in \mathcal{X}$ is *good* if the functor λ_U admits a right adjoint $\Gamma(U; \bullet)$ which commutes with small colimits. Note that if $\mathfrak{X}_U \simeq \operatorname{Spec}^{\acute{e}t} B$ is affine, then λ_U can be identified with the base change functor $\mathcal{C} \to \mathrm{LMod}_B(\mathcal{C})$, which is left adjoint to the forgetful functor $\mathrm{LMod}_B(\mathcal{C}) \to \mathcal{C}$; it follows that U is good.

Note that if we are given a pushout diagram

$$\begin{array}{ccc} U & \longrightarrow & U' \\ \downarrow & & \downarrow \\ V & \longrightarrow & V', \end{array}$$

then we have a pullback diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathfrak{X}_{V'}; f_{V'}^* \mathcal{C}) & \xrightarrow{\phi'} & \mathrm{QCoh}(\mathfrak{X}_V; f_V^* \mathcal{C}) \\ \downarrow \psi' & & \downarrow \psi \\ \mathrm{QCoh}(\mathfrak{X}_{U'}; f_{U'}^* \mathcal{C}) & \xrightarrow{\phi} & \mathrm{QCoh}(\mathfrak{X}_U; f_U^* \mathcal{C}). \end{array}$$

It follows that if U', U, and V are good, then V' is also good, with the functor $\Gamma(V'; \bullet)$ given by the fiber product

$$\Gamma(V; \bullet) \times_{\Gamma(U; \bullet)} \Gamma(U'; \bullet).$$

Using Corollary VIII.2.5.9 and Theorem 1.3.8, we deduce that the final object of $\mathcal{X}$ is good. □

We now show that the global sections functor described in Proposition 1.5.13 is compatible with base change:

Proposition 1.5.14. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{g} & \mathfrak{X} \\ \downarrow f' & & \downarrow f \\ \mathrm{Spec}^{\acute{e}t} A' & \longrightarrow & \mathrm{Spec}^{\acute{e}t} A \end{array}$$

of spectral Deligne-Mumford stacks, $\mathfrak{X}$ (and therefore also $\mathfrak{X}'$) is a quasi-compact quasi-separated spectral algebraic space. Let $\mathcal{C}$ be an A-linear ∞-category and let $\mathcal{C}' = \mathrm{LMod}_{A'}(\mathcal{C})$. Then the diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\lambda} & \mathrm{QCoh}(\mathfrak{X}; f^* \mathcal{C}) \\ \downarrow \otimes_A A' & & \downarrow g^* \\ \mathcal{C}' & \xrightarrow{\lambda'} & \mathrm{QCoh}(\mathfrak{X}'; f'^* \mathcal{C}') \end{array}$$

is right adjointable.

Proof. The existence of right adjoints to λ and λ' follows from Proposition 1.5.13; let us denote these right adjoints by $\Gamma(\mathfrak{X}; \bullet)$ and $\Gamma(\mathfrak{X}'; \bullet)$. To complete the proof, we must show that for every object $M \in \mathcal{C}$, the canonical map

$$A' \otimes_A \Gamma(\mathfrak{X}; M) \to \Gamma(\mathfrak{X}'; g^* M)$$

is an equivalence. Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. For every object $U \in \mathcal{X}$ let $\Gamma(U; M)$ be defined as in the proof of Proposition 1.5.13, and let $\Gamma(g^* U; g^* M)$ be defined similarly. Let $\mathcal{C} \subseteq \mathcal{X}$ be the full subcategory spanned by those objects U for which the canonical map

$$A' \otimes_A \Gamma(U; M) \to \Gamma(g^* U; g^* M)$$

is an equivalence. Lemma VII.6.15 guarantees that every affine object of $\mathcal{X}$ belongs to $\mathcal{C}$, and it follows from the proof of Proposition 1.5.13 that $\mathcal{C}$ is closed under pushouts. Using Corollary VIII.2.5.9 and Theorem 1.3.8, we deduce that the final object of $\mathcal{X}$ belongs to $\mathcal{C}$. □

2 Noetherian Approximation

Let X be a scheme of finite presentation over a commutative ring R. Then there exists a finitely generated subring $R_0 \subseteq R$, an R_0-scheme X_0 of finite presentation, and an isomorphism of schemes

$$X \simeq \operatorname{Spec} R \times_{\operatorname{Spec} R_0} X_0.$$

This observation is the basis of a technique called *Noetherian approximation*: one can often reduce questions about the scheme X to questions about the scheme X_0, which may be easier to answer because X_0 is Noetherian.

We would like to adapt the technique of Noetherian approximation to the setting of spectral algebraic geometry. More specifically, we would like to address questions like the following:

Question 2.0.15. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack over a connective $\mathbb{E}_\infty$-ring R, and suppose that R is given as a filtered colimit $\varinjlim R_\alpha$. Can we find an index α, a spectral Deligne-Mumford stack $\mathfrak{X}_\alpha$ over R_α, and an equivalence of spectral Deligne-Mumford stacks

$$\mathfrak{X} \simeq \operatorname{Spec}^{\text{ét}} R_\alpha \times_{\operatorname{Spec}^{\text{ét}} R} \mathfrak{X}_\alpha?$$

To even have a chance of obtaining an affirmative answer we will need to make some finiteness assumptions on $\mathfrak{X}$. However, even with finiteness assumptions in place, Question 2.0.15 is more subtle than its classical analogue. The main issue is that the data of a spectral Deligne-Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ is infinitary in nature, because the structure sheaf $\mathcal{O}_\mathcal{X}$ potentially has an infinite number of nonzero homotopy sheaves $\pi_m \mathcal{O}_\mathcal{X}$. When looking for "approximations" to $\mathfrak{X}$, we can generally only control finitely many of these homotopy groups at one time. We can attempt to avoid the issue by studying *truncatons* of $\mathfrak{X}$. For each $n \geq 0$, let $\tau_{\leq n} \mathfrak{X}$ denote the spectral Deligne-Mumford stack $(\mathcal{X}, \tau_{\leq n} \mathcal{O}_\mathcal{X})$. A more reasonable version of Question 2.0.15 is the following:

Question 2.0.16. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack over a connective $\mathbb{E}_\infty$-ring R, and suppose that R is given as a filtered colimit $\varinjlim R_\alpha$, and let $n \geq 0$ be an integer. Can we find an index α, a spectral Deligne-Mumford stack $\mathfrak{X}_\alpha$ over R_α, and an equivalence of spectral Deligne-Mumford stacks

$$\tau_{\leq n} \mathfrak{X} \simeq \tau_{\leq n}(\operatorname{Spec}^{\text{ét}} R_\alpha \times_{\operatorname{Spec}^{\text{ét}} R} \mathfrak{X}_\alpha)?$$

In §2.3, we will obtain a positive answer to Question 2.0.16 (Theorem 2.3.2) provided that a mild finiteness condition is satisfied: that is, that $\mathfrak{X}$ be *finitely n-presented* over R. We will study this finiteness condition (and others) in §2.2.

An affirmative answer to Question 2.0.16 raises a host of related questions. Suppose that we are given a morphism $\mathfrak{X} \to \operatorname{Spec}^{\text{ét}} R$ having some special property P. Can we necessarily arrange that the morphism $\mathfrak{X}_\alpha \to \operatorname{Spec}^{\text{ét}} R_\alpha$ of Question 2.0.15 also has the property P? In §2.5, we will verify this for a number of properties of geometric interest. What if we are given a quasi-coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$: can we hope to write $\mathcal{F}$ as the inverse image of a quasi-coherent sheaf on $\mathfrak{X}_\alpha$? This is generally too much to ask for, since there are again infinitely many homotopy groups to control. However, we will show in §2.4 that if $\mathcal{F}$ satisfies some reasonable finiteness conditions, we will see that the truncation $\tau_{\leq n} \mathcal{F}$ can be obtained as the truncation of the pullback of a quasi-coherent sheaf on $\mathfrak{X}_\alpha$ (Theorem 2.4.4).

The proofs of Theorems 2.3.2 and 2.5.3 proceed roughly in two steps: first, we treat the case where $\mathfrak{X}$ is affine. Then, we reduce the general case to the affine case using an affine covering $u : \mathfrak{U}_0 \to \mathfrak{X}$. To carry out the second step, we need to study the groupoid $\mathfrak{U}_\bullet$ given by the Čech nerve of u. It is essential to our arguments that this groupoid be controlled by a finite amount of data: that is, that we only need to consider the objects $\mathfrak{U}_p$ for some finitely many integers p. This is a consequence of some general categorical considerations, which we take up in §2.1.

2.1 Truncated Category Objects

Let $\mathcal{C}$ be an ∞-category which admits finite limits. Recall that a *category object* of $\mathcal{C}$ is a simplicial object $C_\bullet$ of $\mathcal{C}$ satisfying the following condition: for each $n \geq 0$, the diagram of linearly ordered sets

$$\begin{array}{ccccccccccccc} & & \{0,1\} & & & & \cdots & & & & \{n-1,n\} & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ \{0\} & & & & \{1\} & & \cdots & & \{n-1\} & & & & \{n\} \end{array}$$

induces an equivalence

$$C_n \to C_1 \times_{C_0} \cdots \times_{C_0} C_1.$$

Example 2.1.1. Let $C_\bullet$ be a simplicial set. Then $C_\bullet$ is a category object of the category $\mathcal{S}\mathrm{et}$ of sets if and only if $C_\bullet$ is isomorphic to the nerve of a small category $\mathcal{E}$. Moreover, the category $\mathcal{E}$ is determined up to canonical isomorphism: the objects of $\mathcal{E}$ are the elements of the set C_0, the morphisms of $\mathcal{E}$ are the elements of the set C_1, and the composition of morphisms is determined by the map

$$C_1 \times_{C_0} C_1 \overset{\sim}{\leftarrow} C_2 \overset{\rho}{\to} C_1,$$

where ρ is induced by the inclusion of linearly ordered sets $[1] \simeq \{0,2\} \hookrightarrow [2]$.

We would like to call attention to two phenomena at work in Example 2.1.1:

- Let $C_\bullet$ be a category object in sets, so that $C_\bullet \simeq \mathrm{N}(\mathcal{E})$ for some small category $\mathcal{E}$. To recover the category $\mathcal{E}$ (and therefore the entire simplicial set $C_\bullet$), we only need to know the sets C_0, C_1, C_2, and the maps between them.

- When reconstructing the category $\mathcal{E}$ from the simplicial set $C_\bullet$, the main step is to prove that composition of morphisms is associative. The proof of this involves studying the set C_3 and the bijection $C_3 \to C_1 \times_{C_0} C_1 \times_{C_0} C_1$. In particular, it does not make any reference to the sets C_n for $n \geq 4$.

Our goal in this section is to generalize these observations. We begin by introducing some terminology.

Definition 2.1.2. Let $\mathcal{C}$ be an ∞-category which admits finite limits, and let $m \geq 1$. An *m-skeletal simplicial object* of $\mathcal{C}$ is a functor $\mathrm{N}(\mathbf{\Delta}_{\leq m})^{op} \to \mathcal{C}$. If C is an m-skeletal simplicial object and $n \leq m$, we let C_n denote the image in $\mathcal{C}$ of the object $[n] \in \mathbf{\Delta}_{\leq m}$.

An *m-skeletal category object* is a functor $\mathrm{N}(\mathbf{\Delta}_{\leq m})^{op} \to \mathcal{C}$ with the following property: for each $n \leq m$, the diagram of linearly ordered sets

$$\begin{array}{ccccccccccccc} & & \{0,1\} & & & & \cdots & & & & \{n-1,n\} & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ \{0\} & & & & \{1\} & & \cdots & & \{n-1\} & & & & \{n\} \end{array}$$

induces an equivalence $C_n \to C_1 \times_{C_0} \cdots \times_{C_0} C_1$.

We let $\mathrm{CObj}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{C})$ spanned by the category objects, and $\mathrm{CObj}_{\leq m}(\mathcal{C})$ the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{\leq m})^{op}, \mathcal{C})$ spanned by the m-skeletal category objects.

We can now state our main result.

Theorem 2.1.3. *Let $\mathcal{C}$ be an ∞-category which is equivalent to an n-category for some $n \geq -1$ (see Definition T.2.3.4.1) and admits finite limits. Then the restriction functor $\mathrm{CObj}(\mathcal{C}) \to \mathrm{CObj}_{\leq m}(\mathcal{C})$ is fully faithful when $m = n+1$ and an equivalence of ∞-categories when $m \geq n+2$.*

Theorem 2.1.3 is an immediate consequence of the following more precise assertions (and Proposition T.4.3.2.15).

Proposition 2.1.4. *Let $\mathcal{C}$ be an ∞-category which admits finite limits, and let $C_\bullet$ be a category object of $\mathcal{C}$. Assume that the map $C_1 \to C_0 \times C_0$ is $(n-2)$-truncated for some integer $n \geq 0$. Then $C_\bullet$ is a right Kan extension of its restriction to $\mathrm{N}(\mathbf{\Delta}_{\leq n})$.*

Proposition 2.1.5. *Let $\mathcal{C}$ be an ∞-category which admits finite limits, let $n \geq 1$, and let $C_\bullet$ be an n-skeletal category object of $\mathcal{C}$. Assume that the map $C_1 \to C_0 \times C_0$ is $(n-3)$-truncated. Then $C_\bullet$ can be extended to a category object $\overline{C}_\bullet$ of $\mathcal{C}$ (this extension is necessarily a right Kan extension, by virtue of Proposition 2.1.4).*

The proofs of Propositions 2.1.4 and 2.1.5 will require some preliminaries.

Notation 2.1.6. Let $\mathcal{C}$ be an ∞-category which admits finite limits and let $C_\bullet$ be an m-skeletal simplicial object of $\mathcal{C}$. Let K be a simplicial set of dimension $\leq m$, which is isomorphic to a simplicial subset of Δ^n for some n. We let Σ_K denote the partially ordered set of nondegenerate simplices in K. There is an evident forgetful functor $\Sigma_K \to \mathbf{\Delta}_{\leq m}$. We let $C[K]$ denote a limit of the induced diagram

$$\mathrm{N}(\Sigma_K)^{op} \to \mathrm{N}(\mathbf{\Delta}_{\leq m})^{op} \xrightarrow{C_\bullet} \mathcal{C}.$$

Lemma 2.1.7. *Let $\mathcal{C}$ be an ∞-category which admits finite limits, let $C_\bullet$ be an m-skeletal category object of $\mathcal{C}$ for some $m \geq 1$. Then:*

(1) *Let $n \leq m+1$, let $0 < j < n$, and let $A \subseteq \Delta^n$ be the simplicial subset spanned by the edges $\{\Delta^{\{i-1,i\}}\}_{1 \leq i \leq n}$. Then the restriction map $C[\Lambda^n_j] \to C[A]$ is an equivalence.*

(2) *For $0 < j < n \leq m$, the map $C[\Delta^n] \to C[\Lambda^n_j]$ is an equivalence.*

Proof. We prove (1) and (2) by a simultaneous induction on n. Note that if $n \leq m$ and K is defined as in (1), then the composite map $C[\Delta^n] \to C[\Lambda^n_j] \to C[A]$ is an equivalence by virtue of our assumption that $C_\bullet$ is an n-skeletal category object. Consequently, assertion (2) follows from (1) and the two-out-of-three property.

We now prove (1). Let S be the collection of all nondegenerate simplices σ of Δ^n which contain the vertex j together with additional vertices i and k such that $i < j < k$. Write $S = \{\sigma_1, \sigma_2, \ldots, \sigma_b\}$, where $a < a'$ whenever σ_a has dimension larger than $\sigma_{a'}$; in particular, we have $\sigma_1 = \Delta^n$. For $1 \leq a \leq b$, let τ_a be the face of σ_a obtained by the removing the vertex j, and let K_a denote the simplicial subset obtained from Δ^n by removing the simplices $\{\sigma_{a'}, \tau_{a'}\}_{a' \leq a}$. We have a chain of simplicial subsets

$$\Delta^{\{0,\ldots,j\}} \coprod_{\{j\}} \Delta^{\{j,j+1,\ldots,n\}} = K_b \subseteq K_{b-1} \subseteq \cdots \subseteq K_1 = \Lambda^n_j.$$

For $1 \leq a < b$, the inclusion $K_{a+1} \subseteq K_a$ is a pushout of an inner horn inclusion $\Lambda^{n'}_{j'} \subseteq \Delta^{n'}$ for some $0 < j' < n' < n$, so we have a pullback diagram

$$\begin{array}{ccc} C[K_a] & \longrightarrow & C[K_{a+1}] \\ \downarrow & & \downarrow \\ C[\Delta^{n'}] & \longrightarrow & C[\Lambda^{n'}_{j'}] \end{array}$$

The inductive hypothesis implies that the bottom horizontal map is an equivalence, so that $C[K_a] \simeq C[K_{a+1}]$ for $1 \leq a < b$. It follows that the restriction map

$$C[\Lambda^n_j] \to C[K_b] \simeq C[\Delta^{\{0,\ldots,j\}}] \times_{C[\{j\}]} C[\Delta^{\{j,\ldots,n\}}]$$

is an equivalence. Let A_- be the simplicial subset of Δ^n spanned by the edges $\Delta^{\{i-1,i\}}$ for $1 \leq i \leq j$, and let A_+ be the simplicial subset of Δ^n spanned by the edges $\Delta^{\{i-1,i\}}$ for $j < i \leq n$. Since $C_\bullet$ is an m-coskeletal category object, the restriction maps

$$C[\Delta^{\{0,\ldots,j\}}] \to C[A_-] \qquad C[\Delta^{\{j,j+1,\ldots,n\}}] \to C[A_+]$$

are equivalences. It follows that the map

$$C[\Lambda^n_j] \to C[A] \simeq C[A_-] \times_{C[\{j\}]} C[A_+]$$

is an equivalence. □

Proof of Proposition 2.1.4. Let $C_\bullet$ be a category object of $\mathcal{C}$ such that the map $C_1 \to C_0 \times C_0$ is $(n-2)$-truncated. We wish to show that $C_\bullet$ is a right Kan extension of its restriction to $\mathrm{N}(\mathbf{\Delta}_{\leq n})^{op}$. It will suffice to show that for each $m \geq n$, the restriction $C_\bullet | \mathrm{N}(\mathbf{\Delta}_{\leq m})^{op}$ is a right Kan extension of $C_\bullet | \mathrm{N}(\mathbf{\Delta}_{\leq n})^{op}$. Using Proposition T.4.3.2.8 repeatedly, we are reduced to showing that $C_\bullet | \mathrm{N}(\mathbf{\Delta}_{\leq m})^{op}$ is a right Kan extension of $C_\bullet | \mathrm{N}(\mathbf{\Delta}_{\leq m-1})^{op}$ for $m > n$. In other words, we must show that if $m > n$, then the map $C_m \to M_m(C)$ is an equivalence, where $M_m(C)$ denotes the mth matching object of $C_\bullet$ (see Notation T.A.2.9.7). We will show more generally that the map $\beta_m : C_m \to M_m(C)$ is $(n-m-1)$-truncated for each $m \geq 1$. If $m > n$, this implies that β_m is an equivalence. We proceed by induction on m: the case $m = 1$ follows from our hypothesis that $C_1 \to C_0 \times C_0$ is $(n-2)$-truncated. Assume therefore that $m \geq 2$, and choose $0 < j < m$. Lemma 2.1.7 implies that the composite map

$$C[\Delta^m] \simeq C_m \to M_m(C) \simeq C[\partial\,\Delta^m] \to C[\Lambda^m_j]$$

is an equivalence. It will therefore suffice to show that the map $\gamma : C[\partial\,\Delta^m] \to C[\Lambda^m_j]$ is $(n-m)$-truncated. This follows from the inductive hypothesis, since γ is a pullback of the map $C_{m-1} \to M_{m-1}(C)$. □

Proof of Proposition 2.1.5. Let $C_\bullet$ be an n-skeletal category object of $\mathcal{C}$ and assume that the map $C_1 \to C_0 \times C_0$ is $(n-3)$-truncated. Since $\mathcal{C}$ admits finite limits, there exists a simplicial object $\overline{C}_\bullet$ which is a right Kan extension of $C_\bullet$. We wish to show that $\overline{C}_\bullet$ is a category object of $\mathcal{C}$. It will suffice to show that the restriction $\overline{C}_\bullet | \mathrm{N}(\mathbf{\Delta}_{\leq m})^{op}$ is an m-skeletal category object for each $m \geq n$. We proceed by induction on m, the case $m = n$ being trivial. Let $A \subseteq \Delta^m$ be the simplicial subset given by the union of the edges $\{\Delta^{\{i-1,i\}}\}_{1 \leq i \leq m}$; we wish to show that the map $\overline{C}[\Delta^m] \to \overline{C}[A]$ is an equivalence. Since $m > n \geq 1$, we can choose $0 < j < m$. Using the inductive hypothesis and Lemma 2.1.7, we deduce that $\overline{C}[\Lambda^m_j] \to \overline{C}[A]$ is an equivalence. It will therefore suffice to show that the restriction map $\beta : \overline{C}[\Delta^m] \to \overline{C}[\Lambda^m_j]$ is an equivalence. Note that β is a pullback of the map $\beta' : \overline{C}[\Delta^{m-1}] \to \overline{C}[\partial\,\Delta^{m-1}]$. We will show that β' is an equivalence. For this, it suffices to prove the more general claim that for $1 \leq k < m$, the map $\beta'_k : \overline{C}[\Delta^k] \to \overline{C}[\partial\,\Delta^k]$ is $(n-k-2)$-truncated.

As in the proof of Proposition 2.1.4, we proceed by induction on k, the case $k = 1$ being true by virtue of our hypothesis. Assume therefore that $k \geq 2$, and choose $0 < i < k$. Since $k < m$, the restriction $\overline{C}_\bullet | \mathrm{N}(\mathbf{\Delta}_{\leq k})$ is a k-skeletal category object so that Lemma 2.1.7 implies that the composite map

$$\overline{C}[\Delta^k] \xrightarrow{\beta'_k} \overline{C}[\partial\,\Delta^k] \xrightarrow{\gamma} \overline{C}[\Lambda^k_i]$$

is an equivalence. It is therefore sufficient to show that γ is $(n-k-1)$-truncated. This follows from the inductive hypothesis, since γ is a pullback of β'_{k-1}. □

2.2 Finitely n-Presented Morphisms

In §IX.8, we studied a finiteness condition on morphisms of spectral Deligne-Mumford stacks: the property of being *locally of finite presentation to order n*. Very roughly speaking, a morphism $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is locally of finite presentation to order n if the homotopy groups $\pi_i\,\mathcal{O}_\mathcal{X}$ are controlled by $\mathcal{O}_\mathcal{Y}$ for $i < n$. In practice, it is often useful to use this notion in conjunction with another hypothesis which controls the homotopy groups $\pi_i\,\mathcal{O}_\mathcal{X}$ for $i \geq n$.

Definition 2.2.1. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. We will say that $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is *locally finitely n-presented* over $\mathfrak{Y}$ if the following conditions are satisfied:

(*i*) The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is n-truncated.

(*ii*) The map f is locally of finite presentation to order $(n+1)$ (see Definition IX.8.16).

In this case, we will also say that the morphism f is locally finitely n-presented.

Example 2.2.2. Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a map of spectral Deligne-Mumford stacks. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are 1-localic. Then f is locally finitely 0-presented if and only if the following conditions are satisfied:

(*i*) The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is discrete.

(*ii*) The induced map $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathcal{O}_{\mathcal{Y}})$, when viewed as a map of ordinary Deligne-Mumford stacks (see Proposition VII.8.36), is locally of finite presentation in the sense of classical algebraic geometry.

We now summarize some of the formal properties of Definition 2.2.1.

Proposition 2.2.3. *Fix an integer $n \geq 0$.*

(1) *The condition that a map $f : \mathfrak{X} \to \mathfrak{Y}$ be locally finitely n-presented is local on the source with respect to the étale topology.*

(2) *The condition that a map $f : \mathfrak{X} \to \mathfrak{Y}$ be locally finitely n-presented is local on the target with respect to the flat topology.*

(3) *Suppose given a pair of maps $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{Z}$. Assume that g is locally finitely n-presented. Then f is locally finitely n-presented if and only if $g \circ f$ is locally finitely n-presented.*

Proof. Assertion (1) follows from Proposition IX.8.18 and Example VIII.1.5.25, assertion (2) follows from Proposition IX.8.24 and Example VIII.1.5.25, and assertion (3) follows from Proposition IX.8.10. □

Proposition 2.2.4. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. Assume that $\mathfrak{Y}$ is locally Noetherian. Then f is locally finitely n-presented if and only if the following conditions are satisfied:*

(1) *The morphism f is locally of finite presentation to order* 0.

(2) *The spectral Deligne-Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is locally Noetherian.*

(3) *The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is n-truncated.*

Proof. We may assume without loss of generality that $\mathfrak{Y} \simeq \mathrm{Spec}^{\text{ét}} A$ and $\mathfrak{X} \simeq \mathrm{Spec}^{\text{ét}} B$ are affine. Assume first that f is locally finitely n-presented. Conditions (1) and (3) are obvious. To prove (2), we note that B is a compact object of $\tau_{\leq n} \mathrm{CAlg}_A$ (Remark IX.8.7), so that $B \simeq \tau_{\leq n} B'$ for some A-algebra B' which is of finite presentation over A. It follows from Proposition A.7.2.5.31 that B' is Noetherian, so that B is also Noetherian.

Now suppose that (1), (2), and (3) are satisfied. Using (1), (2), and Proposition A.7.2.5.31, we deduce that the map $A \to B$ is locally almost of finite presentation, and in particular of finite presentation to order $(n+1)$ over A. Combining this with (3), we deduce that f is finitely n-presented as desired. □

Warning 2.2.5. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks which is locally finitely n-presented. If $\mathfrak{Y}$ is not locally Noetherian, then f need not be locally finitely m-presented for $m > n$.

We now combine the local finiteness condition of Definition 2.2.1 with some global considerations.

Definition 2.2.6. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks and let $n \geq 0$ be an integer. We will say that f is *finitely n-presented* if the following conditions are satisfied:

(1) The map f is locally finitely n-presented (see Definition 2.2.1).

(2) For every map $\mathrm{Spec}^{\text{ét}} A \to \mathfrak{Y}$, the fiber product $\mathrm{Spec}^{\text{ét}} A \times_{\mathfrak{Y}} \mathfrak{X}$ is a spectral Deligne-Mumford m-stack for some integer $m \geq 0$.

(3) The morphism f is ∞-quasi-compact (see Definition VIII.1.4.5).

Remark 2.2.7. In the situation of Definition 2.2.6, assume that $\mathfrak{Y}$ is quasi-compact. Then condition (2) is equivalent to the following:

(2′) The map $f : \mathfrak{X} \to \mathfrak{Y}$ is a relative spectral Deligne-Mumford m-stack for some $m \geq 0$.

In this case, Remark 1.3.5 implies that condition (3) is equivalent to the following apparently weaker condition:

(3′) The morphism f is $(m+1)$-quasi-compact.

Remark 2.2.8. Suppose we are given maps of spectral Deligne-Mumford stacks

$$\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}.$$

Assume that g is finitely n-presented. Then f is finitely n-presented if and only if g is finitely n-presented (combine Proposition 2.2.3 with Corollary VIII.1.4.16).

The property of being finitely n-presented is not stable under arbitrary base change. Given a pullback diagram

$$\begin{array}{ccc} \mathfrak{Y}' & \longrightarrow & \mathfrak{Y} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{X}' & \xrightarrow{g} & \mathfrak{X} \end{array}$$

where f is finitely n-presented, the morphism f' need not be finitely n-presented without some flatness assumption on the morphism g. We can correct this difficulty by truncating the structure sheaf of the spectral Deligne-Mumford stack $\mathfrak{Y}'$.

Notation 2.2.9. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. We let $\tau_{\leq n} \mathfrak{X}$ denote the spectral Deligne-Mumford stack $(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$. We will refer to $\tau_{\leq n} \mathfrak{X}$ as the *n-truncation* of $\mathfrak{X}$.

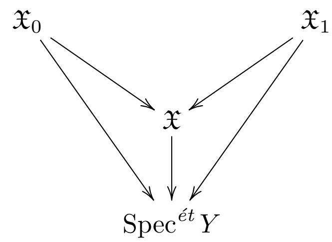
Proposition 2.2.10. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \longrightarrow & \mathfrak{Y} \end{array}$$

of spectral Deligne-Mumford stacks. If $\tau_{\leq n} \mathfrak{X}$ is finitely n-presented over $\mathfrak{Y}$, then $\tau_{\leq n} \mathfrak{X}'$ is finitely n-presented over $\mathfrak{Y}'$.

Proof. We may assume without loss of generality that $\mathfrak{Y} \simeq \mathrm{Spec}^{\text{ét}} A$ and $\mathfrak{Y}' \simeq \mathrm{Spec}^{\text{ét}} A'$ are affine. Then $\mathfrak{X}$ is an ∞-quasi-compact spectral Deligne-Mumford m-stack for some $m \geq 0$. It follows that $\mathfrak{X}'$ is also an ∞-quasi-compact spectral Deligne-Mumford m-stack (see Remark VIII.1.3.9 and Corollary VIII.1.4.18). To complete the proof, it will suffice to show that f' is locally finitely n-presented. Replacing $\mathfrak{X}$ by $\tau_{\leq n} \mathfrak{X}$, we may assume that $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is locally of finite presentation to order $(n+1)$ over A. It follows that $\mathfrak{X}' = (\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is locally of finite presentation to order $(n+1)$ over $\mathrm{Spec}^{\text{ét}} A'$. Using Remark IX.8.6, we deduce that $(\mathcal{X}', \tau_{\leq n} \mathcal{O}_{\mathcal{X}'})$ is also locally of finite presentation to order $(n+1)$ over $\mathrm{Spec}^{\text{ét}} A'$, hence loally finitely n-presented over $\mathrm{Spec}^{\text{ét}} A'$. □

Corollary 2.2.11. *Suppose we are given a commutative diagram of spectral Deligne-Mumford stacks*



where $\mathfrak{X}_0$, $\mathfrak{X}_1$, and $\mathfrak{X}$ are finitely n-presented over $\mathfrak{Y}$. Then $\tau_{\leq n}(\mathfrak{X}_0 \times_{\mathfrak{X}} \mathfrak{X}_1)$ is finitely n-presented over $\mathfrak{Y}$.

Proof. Using Remark 2.2.8, we see that $\mathfrak{X}_0$ is finitely n-presented over $\mathfrak{X}$. It follows from Proposition 2.2.10 that $\tau_{\leq n}(\mathfrak{X}_0 \times_{\mathfrak{X}} \mathfrak{X}_1)$ is finitely n-presnented over $\mathfrak{X}_1$, and hence also finitely n-presented over $\mathfrak{X}$ (Remark 2.2.8). □

2.3 Approximation of Spectral Deligne-Mumford Stacks

Suppose we are given a filtered diagram $\{A_\alpha\}$ of connective $\mathbb{E}_\infty$-rings having colimit $A = \varinjlim A_\alpha$, and let $f : \mathfrak{X} \to \mathrm{Spec}^{\text{ét}} A$ be a finitely n-presented morphism. Our goal in this section is to prove that there exists an index α, a finitely n-presented morphism $\mathfrak{X}_\alpha \to \mathrm{Spec}^{\text{ét}} A_\alpha$, and an equivalence of spectral Deligne-Mumford stacks

$$\mathfrak{X} \simeq \tau_{\leq n}(\mathrm{Spec}^{\text{ét}} A_\alpha \times_{\mathrm{Spec}^{\text{ét}} A} \mathfrak{X}_\alpha).$$

Moreover, we will show that the spectral Deligne-Mumford stack $\mathfrak{X}_\alpha$ is essentially unique, up to changes in the index α. To formulate this last condition more precisely, it will be convenient to introduce some notation.

Construction 2.3.1. Fix an integer $n \geq 0$. Let $\mathcal{C}$ denote the full subcategory of

$$\mathrm{Fun}(\Delta^1, \mathrm{Stk}) \times_{\mathrm{Fun}(\{1\}, \mathrm{Stk})} (\mathrm{CAlg}^{\mathrm{cn}})^{op}$$

spanned by those morphisms $\mathfrak{X} \to \mathrm{Spec}^{\text{ét}} A$ where A is a connective $\mathbb{E}_\infty$-ring and $\mathfrak{X}$ is a spectral Deligne-Mumford stack which is finitely n-presented over $\mathrm{Spec}^{\text{ét}} A$. It follows from Proposition 2.2.10 that the projection map $\theta : \mathcal{C} \to (\mathrm{CAlg}^{\mathrm{cn}})^{op}$ is a Cartesian fibration. We let $\mathrm{DM}_n^{\mathrm{fp}} : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ denote a functor classifying the Cartesian fibration θ.

More informally, the functor $\mathrm{DM}_n^{\mathrm{fp}}$ associates to every connective $\mathbb{E}_\infty$-ring A the full subcategory

$$\mathrm{DM}_n^{\mathrm{fp}}(A) \subseteq \mathrm{Stk}_{/ \mathrm{Spec}\, A}$$

spanned by those maps $\mathfrak{X} \to \mathrm{Spec}\, A$ which exhibit $\mathfrak{X}$ as finitely n-presented over $\mathrm{Spec}\, A$. To every morphism $f : A \to B$ of connective $\mathbb{E}_\infty$-rings, $\mathrm{DM}_n^{\mathrm{fp}}$ associates the functor $\mathrm{DM}_n^{\mathrm{fp}}(A) \to \mathrm{DM}_n^{\mathrm{fp}}(B)$ given by

$$\mathfrak{X} \mapsto \tau_{\leq n}(\mathrm{Spec}^{\text{ét}} B \times_{\mathrm{Spec}^{\text{ét}} A} \mathfrak{X}).$$

We can now state our main result.

Theorem 2.3.2. *Let $n \geq 0$ be an integer, and let $\mathrm{DM}_n^{\mathrm{fp}} : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Construction 2.3.1. Then:*

(1) *For every connective $\mathbb{E}_\infty$-ring R, the ∞-category $\mathrm{DM}_n^{\mathrm{fp}}(R)$ is essentially small.*

(2) *The functor $\mathrm{DM}_n^{\mathrm{fp}}$ commutes with filtered colimits.*

The proof of Theorem 2.3.2 will require some preliminaries.

Lemma 2.3.3. *Let A be a connective $\mathbb{E}_\infty$-ring, let $\mathfrak{X} \to \operatorname{Spec}^{ét} A$ be a finitely n-presented morphism, and let $F : \tau_{\leq n} \operatorname{CAlg}^{\text{cn}}_A \to \mathcal{S}$ be the functor represented by $\mathfrak{X}$, given informally by*

$$F(R) = \operatorname{Map}_{\operatorname{Stk}_{/ \operatorname{Spec}^{ét} A}}(\operatorname{Spec}^{ét} R, \mathfrak{X}).$$

Then F commutes with filtered colimits.

We defer the proof of Lemma 2.3.3 for the moment; a proof will appear in [48].

Lemma 2.3.4. *Let $\mathcal{C}$ be a presentable ∞-category, let $\widetilde{\mathcal{C}}$ be the full subcategory of $\operatorname{Fun}(\Delta^1, \mathcal{C})$ spanned by those morphisms $C \to C'$ in $\mathcal{C}$ which exhibit C' as a compact object of $\mathcal{C}^{C/}$, and let $e : \widetilde{\mathcal{C}} \to \mathcal{C}$ be given by evaluation at (1). Then e is a coCartesian fibration, classified by a functor $\chi : \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$. Moreover, for every filtered diagram $F : \mathcal{J} \to \mathcal{C}$, the canonical map $\varinjlim_J \chi(F(J)) \to \chi(\varinjlim F)$ is fully faithful.*

Proof. We first show that e is a coCartesian fibration. For this, it suffices to show that for every map $\beta : C \to D$ in $\mathcal{C}$, the associated functor $\beta_! : \mathcal{C}^{C/} \to \mathcal{C}^{D/}$ preserves compact objects. In view of Proposition T.5.5.7.2, it suffices to show that the pullback functor $\beta^* : \mathcal{C}^{D/} \to \mathcal{C}^{C/}$ preserves filtered colimits, which follows immediately from Proposition T.1.2.13.8. Since each $\mathcal{C}^{C/}$ is a presentable ∞-category, the full subcategory spanned by the compact objects is essentially small, so that e is classified by a functor $\chi : \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$.

Now suppose that $F : \mathcal{J} \to \mathcal{C}$ is a filtered diagram. We wish to show that the canonical functor $\varinjlim_J \chi(F(J)) \to \chi(\varinjlim F)$ is fully faithful. Fix a pair of objects $X, Y \in \varinjlim_J \chi(F(J))$. Since $\mathcal{J}$ is filtered, we may assume that X and Y are the images of objects of $\chi(F(J))$ for some $J \in \mathcal{J}$, corresponding to a pair of compact objects $X_0, Y_0 \in \mathcal{C}^{F(J)/}$. Let $C = \varinjlim F$. Unwinding the definitions, we must show that the canonical map

$$\varinjlim_{J' \in \mathcal{J}_{J/}} \operatorname{Map}_{\mathcal{C}^{F(J')/}}(X_0 \amalg_{F(J)} F(J'), Y_0 \amalg_{F(J)} F(J')) \to \operatorname{Map}_{\mathcal{C}^{C/}}(X_0 \amalg_{F(J)} C, Y_0 \amalg_{F(J)} C)$$

is a homotopy equivalence. We can identify α with the map

$$\varinjlim_{J' \in \mathcal{J}_{J/}} (\operatorname{Map}_{\mathcal{C}^{F(J)/}}(X_0, Y_0 \amalg_{F(J)} F(J')) \to \operatorname{Map}_{\mathcal{C}^{F(J)/}}(X_0, Y_0 \amalg_{F(J)} C),$$

which is a homotopy equivalence since X_0 is a compact object of $\mathcal{C}^{F(J)/}$. □

Proof of Theorem 2.3.2. Let us say that a map $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford is a *quasi-monomorphism* if, for every discrete commutative ring A, the induced map

$$\operatorname{Map}_{\operatorname{Stk}}(\operatorname{Spec} A, \mathfrak{X}) \to \operatorname{Map}_{\operatorname{Stk}}(\operatorname{Spec} A, \mathfrak{Y})$$

is (-1)-truncated. For $m \geq 1$ and any connective $\mathbb{E}_\infty$-ring R, let $\operatorname{DM}^{\text{fp}}_{n,m}(R)$ denote the full subcategory of $\operatorname{DM}^{\text{fp}}_n(R)$ spanned by those maps $f : \mathfrak{X} \to \operatorname{Spec} R$ which are finitely n-presented, where $\mathfrak{X}$ is a spectral Deligne-Mumford m-stack. Let $\operatorname{DM}^{\text{fp}}_{n,0}(R) \subseteq \operatorname{DM}^{\text{fp}}_n(R)$ be the full subcategory spanned by those morphisms $f : \mathfrak{X} \to \operatorname{Spec} R$ where $\mathfrak{X}$ is a spectral algebraic space. Let $\operatorname{DM}^{\text{fp}}_{n,-1}(R) \subseteq \operatorname{DM}^{\text{fp}}_n(R)$ be the full subcategory spanned by those maps $f : \mathfrak{X} \to \operatorname{Spec} R$ which fit into a commutative diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{u} & \operatorname{Spec} A \\ & {\scriptstyle f} \searrow \quad \swarrow & \\ & \operatorname{Spec} R & \end{array}$$

where u is a quasi-monomorphsim and A is of finite presentation to order $(n+1)$ over R. Let $\operatorname{DM}^{\text{fp}}_{n,-2}(R) \subseteq \operatorname{DM}^{\text{fp}}_n(R)$ denote the full subcategory spanned by those maps $f : \mathfrak{X} \to \operatorname{Spec} R$ where $\mathfrak{X}$ is affine. Note that if $\mathfrak{X}$ belongs to $\operatorname{Stk}^{\text{fp}}_{n,m}(R)$, then $\tau_{\leq n}(\mathfrak{X} \times_{\operatorname{Spec}^{\text{ét}} R} \operatorname{Spec}^{\text{ét}} R')$, then belongs to $\operatorname{DM}^{\text{fp}}_{n,m}(R')$. Consequently, we have functors $\operatorname{DM}^{\text{fp}}_{n,m} : \operatorname{CAlg}^{\text{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ for each $m \geq -2$, and $\operatorname{DM}^{\text{fp}}_n \simeq \varinjlim_m \operatorname{DM}^{\text{fp}}_{n,m}$. It will therefore suffice to prove the following variants of (1) and (2), for each $m \geq -2$.

(1′) For every connective $\mathbb{E}_\infty$-ring R, the ∞-category $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ is essentially small.

(2′) The functor $\mathrm{DM}^{\mathrm{fp}}_{n,m}$ commutes with filtered colimits.

The proofs of (1′) and (2′) proceed by induction on m. We begin with the case $m = -2$. Note that $\mathrm{DM}^{\mathrm{fp}}_{n,-2}(R)$ can be identified with the opposite of the full subcategory of $\tau_{\leq n}\,\mathrm{CAlg}^{\mathrm{cn}}_R$ spanned by the compact objects. Assertion (1′) follows from the observation that $\tau_{\leq n}\,\mathrm{CAlg}^{\mathrm{cn}}_R$ is a presentable ∞-category. To prove (2′), we note that the canonical map $\mathrm{DM}^{\mathrm{fp}}_{n,-2}(R) \to \mathrm{DM}^{\mathrm{fp}}_{n,-2}(\tau_{\leq n}R)$ is an equivalence of ∞-categories. It will therefore suffice to show that if $\{R_\alpha\}$ is a filtered diagram of n-truncated connective $\mathbb{E}_\infty$-rings having colimit R, then the canonical map

$$\theta : \varinjlim \mathrm{DM}^{\mathrm{fp}}_{n,-2}(R_\alpha) \to \mathrm{DM}^{\mathrm{fp}}_{n,-2}(R)$$

is an equivalence of ∞-categories. Lemma 2.3.4 implies that θ is fully faithful. We can identify (the opposite of) the essential image with a full subcategory $\mathcal{C} \subseteq \tau_{\leq n}\,\mathrm{CAlg}^{\mathrm{cn}}_R$ which consists of compact objects and is closed under finite colimits. We therefore obtain a fully faithful embedding $F : \mathrm{Ind}(\mathcal{C}) \hookrightarrow \tau_{\leq n}\,\mathrm{CAlg}^{\mathrm{cn}}_R$. The functor F admits small colimits and therefore admits a right adjoint G (Corollary T.5.5.2.9). If $\alpha : A \to B$ is a morphism in $\tau_{\leq n}\,\mathrm{CAlg}^{\mathrm{cn}}_R$ such that $G(\alpha)$ is an equivalence, then (since $\mathcal{C}$ contains $\tau_{\leq n}R\{x\}$) α induces a homotopy equivalence

$$\Omega^\infty A \simeq \mathrm{Map}_{\mathrm{CAlg}_R}(\tau_{\leq n}R\{x\}, A) \to \mathrm{Map}_{\mathrm{CAlg}_R}(\tau_{\leq n}R\{x\}, B) \simeq \Omega^\infty B.$$

Since A and B are connective we deduce that α is an equivalence. In other words, the functor G is conservative. It follows that F and G are inverse equivalences. Consequently, F exhibits the ∞-category of compact objects of $\tau_{\leq n}\,\mathrm{CAlg}^{\mathrm{cn}}_R$ as an idempotent completion of $\mathcal{C}$. In particular, every compact object $R' \in \tau_{\leq n}\,\mathrm{CAlg}^{\mathrm{cn}}_R$ is the colimit of a diagram $p : \mathrm{Idem} \to \tau_{\leq n}\,\mathrm{CAlg}^{\mathrm{cn}}_R$, where Idem is the simplicial set of Definition T.4.4.5.2. Since $\tau_{\leq n}\,\mathrm{CAlg}^{\mathrm{cn}}_R$ is equivalent to an $(n+1)$-category, R' is the colimit of the restriction of p to the $(n+1)$-skeleton of Idem, which is a finite simplicial set. Since $\mathcal{C}$ is closed under finite colimits, we conclude that $R' \in \mathcal{C}$. This completes the proof of (2′) in the case $m = -2$.

Now suppose that $m > -2$, and let $f : \mathfrak{X} \to \mathrm{Spec}\,R$ be an object of $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ for some $\mathbb{E}_\infty$-ring R. Then $\mathfrak{X}$ is quasi-compact, so we can choose an étale surjection $u : \mathfrak{X}_0 \to \mathfrak{X}$ where $\mathfrak{X}_0$ is affine. Let $\mathfrak{X}_\bullet$ denote the Čech nerve of u in the ∞-category of n-truncated spectral Deligne-Mumford stacks. We claim that each $\mathfrak{X}_i$ belongs to $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R)$. For $m > 0$ this is clear. When $m = 0$, we let $\mathfrak{Y}$ denote n-truncation of the i-fold fiber power of $\mathfrak{X}_0$ over $\mathrm{Spec}\,R$. Then $\mathfrak{Y}$ is affine, and the canonical map $\mathfrak{X}_i \to \mathfrak{Y}$ is a quasi-monomorphism. If $m = -1$, we can choose a commutative diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{v} & \mathrm{Spec}\,A \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & \mathrm{Spec}\,R & \end{array}$$

where v is a quasi-monomorphism and $\mathrm{Spec}\,A$ is finitely n-presented over $\mathrm{Spec}\,R$. Let $\mathfrak{X}'_\bullet$ be the Čech nerve of the composite map $(v \circ u) : \mathfrak{X}_0 \to \mathrm{Spec}\,A$. Since v is a quasi-monomorphism, the induced map $\mathfrak{X}_\bullet \to \mathfrak{X}'_\bullet$ induces an equivalence of 0-truncations. Since each $\mathfrak{X}'_i$ is affine, each $\mathfrak{X}_i$ is affine by Theorem VII.8.42.

We now prove (1′). Fix a connective $\mathbb{E}_\infty$-ring R. Since $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R)$ is essentially small by the inductive hypothesis, the ∞-category of simplicial objects of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R)$ is also essentially small. The above argument shows that every object of $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ can be obtained as the geometric realization (in $\mathrm{Stk}_{/\,\mathrm{Spec}\,R}$) of a simplicial object $\mathfrak{X}_\bullet$ of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R)$, so that $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ is essentially small.

We now prove (2′). Choose a filtered diagram of connective $\mathbb{E}_\infty$-rings $\{R_\alpha\}$ having colimit R, and consider the functor

$$\theta : \varinjlim_{\alpha} \mathrm{DM}^{\mathrm{fp}}_{n,m}(R_\alpha) \to \mathrm{DM}^{\mathrm{fp}}_{n,m}(R).$$

We first show that θ is fully faithful. We may assume without loss of generality that the diagram $\{R_\alpha\}$ is indexed by the nerve of a filtered partially ordered set P (Proposition T.5.3.1.16). Fix objects $\mathfrak{X}_\alpha, \mathfrak{Y}_\alpha \in \mathrm{DM}^{\mathrm{fp}}_{n,m}(R_\alpha)$. For $\beta \geq \alpha$, let $\mathfrak{X}_\beta$ and $\mathfrak{Y}_\beta$ denote the images of $\mathfrak{X}_\alpha$ and $\mathfrak{Y}_\alpha$ in $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R_\beta)$, and let $\mathfrak{X}$ and $\mathfrak{Y}$ denote the images of $\mathfrak{X}_\alpha$ and $\mathfrak{Y}_\alpha$ in $\mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$. We wish to show that the canonical map

$$\varinjlim_{\beta \geq \alpha} \mathrm{Map}_{\mathrm{Stk}_{/\operatorname{Spec} R_\beta}}(\mathfrak{X}_\beta, \mathfrak{Y}_\beta) \to \mathrm{Map}_{\mathrm{Stk}_{/\operatorname{Spec} R}}(\mathfrak{X}, \mathfrak{Y})$$

is a homotopy equivalence. Note that the left hand side can be identified with $\varinjlim_{\beta \geq \alpha} \mathrm{Map}_{\mathrm{Stk}_{/\operatorname{Spec} R}}(\mathfrak{X}_\beta, \mathfrak{Y})$. We will regard $\mathfrak{Y} \to \operatorname{Spec} R$ as fixed and prove the following:

$(*)$ For every object $\mathfrak{X}_\alpha \in \mathrm{DM}^{\mathrm{fp}}_{n,m'}(R_\alpha)$, the canonical map

$$\varinjlim_{\beta \geq \alpha} \mathrm{Map}_{\mathrm{Stk}_{/\operatorname{Spec} R}}(\mathfrak{X}_\beta, \mathfrak{Y}) \to \mathrm{Map}_{\mathrm{Stk}_{/\operatorname{Spec} R}}(\mathfrak{X}, \mathfrak{Y})$$

is a homotopy equivalence.

The proof of $(*)$ proceeds by induction on m'. Suppose first that $m = -2$, so that $\mathfrak{X}_\alpha \simeq \operatorname{Spec} A_\alpha$ for some connective $\mathbb{E}_\infty$-algebra A_α over R_α. Unwinding the definitions, we see that $\mathfrak{X}_\beta \simeq \tau_{\leq n}(A_\alpha \otimes_{R_\alpha} R_\beta)$ and that $\mathfrak{X} \simeq \tau_{\leq n}(A_\alpha \otimes_{R_\alpha} R)$. Since $R \simeq \varinjlim R_\beta$, we conclude that the canonical map

$$\varinjlim_{\beta \geq \alpha} \tau_{\leq n}(A_\alpha \otimes_{R_\alpha} R_\beta) \to \tau_{\leq n}(A_\alpha \otimes_{R_\alpha} R)$$

is an equivalence. Assertion $(*)$ now follows from Lemma 2.3.3.

Now suppose that $m' > -2$. Choose an étale surjection $u : \mathfrak{X}_{\alpha,0} \to \mathfrak{X}_\alpha$ where $\mathfrak{X}_{\alpha,0}$ is affine, and let $\mathfrak{X}_{\alpha,\bullet}$ be the Čech nerve of u. Define $\mathfrak{X}_{\beta,\bullet}$ and $\mathfrak{X}_\bullet$ as above. We wish to show that the canonical map

$$\phi : \varinjlim_{\beta \geq \alpha} \varprojlim_{[p] \in \mathbf{\Delta}} \mathrm{Map}_{\mathrm{Stk}_{/\operatorname{Spec} R}}(\mathfrak{X}_{\beta,p}, \mathfrak{Y}) \to \varprojlim_{[p] \in \mathbf{\Delta}} \mathrm{Map}_{\mathrm{Stk}_{/\operatorname{Spec} R}}(\mathfrak{X}_p, \mathfrak{Y})$$

is a homotopy equivalence. Choose an integer $k \geq m, n$, so all of the mapping spaces above are k-truncated (Lemma VIII.1.3.6). Arguing as in the proof of Lemma A.1.3.2.9, we can identify ϕ with the map

$$\varinjlim_{\beta \geq \alpha} \varprojlim_{[p] \in \mathbf{\Delta}_{\leq k+1}} (\mathrm{Map}_{\mathrm{Stk}_{/\operatorname{Spec} R}}(\mathfrak{X}_{\beta,p}, \mathfrak{Y}) \to \varprojlim_{[p] \in \mathbf{\Delta}_{\leq k+1}} \mathrm{Map}_{\mathrm{Stk}_{/\operatorname{Spec} R}}(\mathfrak{X}_p, \mathfrak{Y}).$$

Since filtered colimits in $\mathcal{S}$ commute with finite limits, ϕ is a finite limit of morphisms

$$\phi_p : \varinjlim_{\beta \geq \alpha} \mathrm{Map}_{\mathrm{Stk}_{/\operatorname{Spec} R}}(\mathfrak{X}_{\beta,p}, \mathfrak{Y}) \to \mathrm{Map}_{\mathrm{Stk}_{/\operatorname{Spec} R}}(\mathfrak{X}_p, \mathfrak{Y}).$$

Since $\mathfrak{X}_{\alpha,p} \in \mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R_\alpha)$, the map ϕ_p is an equivalence by the inductive hypothesis. This completes the proof that θ is fully faithful.

It remains to prove that θ is essentially surjective. Fix an object $\mathfrak{X} \in \mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ and choose an étale surjection $u : \mathfrak{X}_0 \to \mathfrak{X}$, where u is an étale surjection. Let $\mathfrak{X}_\bullet$ be the Čech nerve of u. Choose $k \geq m, n$, so that the ∞-category $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R)$ is equivalent to a $(k+1)$-category (Lemma VIII.1.3.6). It follows that $\mathfrak{X}_\bullet$ is a right Kan extension of $\mathfrak{X}^t_\bullet = \mathfrak{X}_\bullet \,|\, \mathrm{N}(\mathbf{\Delta}^{op}_{\leq k+3})$ (Proposition 2.1.4), which is an $(k+2)$-skeletal category object of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R)$ (see Definition 2.1.2). Since $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R) \simeq \varinjlim_\alpha \mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R_\alpha)$ by the inductive hypothesis and the simplicial set $\mathrm{N}(\mathbf{\Delta}^{op}_{\leq k+3})$ is finite, $\mathfrak{X}^t_\bullet$ is the image of a $(k+3)$-skeletal simplicial object $\overline{\mathfrak{X}}^t_\bullet$ of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R_\alpha)$. Enlarging α if necessary, we may assume that $\overline{\mathfrak{X}}^t_\bullet$ is a $(k+3)$-skeletal category object of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R_\alpha)$. Let $\overline{\mathfrak{X}}_\bullet : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{DM}^{\mathrm{fp}}_{n,m-1}$ be a right Kan extension of $\overline{\mathfrak{X}}^t_\bullet$. Using Lemma 2.3.3 and Proposition 2.1.5, we deduce that $\overline{\mathfrak{X}}_\bullet$ is a category object of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}$. Enlarging α if necessary, we may assume that $\overline{\mathfrak{X}}_\bullet$ is a groupoid object of $\mathrm{DM}^{\mathrm{fp}}_{n,m-1}(R_\alpha)$, and that the projection maps $\overline{\mathfrak{X}}_1 \to \overline{\mathfrak{X}}_0$ are étale. Then $\overline{\mathfrak{X}}_\bullet$ has a geometric realization $\overline{\mathfrak{X}}$ in $\mathrm{Stk}_{/\operatorname{Spec} R_\alpha}$. We will prove that, after enlarging α if necessary, we have $\overline{\mathfrak{X}} \in \mathrm{DM}^{\mathrm{fp}}_{n,m}(R_\alpha)$. It will then follow that $\overline{\mathfrak{X}}$ is a preimage of $\mathfrak{X} \in \mathrm{DM}^{\mathrm{fp}}_{n,m}(R)$ and the proof will be complete.

Since we have an étale surjection $\overline{\mathfrak{X}}_0 \to \overline{\mathfrak{X}}$, it is clear that $\overline{\mathfrak{X}}$ is locally finitely n-presented over $\operatorname{Spec} R_\alpha$. We next prove that the underlying ∞-topos $\mathcal{X}$ of $\overline{\mathfrak{X}}$ is coherent. The étale map $\overline{\mathfrak{X}}_0 \to \overline{\mathfrak{X}}$ corresponds to an object $U \in \mathcal{X}$. Since $\overline{\mathfrak{X}}_0$ and $\overline{\mathfrak{X}}_1$ belong to $\mathrm{DM}^{\mathrm{fp}}_{n}(R_\alpha)$, U and $U \times U$ are coherent objects of $\mathcal{X}$. Example VII.3.8 shows that the projection map $U \times U \to U$ is relatively i-coherent for every integer i. Let $\mathbf{1}$ denote the final object of $\mathcal{X}$. Since $p : U \to \mathbf{1}$ is an effective epimorphism, we deduce that p is relatively i-coherent for every integer i (Corollary VII.3.11). Using the i-coherence of U, we deduce that $\mathcal{X}$ is i-coherent. Since i is arbitrary, we deduce that $\mathcal{X}$ is coherent.

Assume now that $m \geq 0$. To complete the proof that $\overline{\mathfrak{X}} \in \mathrm{DM}^{\mathrm{fp}}_{n,m}$, we must show that for every discrete commutative ring A, the mapping space

$$\mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} A, \overline{\mathfrak{X}})$$

is m-truncated. For every étale map $A \to A'$, set

$$\mathcal{F}(A') = \mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} A', \overline{\mathfrak{X}}) \qquad \mathcal{F}_\bullet(A') = \mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} A', \overline{\mathfrak{X}}_\bullet).$$

Then $\mathcal{F}$ is an étale sheaf; we will prove that it is m-truncated. The projection $\overline{\mathfrak{X}}_0 \to \overline{\mathfrak{X}}$ induces an effective epimorphism of étale sheaves $\mathcal{F}_0 \to \mathcal{F}$. Since $\mathfrak{X}_0$ is affine, we may assume (enlarging α if necessary) that $\overline{\mathfrak{X}}_0$ is affine, so that $\mathcal{F}_0$ is 0-truncated. If $m > 0$, it suffices to prove that $\mathcal{F}_1 = \mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ is $(m-1)$-truncated, which follows from the fact that $\overline{\mathfrak{X}}_1 \in \mathrm{DM}^{\mathrm{fp}}_{n,m-1}$. If $m = 0$, we must work a bit harder: to show that $\mathcal{F}$ is discrete, we must show that $\mathcal{F}_1$ is an equivalence relation on $\mathcal{F}_0$ (note that each $\mathcal{F}_i$ is a *discrete* étale sheaf on A when $m \leq 1$). For this, it suffices to show that the diagonal map

$$v : \mathcal{F}_1 \to \mathcal{F}_1 \times_{\mathcal{F}_0 \times \mathcal{F}_0} \mathcal{F}_1$$

is an equivalence. Consider the diagonal map

$$\overline{\delta} : \overline{\mathfrak{X}}_1 \to \overline{\mathfrak{X}}_1 \times_{\overline{\mathfrak{X}}_0 \times \overline{\mathfrak{X}}_0} \overline{\mathfrak{X}}_1.$$

Since $\mathfrak{X}$ is a spectral algebraic space, the map

$$\delta : \mathfrak{X}_1 \to \mathfrak{X}_1 \times_{\mathfrak{X}_0 \times \mathfrak{X}_0} \mathfrak{X}_1$$

induces an equivalence of 0-truncations. Since $\mathrm{DM}^{\mathrm{fp}}_{0,-1}(R) \simeq \varinjlim_{\beta \geq \alpha} \mathrm{DM}^{\mathrm{fp}}_{0,-1}(R_\beta)$, we may assume (after enlarging α if necessary) that $\overline{\delta}$ also induces an equivalence of 0-truncations, from which it follows that v is an equivalence.

We now consider the case $m = -1$. Choose a commutative diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{u} & \operatorname{Spec} A \\ & {}_{f}\searrow \quad \swarrow & \\ & \operatorname{Spec} R & \end{array}$$

where u is a quasi-monomorphism and A is locally of finite presentation to order $n+1$ over R. Replacing A by $\tau_{\leq n} A$, we may assume that $\operatorname{Spec} A \in \mathrm{DM}^{\mathrm{fp}}_{n,-2}(R)$. Using the inductive hypothesis (and enlarging α if necessary), we may assume that $A = \tau_{\leq n}(\overline{A} \otimes_{R_\alpha} R)$ for some n-truncated $\overline{A}$ which is of finite presentation to order $n+1$ over R_α. The proof of $(*)$ shows that we may assume (after enlarging α if necessary) that u

is the image of a map $\overline{u} : \overline{\mathfrak{X}} \to \operatorname{Spec} \overline{A}$ in $\mathrm{Stk}_{/ \operatorname{Spec} A_\alpha}$. To complete the proof, it will suffice to show that $\overline{u}$ is a quasi-monomorphism. That is, we must show that the diagonal map

$$\overline{\mathfrak{X}} \to \overline{\mathfrak{X}} \times_{\operatorname{Spec} \overline{A}} \overline{\mathfrak{X}}$$

induces an equivalence of 0-truncations. This assertion is local on $\overline{\mathfrak{X}}$: it therefore suffices to show that the map

$$\overline{\mathfrak{X}}_1 \simeq \overline{\mathfrak{X}}_0 \times_{\overline{\mathfrak{X}}} \overline{\mathfrak{X}}_0 \to \overline{\mathfrak{X}}_0 \times_{\operatorname{Spec} \overline{A}} \overline{\mathfrak{X}}_0$$

induces an equivalence of 0-truncations. Note that both sides are affine. Since

$$\mathrm{DM}^{\mathrm{fp}}_{0,-2}(R) \simeq \varinjlim_{\beta \geq \alpha} \mathrm{DM}^{\mathrm{fp}}_{0,-2}(R_\beta),$$

it suffices to show that the map $\mathfrak{X}_1 \simeq \mathfrak{X}_0 \times_{\mathfrak{X}} \mathfrak{X}_0 \to \mathfrak{X}_0 \times_{\operatorname{Spec} A} \mathfrak{X}_0$ induces an equivalence of 0-truncations, which follows from our assumption that u is a quasi-monomorphism. □

2.4 Approximation of Quasi-Coherent Sheaves

Let $\mathfrak{X}$ be spectral Deligne-Mumford stack which is finitely n-presented over a connective $\mathbb{E}_\infty$-ring R which is given as a filtered colimit $\varinjlim R_\alpha$. In §2.3, we showed that $\mathfrak{X}$ can be always be written as

$$\tau_{\leq n}(\operatorname{Spec}^{\text{ét}} R \times_{\operatorname{Spec}^{\text{ét}} R_\alpha} \mathfrak{X}_\alpha)$$

for some index α and some spectral Deligne-Mumford stack $\mathfrak{X}_\alpha$ which is finitely n-presented over R_α (Theorem 2.3.2). Our goal in this section is to prove an analogous result for quasi-coherent sheaves on $\mathfrak{X}$. As in §2.3, it will be necessary to demand that our quasi-coherent sheaves satisfy some finiteness conditions.

Definition 2.4.1. Let R be a connective $\mathbb{E}_\infty$-ring and $n \geq 0$ an integer. Recall that a connective R-module M is said to be *finitely n-presented* if it is a compact object of the ∞-category $(\mathrm{Mod}_R)_{\leq n}$: equivalently, M is finitely n-presented if it is n-truncated and perfect to order $(n+1)$ (Definition VIII.2.6.10). If $\mathfrak{X}$ is a spectral Deligne-Mumford stack and $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, we see that $\mathcal{F}$ is *finitely n-presented* if, for every étale map $\eta : \operatorname{Spec}^{\text{ét}} R \to \mathfrak{X}$, the pullback $\eta^* \mathcal{F}$ is finitely n-presented when regarded as an object of $\mathrm{Mod}_R \simeq \mathrm{QCoh}(\operatorname{Spec}^{\text{ét}} R)$ (see Definition VIII.2.6.17).

The condition that a quasi-coherent sheaf be finitely n-presented is not stable under base change. However, we do have the following analogue of Proposition 2.2.10.

Proposition 2.4.2. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks, and let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{Y})$ be finitely n-presented. Then $\tau_{\leq n} f^* \mathcal{F}$ is finitely n-presented.*

Proof. Since $\mathcal{F}$ is connective and perfect to order $(n+1)$, we deduce that $f^* \mathcal{F}$ is connective and perfect to order $(n+1)$. It follows that $\tau_{\leq n} f^* \mathcal{F}$ is also connective and perfect to order $(n+1)$ (see Remark VIII.2.6.6). Since $\tau_{\leq n} f^* \mathcal{F}$ is obviously n-truncated, we deduce that $\tau_{\leq n} f^* \mathcal{F}$ is finitely n-presented. □

Construction 2.4.3. The functor $\mathfrak{X} \mapsto \mathrm{QCoh}(\mathfrak{X})$ classifies a Cartesian fibration $\theta : \mathcal{C} \to \mathrm{Stk}$. We can identify objects of $\mathcal{C}$ with pairs $(\mathfrak{X}, \mathcal{F})$, where $\mathfrak{X}$ is a spectral Deligne-Mumford stack and $\mathcal{F}$ is a quasi-coherent sheaf on $\mathfrak{X}$. Let $n \geq 0$ be an integer, and let $\mathcal{C}^{n-fp}$ denote the full subcategory of $\mathcal{C}$ spanned by those pairs $(\mathfrak{X}, \mathcal{F})$, where $\mathcal{F}$ is finitely n-presented. Using Proposition 2.4.2, we deduce that θ restricts to a Cartesian fibration $\mathcal{C}^{n-fp} \to \mathrm{Stk}$. This Cartesian fibration is classified by a functor $\mathrm{QCoh}^{n-fp} : \mathrm{Stk}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. We can describe this functor concretely as follows:

(a) To every spectral Deligne-Mumford stack $\mathfrak{X}$, $\mathrm{QCoh}^{n-fp}(\mathfrak{X})$ can be identified with the full subcategory of $\mathrm{QCoh}(\mathfrak{X})$ spanned by those quasi-coherent sheaves $\mathcal{F}$ which are finitely n-presented.

(b) To every map $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks, the functor

$$\mathrm{QCoh}^{n-fp}(f) : \mathrm{QCoh}^{n-fp}(\mathfrak{Y}) \to \mathrm{QCoh}^{n-fp}(\mathfrak{X})$$

is given on objects by the construction $\mathcal{F} \mapsto \tau_{\leq n} f^* \mathcal{F}$.

We can now formula a linear analogue of Theorem 2.3.2.

Theorem 2.4.4. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathfrak{X}$ be a spectral Deligne-Mumford m-stack over R, for some integer $m < \infty$. Assume that $\mathfrak{X}$ is ∞-quasi-compact, and let $n \geq 0$ be an integer. Then:*

(1) *The ∞-category $\mathrm{QCoh}^{n-fp}(\mathfrak{X})$ is essentially small.*

(2) *Suppose we are given a filtered diagram $\{R_\alpha\}$ of connective $\mathbb{E}_\infty$-algebras over R having colimit R'. Let $\mathfrak{X}_\alpha = \mathfrak{X} \times_{\mathrm{Spec}^{\acute{e}t} R} \mathrm{Spec}^{\acute{e}t} R_\alpha$, and let $\mathfrak{X}' = \mathfrak{X} \times_{\mathrm{Spec}^{\acute{e}t} R} \mathrm{Spec}^{\acute{e}t} R'$. Then the canonical functor*

$$\theta : \varinjlim_\alpha \mathrm{QCoh}^{n-fp}(\mathfrak{X}_\alpha) \to \mathrm{QCoh}^{n-fp}(\mathfrak{X}')$$

is an equivalence of ∞-categories.

The proof of Theorem 2.4.4 will require the following general observation.

Lemma 2.4.5. *Filtered colimits are left exact in the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of small ∞-categories.*

Proof. Let $G : \mathcal{C}\mathrm{at}_\infty \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{S})$ be the fully faithful embedding of Proposition A.A.7.10, given by $G(\mathcal{C})([n]) = \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\Delta^n, \mathcal{C})$. Since filtered colimits in $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{S})$ are left exact (Example T.7.3.4.7), it will suffice to show that G preserves finite limits and filtered colimits. The first assertion is obvious, and the second follows from the observation that each Δ^n is a compact object of $\mathcal{C}\mathrm{at}_\infty$. □

Proof of Theorem 2.4.4. Consider the following hypothesis for $m \geq -2$:

($*_m$) If $m \geq 0$, then $\mathfrak{X}$ is a spectral Deligne-Mumford m-stack. If $m = -1$, then there exists a quasi-monomorphism $\mathfrak{X} \to \mathrm{Spec}^{\text{ét}} A$ for some connective $\mathbb{E}_\infty$-algebra A over R (see the proof of Theorem 2.3.2). If $m = -2$, then $\mathfrak{X}$ is affine.

By assumption, condition ($*_m$) holds for m sufficiently large. We proceed by induction on m, beginning with the case $m = -2$. In this case, we can write $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} A$ for some connective $\mathbb{E}_\infty$-ring A. Using Remark VIII.2.6.7, we deduce that $\mathrm{QCoh}^{n-fp}(\mathfrak{X})$ is equivalent to the ∞-category of compact objects of the presentable ∞-category $(\mathrm{Mod}_A^{\mathrm{cn}})_{\leq n}$, which proves (1). To prove (2), let $A_\alpha = A \otimes_R R_\alpha$ and let $A' = A \otimes_R R'$, so that $A' \simeq \varinjlim_\alpha A_\alpha$. Note that the ∞-category $\mathcal{C} = (\mathrm{Mod}_A^{\mathrm{cn}})_{\leq n}$ is tensored over the ∞-category $\mathrm{Mod}_A^{\mathrm{cn}}$, and we have equivalences

$$(\mathrm{Mod}_{A_\alpha}^{\mathrm{cn}})_{\leq n} \simeq \mathrm{Mod}_{A_\alpha}(\mathcal{C}) \qquad (\mathrm{Mod}_{A'}^{\mathrm{cn}})_{\leq n} \simeq \mathrm{Mod}_{A'}(\mathcal{C}).$$

Note that the forgetful functor $\mathrm{CAlg}_A^{\mathrm{cn}} \to \mathrm{Alg}_A^{\mathrm{cn}}$ preserves sifted colimits. Combining this with Theorem A.6.3.5.10, we deduce that $(\mathrm{Mod}_{A'}^{\mathrm{cn}})_{\leq n}$ is the colimit of the diagram $(\mathrm{Mod}_{A_\alpha}^{\mathrm{cn}})_{\leq n}$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories. Equivalently, the functor θ appearing in (2) induces an equivalence of ∞-categories

$$\mathrm{Ind}(\varinjlim_\alpha \mathrm{QCoh}^{n-fp}(\mathfrak{X}_\alpha)) \to \mathrm{Ind}(\mathrm{QCoh}^{n-fp}(\mathfrak{X}')).$$

To prove that θ is an equivalence, it suffices to show that the domain and codomain of θ are idempotent complete. This is clear, since the domain and codomain of θ are equivalent to $(n+1)$-categories and admit finite colimits.

Now suppose that $m \geq -1$. Since $\mathfrak{X}$ is coherent, we can choose an étale surjection $u : \mathfrak{X}_0 \to \mathfrak{X}$, where $\mathfrak{X}_0$ is affine. Let $\mathfrak{X}_\bullet$ be the Čech nerve of u. Then each $\mathfrak{X}_p$ satisfies ($*_{m-1}$). Using the equivalence of ∞-categories

$\mathrm{QCoh}^{n-fp}(\mathfrak{X}) \simeq \varprojlim \mathrm{QCoh}^{n-fp}(\mathfrak{X}_\bullet)$ and the inductive hypothesis, we deduce that $\mathrm{QCoh}^{n-fp}(\mathfrak{X})$ is essentially small; this proves (1). To prove (2), we let $\mathfrak{X}_{\bullet,\alpha}$ denote the simplicial spectral Deligne-Mumford stack given by $\mathfrak{X}_\bullet \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, R_\alpha$, and $\mathfrak{X}'_\bullet$ the simplicial spectral Deligne-Mumford stack given by $\mathfrak{X}_\bullet \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, R'$. We have a commutative diagram

$$\begin{array}{ccc}
\varinjlim_\alpha \mathrm{QCoh}^{n-fp}(\mathfrak{X}_\alpha) & \xrightarrow{\theta} & \mathrm{QCoh}^{n-fp}(\mathfrak{X}') \\
\downarrow & & \downarrow \\
\varinjlim_\alpha \varprojlim_{[k]\in\mathbf{\Delta}} \mathrm{QCoh}^{n-fp}(\mathfrak{X}_{k,\alpha}) & \longrightarrow & \varprojlim_{[k]\in\mathbf{\Delta}} \mathrm{QCoh}^{n-fp}(\mathfrak{X}'_k) \\
\downarrow & & \downarrow \\
\varinjlim_\alpha \varprojlim_{[k]\in\mathbf{\Delta}_{\leq n+2}} \mathrm{QCoh}^{n-fp}(\mathfrak{X}_{k,\alpha}) & \xrightarrow{\phi} & \varprojlim_{[k]\in\mathbf{\Delta}_{\leq n+2}} \mathrm{QCoh}^{n-fp}(\mathfrak{X}'_k).
\end{array}$$

Here the vertical maps are equivalences by virtue of the observation that the functor QCoh^{n-fp} takes values in the full subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ spanned by those ∞-categories which are equivalent to $(n+1)$-categories (since $\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}$ is equivalent to an $(n+1)$-category, for every spectral Deligne-Mumford stack $\mathfrak{Y}$), and that this subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ is itself equivalent to an $(n+2)$-category. Consequently, to prove that θ is an equivalence of ∞-categories, it will suffice to show that ϕ is an equivalence of ∞-categories. The functor ϕ fits into a commutative diagram

$$\begin{array}{ccc}
\varinjlim_\alpha \varprojlim_{[k]\in\mathbf{\Delta}_{\leq n+2}} \mathrm{QCoh}^{n-fp}(\mathfrak{X}_{k,\alpha}) & & \\
& \searrow^{\phi'} & \\
\downarrow \phi & & \varprojlim_{[k]\in\mathbf{\Delta}_{\leq n+2}} \varinjlim_\alpha \mathrm{QCoh}^{n-fp}(\mathfrak{X}_{k,\alpha}). \\
& \swarrow_{\phi''} & \\
\varprojlim_{[k]\in\mathbf{\Delta}_{\leq n+2}} \mathrm{QCoh}^{n-fp}(\mathfrak{X}'_k) & &
\end{array}$$

Here ϕ' is an equivalence of ∞-categories by Lemma 2.4.5, and ϕ'' is an equivalence of ∞-categories by the inductive hypothesis. □

2.5 Descent of Properties along Filtered Colimits

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a a map of spectral Deligne-Mumford stack which are finitely n-presented over a filtered colimit $A \simeq \varinjlim A_\alpha$ of connective $\mathbb{E}_\infty$-rings. Using Theorem 2.3.2, we deduce the existence of an index α and a map $f_\alpha : \mathfrak{X}_\alpha \to \mathfrak{Y}_\alpha$ of spectral Deligne-Mumford stacks which are finitely n-presented over A_α, such that f is equivalent to the induced map

$$\tau_{\leq n}(\mathrm{Spec}^{\text{ét}} A \times_{\mathrm{Spec}^{\text{ét}} A_\alpha} \mathfrak{X}_\alpha) \to \tau_{\leq n}(\mathrm{Spec}^{\text{ét}} A \times_{\mathrm{Spec}^{\text{ét}} A_\alpha} \mathfrak{Y}_\alpha).$$

Our goal in this section is to prove a variety of results which assert that if f has some property P, then we can arrange that f_α also has the property P. More precisely, we show the following:

(a) If f is affine, then f_α can be chosen to be affine (Proposition 2.5.1).

(b) If f is étale, then f_α can be chosen to be étale (Proposition 2.5.2).

(c) If f is an open immersion, then f_α can be chosen to be an open immersion (Corollary 2.5.3).

(*d*) If f is a closed immersion, then f_α can be chosen to be a closed immersion (Proposition 2.5.7).

(*e*) If f is strongly separated, then f_α can be chosen to be strongly separated (Corollary 2.5.8).

(*f*) If f is surjective, then f_α can be chosen to be surjective (Proposition 2.5.9).

We begin with the proof of (a).

Proposition 2.5.1. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geq 0$ be an integer, let $f_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ be a morphism in $\mathrm{DM}_n^{\mathrm{fp}}(A_0)$. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Suppose that f is affine. Then there exists an index α such that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is affine.*

Proof. Since $\mathfrak{Y}_0$ is quasi-compact, we can choose an étale surjection $\mathfrak{Y}_0' \to \mathfrak{Y}_0$, where $\mathfrak{Y}_0' \simeq \mathrm{Spec}^{\text{ét}} B_0$ is affine. Replacing $\mathfrak{Y}_0$ by $\mathfrak{Y}_0'$, we can reduce to the case where $\mathfrak{Y}_0$ is affine, so that $\mathfrak{X}$ is also affine. The proof of Theorem 2.3.2 shows that there exists an index α and an affine spectral Deligne-Mumford stack $\mathfrak{Z} \in \mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ having image $\mathfrak{X}$ in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Let $\mathfrak{X}_\alpha$ denote the image of $\mathfrak{X}_0$ in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$. Then $\mathfrak{X}_\alpha$ and $\mathfrak{Z}$ have equivalent images in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Changing α if necessary, we may assume that $\mathfrak{X}_\alpha \simeq \mathfrak{Z}$ is affine, so that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is affine. □

Our next result is somewhat more difficult.

Proposition 2.5.2. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geq 0$ be an integer, let $f_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ be a morphism in $\mathrm{DM}_n^{\mathrm{fp}}(A_0)$. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Suppose that f is étale. Then there exists an index α such that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is étale.*

Corollary 2.5.3. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geq 0$ be an integer, let $f_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ be a morphism in $\mathrm{DM}_n^{\mathrm{fp}}(A_0)$. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Suppose that f is an open immersion. Then there exists an index α such that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is an open immersion.*

Proof. Using Proposition 2.5.2, we can reduce to the case where f_0 is étale. Let δ_0 denote the diagonal map $\mathfrak{X}_0 \to \mathfrak{X}_0 \times_{\mathfrak{Y}_0} \mathfrak{X}_0$. Then the image of δ_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$ is an equivalence. Theorem 2.3.2 implies that there exists an index α such that the image of δ_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is an equivalence. It follows that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is an open immersion. □

The proof of Proposition 2.5.2 will require some preliminaries.

Remark 2.5.4. Let B be a connective $\mathbb{E}_\infty$-ring and let $n \geq 0$ be an integer. The truncation map $B \to \tau_{\leq n} B$ is $(n+1)$-connective. Consequently, Theorem A.7.4.3.1 supplies an $(2n+3)$-connective map

$$(\tau_{\leq n} B \otimes_B \tau_{\geq n+1} B) \to L_{\tau_{\leq n} B/B}[-1].$$

The map $\tau_{\geq n+1} B \to (\tau_{\leq n} B \otimes_B \tau_{\geq n+1} B)$ is $(2n+2)$-connective, so that the composite map

$$\tau_{\geq n+1} B \to L_{\tau_{\leq n} B/B}[-1]$$

determines bijections $\theta_m : \pi_m B \to \pi_{m+1} L_{\tau_{\leq n} B/B}$ which for $n < m < 2n+2$ and a surjectivion when $m = 2n+2$.

Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings, and let $\partial : L_{\tau_{\leq n} B/B}[-1] \to L_{B/A}$ be the associated boundary map. Unwinding the definitions, we see that for $m > n$, the composition

$$\pi_m B \xrightarrow{\theta_m} \pi_{m+1} L_{\tau_{\leq n} B/B} \xrightarrow{\partial} \pi_m L_{B/A}$$

is induced by the universal A-linear derivation $d : B \to L_{B/A}$. In particular (taking $n = 0$), we conclude that the induced maps $\pi_m B \to \pi_m L_{B/A}$ is $\pi_0 B$-linear for $m > 0$.

Lemma 2.5.5. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings, and let $n \geq 0$. The induced map $\tau_{\leq n} A \to \tau_{\leq n} B$ is étale if and only if the following conditions are satisfied:*

(1) *The commutative ring $\pi_0 B$ is finitely presented over $\pi_0 A$.*

(2) *The relative cotangent complex $L_{B/A}$ is $(n+1)$-connective.*

(3) *Let $d : B \to L_{B/A}$ be the universal derivation (so that d is a map of A-module spectra). Then d induces a surjection $\pi_{n+1} B \to \pi_{n+1} L_{B/A}$.*

Proof. Suppose first that $\tau_{\leq n} B$ is étale over $\tau_{\leq n} A$. Then $\pi_0 B$ is étale over $\pi_0 A$, which immediately implies (1). Using Theorem A.7.5.0.6, we can choose an étale A-algebra A' and an isomorphism $\alpha : \pi_0 A' \simeq \pi_0 B$. Theorem A.7.5.4.2 implies that α can be lifted to a map of A-algebras $\overline{\alpha} : A' \to B$. Since $L_{A'/A} \simeq 0$, we conclude that the canonical map $L_{B/A} \to L_{B/A'}$ is an equivalence. Note that $\overline{\alpha}$ induces an equivalence $\tau_{\leq n} A' \to \tau_{\leq n} B$, and is therefore n-connective. Using Corollary A.7.4.3.2, we deduce that $L_{B/A} \simeq L_{B/A'}$ is $(n+1)$-connective, thereby proving (2). To prove (3), we note that the composite map $A' \to B \to \tau_{\leq n} B$ is $(n+1)$-connective, so that $L_{\tau_{\leq n} B/A} \simeq L_{\tau_{\leq n} B/A'}$ is $(n+2)$-connective. We have a fiber sequence

$$(\tau_{\leq n} B) \otimes_B L_{B/A} \to L_{\tau_{\leq n} B/A} \to L_{\tau_{\leq n} B/B}.$$

The vanishing of $\pi_{n+1} L_{\tau_{\leq n} B/A}$ implies that the boundary map

$$\theta : \pi_{n+2} L_{\tau_{\leq n} B/B} \to \pi_{n+1}(\tau_{\leq n} B \otimes_B L_{B/A}) \simeq \pi_{n+1} L_{B/A}$$

is surjective. Using Remark 2.5.4, we deduce that the universal derivation $d : B \to L_{B/A}$ induces a surjection $\pi_{n+1} B \to \pi_{n+1} L_{B/A}$, so that (3) is satisfied.

Now suppose that conditions (1), (2), and (3) hold. We wish to prove that $\tau_{\leq n} B$ is étale over $\tau_{\leq n} A$. Consider the fiber sequence

$$(\tau_{\leq n} B) \otimes_B L_{B/A} \to L_{\tau_{\leq n} B/A} \to L_{\tau_{\leq n} B/B}.$$

It follows from condition (2) that $(\tau_{\leq n} B) \otimes_B L_{B/A}$ is $(n+1)$-connective, and we have a canonical isomorphism $\pi_{n+1}(\tau_{\leq n} B \otimes_B L_{B/A}) \simeq \pi_{n+1} L_{B/A}$. Using Remark 2.5.4, we deduce that $L_{\tau_{\leq n} B/B}$ is $(n+2)$-connective and obtain a canonical isomorphism $\pi_{n+2} L_{\tau_{\leq n} B/B} \simeq \pi_{n+1} B$. It follows that $L_{\tau_{\leq n} B/A}$ is $(n+1)$-connective, and we have a short exact sequence of abelian groups

$$\pi_{n+1} B \to \pi_{n+1} L_{B/A} \to \pi_{n+1} L_{\tau_{\leq n} B/A} \to 0.$$

Using condition (3), we conclude that $L_{\tau_{\leq n} B/A}$ is $(n+2)$-connective. Invoking Lemma VII.8.8, we see that f factors as a composition

$$A \xrightarrow{f'} A' \xrightarrow{f''} B$$

where f' is étale and f'' is $(n+1)$-connective. It follows that $\tau_{\leq n} B \simeq \tau_{\leq n} A'$ is étale over $\tau_{\leq n} A$, as desired. □

We now prove Proposition 2.5.2 in the affine case.

Lemma 2.5.6. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $f : A_0 \to B_0$ be a map of connective $\mathbb{E}_\infty$-rings which is of finite presentation to order $n+1$ for some $n \geq 0$. Let $B_\alpha = A_\alpha \otimes_{A_0} B_0$ and let $B = \varinjlim B_\alpha \simeq A \otimes_{A_0} B_0$. Suppose that $\tau_{\leq n} B$ is étale over $\tau_{\leq n} A$. Then there exists an index α such that $\tau_{\leq n} B_\alpha$ is étale over $\tau_{\leq n} A_\alpha$.*

Proof. Using Lemma 2.5.5, we see that $L_{B/A} \simeq B \otimes_{B_0} L_{B_0/A_0}$ is $(n+1)$-connective. Using Theorem 2.4.4, we deduce that there exists an index α such that $B_\alpha \otimes_{B_0} L_{B_0/A_0}$ is $(n+1)$-connective. Since B_0 is of finite presentation to order $(n+1)$ over A_0, the relative cotangent complex L_{B_0/A_0} is perfect to order $(n+1)$ so that $\pi_{n+1} L_{B_0/A_0}$ is finitely generated as a module over $\pi_0 B_0$. Choose a finite collection of generators

$x_1, \ldots, x_k$ for $\pi_{n+1} L_{B_0/A_0}$, and let $x'_1, \ldots, x'_k$ denote their images in $\pi_{n+1} L_{B/A}$. Lemma 2.5.5 implies that the universal derivation $B \to L_{B/A}$ induces a surjection

$$\phi : \varinjlim \pi_{n+1} B_\alpha \simeq \pi_{n+1} B \to \pi_{n+1} L_{B/A}.$$

It follows that there exists an index α and elements $y_1, \ldots, y_k \in \pi_{n+1} B_\alpha$ such that $\phi(y_i) = x'_i$ for $1 \leq i \leq k$. Let x''_i denote the image of x_i in $\pi_{n+1} L_{B_\alpha/A_\alpha}$. Enlarging α if necessary, we may assume that the universal derivation $d_\alpha : B_\alpha \to L_{B_\alpha/A_\alpha}$ carries each y_i to x''_i. Note that $\pi_{n+1} L_{B_\alpha/A_\alpha} \simeq \mathrm{Tor}_0^{\pi_0 B_0}(\pi_0 B, \pi_{n+1} L_{B_0/A_0})$, so that the elements x''_i generate $\pi_{n+1} L_{B_\alpha/A_\alpha}$ as a module over $\pi_0 B_\alpha$. Since the universal derivation induces a $\pi_0 B_\alpha$-linear map $\pi_{n+1} B_\alpha \to \pi_{n+1} L_{B_\alpha/A_\alpha}$ (see Remark 2.5.4), we deduce that this map is surjective. Applying Lemma 2.5.5, we conclude that $\tau_{\leq n} B_\alpha$ is étale over $\tau_{\leq n} A_\alpha$, as desired. □

Proof of Proposition 2.5.2. Since $\mathfrak{Y}_0$ is quasi-compact, we can choose an étale surjection $\mathfrak{Y}'_0 \to \mathfrak{Y}_0$, where $\mathfrak{Y}'_0$ is affine. Replacing $\mathfrak{Y}_0$ by $\mathfrak{Y}'_0$ and $\mathfrak{X}_0$ by the fiber product $\mathfrak{X}_0 \times_{\mathfrak{Y}_0} \mathfrak{Y}'_0$, we can assume that $\mathfrak{Y}_0$ is affine. Since $\mathfrak{X}_0$ is quasi-compact, we can choose an étale surjection $\mathfrak{X}'_0 \to \mathfrak{X}_0$, where $\mathfrak{X}'_0$ is affine. We may therefore replace $\mathfrak{X}_0$ by $\mathfrak{X}'_0$ and thereby reduce to the case where $\mathfrak{X}_0$ is also affine. The desired result now follows immediately from Theorem VIII.1.2.1 and Lemma 2.5.6. □

Proposition 2.5.7. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geq 0$ be an integer, let $f_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ be a morphism in $\mathrm{DM}_n^{\mathrm{fp}}(A_0)$. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Suppose that f is a closed immersion. Then there exists an index α such that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is a closed immersion.*

Proof. Since $\mathfrak{Y}_0$ is quasi-compact, we can choose an étale surjection $\mathfrak{Y}'_0 \to \mathfrak{Y}_0$, where $\mathfrak{Y}'_0 \simeq \mathrm{Spec}^{\text{ét}} B_0$ is affine. Replacing $\mathfrak{Y}_0$ by $\mathfrak{Y}'_0$, we can reduce to the case where $\mathfrak{Y}_0$ is affine. Using Proposition 2.5.1, we may assume that $\mathfrak{X}_0 \simeq \mathrm{Spec}^{\text{ét}} C_0$ is also affine. The condition that f is a closed immersion guarantees that the map

$$\varinjlim \mathrm{Tor}_0^{\pi_0 A_0}(\pi_0 A_\alpha, \pi_0 B_0) \to \varinjlim \mathrm{Tor}_0^{\pi_0 A_0}(\pi_0 A_\alpha, \pi_0 C_0)$$

is surjective. Since $\pi_0 C_0$ is a finitely presented algebra over $\pi_0 B_0$, we deduce that there is an index α such that the image of the map

$$\mathrm{Tor}_0^{\pi_0 A_0}(\pi_0 A_\alpha, \pi_0 B_0) \to \mathrm{Tor}_0^{\pi_0 A_0}(\pi_0 A_\alpha, \pi_0 C_0)$$

contains the image of $\pi_0 C_0$, and is therefore surjective. It follows that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is a closed immersion. □

Corollary 2.5.8. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geq 0$ be an integer, let $f_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ be a morphism in $\mathrm{DM}_n^{\mathrm{fp}}(A_0)$. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Suppose that f is strongly separated. Then there exists an index α such that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is strongly separated.*

Proof. Apply Proposition 2.5.7 to the diagonal map $\mathfrak{X}_0 \to \mathfrak{X}_0 \times_{\mathfrak{Y}_0} \mathfrak{X}_0$. □

Our final goal in this section is to prove the following:

Proposition 2.5.9. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geq 0$ be an integer, let $f_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ be a morphism in $\mathrm{DM}_n^{\mathrm{fp}}(A_0)$. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. Suppose that f is surjective. Then there exists an index α such that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is surjective.*

The proof will require the following fact from commutative algebra:

Theorem 2.5.10 (Chevalley's Constructibility Theorem)**.** *Let $f : R \to R'$ be a map of commutative rings such that R' is finitely presented as an R-algebra. Then the induced map $\mathrm{Spec}^{\mathrm{Z}} R' \to \mathrm{Spec}^{\mathrm{Z}} R$ has constructible image in $\mathrm{Spec}^{\mathrm{Z}} R$.*

Combining Theorem 2.5.10 with Corollary A.3.36, we obtain the following result:

Lemma 2.5.11. *Let A_0 be a commutative ring, and let $\{A_\alpha\}$ be a filtered diagram of commutative A_0-algebras having colimit A. Suppose that B_0 is an A_0-algebra of finite presentation, set $B = \pi_0(B_0 \otimes_{A_0} A)$, and assume that the map $\mathrm{Spec}^{\mathrm{Z}} B \to \mathrm{Spec}^{\mathrm{Z}} A$ is surjective. Then there exists an index α such that the map $\mathrm{Spec}^{\mathrm{Z}} B_\alpha \to \mathrm{Spec}^{\mathrm{Z}} A_\alpha$ is surjective, where $B_\alpha = \pi_0(B_0 \otimes_{A_0} A_\alpha)$.*

Proof of Proposition 2.5.9. We may assume without loss of generality that our diagram is indexed by a filtered partially ordered set. Since $\mathfrak{Y}_0$ is quasi-compact, we can choose an étale surjection $\mathfrak{Y}'_0 \to \mathfrak{Y}_0$, where $\mathfrak{Y}'_0$ is quasi-compact. Replacing $\mathfrak{Y}_0$ by $\mathfrak{Y}'_0$, we can reduce to the case where $\mathfrak{Y}_0 \simeq \mathrm{Spec}^{\text{ét}} B_0$ is affine. Similarly, we can assume that $\mathfrak{X}_0 \simeq \mathrm{Spec}^{\text{ét}} C_0$ is affine. Our assumption implies that the map of topological spaces

$$\mathrm{Spec}^{\mathrm{Z}} \mathrm{Tor}_0^{\pi_0 A_0}(\pi_0 C_0, \pi_0 A) \to \mathrm{Spec}^{\mathrm{Z}} \mathrm{Tor}_0^{\pi_0 A_0}(\pi_0 B_0, \pi_0 A)$$

is surjective. Using Lemma 2.5.11, we deduce the existence of an index α such that the map

$$\mathrm{Spec}^{\mathrm{Z}} \mathrm{Tor}_0^{\pi_0 A_0}(\pi_0 C_0, \pi_0 A_\alpha) \to \mathrm{Spec}^{\mathrm{Z}} \mathrm{Tor}_0^{\pi_0 A_0}(\pi_0 B_0, \pi_0 A_\alpha)$$

is surjective. It follows that the image of f_0 in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$ is a surjective map of spectral Deligne-Mumford stacks. □

3 Properness

Let $f : X \to Y$ be a map of schemes. Recall that f is said to be *proper* if it is of finite type, separated, and *universally closed*: that is, if for every pullback diagram of schemes

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y, \end{array}$$

the morphism f' induces a closed map between the underlying topological spaces of X' and Y'. Our goal in this section is to study the analogous condition in the setting of spectral algebraic geometry. We begin in §3.1 by introducing the notion of a *strongly proper* morphism $f : \mathfrak{X} \to \mathfrak{Y}$ between spectral Deligne-Mumford stacks. Here the word “strong” is included to indicate that we require in particular that that the diagonal $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ be a closed immersion (so that we exclude examples such as moduli stacks of semistable curves from our considerations).

The main result of this section is the following version of the proper direct image theorem which we prove in §3.2: if $f : \mathfrak{X} \to \mathfrak{Y}$ is a strongly proper morphism which is locally almost of finite presentation, then the pushforward functor $f_* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{Y})$ carries almost perfect objects of $\mathrm{QCoh}(\mathfrak{X})$ to almost perfect objects of $\mathrm{QCoh}(\mathfrak{Y})$ (Theorem 3.2.2). One pleasant feature of our setting is that the statement of this result does not require any Noetherian hypotheses on $\mathfrak{X}$ or $\mathfrak{Y}$. However, our proof will proceed by reduction to the Noetherian case (using the techniques of §2), followed by reduction to the usual direct image theorem in classical algebraic geometry.

In §3.3, we introduce the notion of a *proper* R-linear ∞-category, and the related notion of a *locally proper* quasi-coherent stack $\mathcal{C}$ on a spectral Deligne-Mumford stack $\mathfrak{X}$. We then prove a categorified analogue of the proper direct image theorem: if $f : \mathfrak{X} \to \mathfrak{Y}$ is strongly proper, almost of finite presentation, and flat, then the pushforward functor $f_* : \mathrm{QStk}(\mathfrak{X}) \to \mathrm{QStk}(\mathfrak{Y})$ carries locally proper quasi-coherent stacks on $\mathfrak{X}$ to locally proper quasi-coherent stacks on $\mathfrak{Y}$ (Theorem 3.3.11).

The condition that a map of spectral algebraic spaces $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be strongly proper (or strongly separated) depends only on the underlying map of ordinary algebraic spaces $f_0 : (\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathcal{O}_{\mathcal{Y}})$. That is, f is strongly proper (strongly separated) if and only if f_0 is proper (separated) in the sense of classical algebraic geometry. Consequently, many basic facts about strongly proper and strongly separated morphisms can be deduced immediately from the corresponding assertions in classical algebraic geometry. In particular, the valuative criteria for separatedness and properness carry over to the setting of spectral algebraic geometry without essential change. We will give precise formulations (and proofs) in §3.4.

3.1 Strongly Proper Morphisms

In this section, we will study the theory of *proper morphisms* $f : \mathfrak{X} \to \mathfrak{Y}$ between spectral Deligne-Mumford stacks. For simplicity, we will confine our attention to the case where f is a relative 0-stack (so that the fibers of f are spectral algebraic spaces).

Definition 3.1.1. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of spectral Deligne-Mumford stacks. We will say that f is *strongly proper* if the following conditions are satisfied:

(i) The morphism f is strongly separated (in particular, f is a relative Deligne-Mumford 0-stack).

(ii) The morphism f is quasi-compact.

(iii) The morphism f is locally of finite presentation to order 0 (see Definition IX.8.16).

(iv) For every pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}^{\text{ét}} R & \longrightarrow & \mathfrak{Y}, \end{array}$$

the induced map of topological spaces $|\,\mathfrak{X}'\,| \to |\operatorname{Spec}^{\text{ét}} R| \simeq \operatorname{Spec}^{\mathrm{Z}} R$ is closed.

Remark 3.1.2. The condition that a morphism $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be strongly proper depends only on the induced map of 0-truncated spectral algebraic spaces $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathcal{O}_{\mathcal{Y}})$. Note that if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ are 0-truncated spectral algebraic spaces, then f is strongly proper if and only if it is proper when regarded as a map between ordinary algebraic spaces (as defined in [31]).

Example 3.1.3. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a closed immersion of spectral Deligne-Mumford stacks. Then f is strongly proper.

Example 3.1.4. Let R be an ordinary commutative ring, let $n \geq 0$ be an integer, and let $\mathbf{P}^n_R$ denote the projective space of dimension n over R. We can identify $\mathbf{P}^n_R$ with a 0-truncated spectral algebraic space which is strongly proper over $\operatorname{Spec}^{\text{ét}} R$.

Remark 3.1.5. Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \longrightarrow & \mathfrak{Y}\,. \end{array}$$

If f is strongly proper, then f' is strongly proper.

Remark 3.1.6. Suppose we are given a collection of strongly proper morphisms $\{f_\alpha : \mathfrak{X}_\alpha \to \mathfrak{Y}_\alpha\}$. Then the induced map $\coprod \mathfrak{X}_\alpha \to \coprod \mathfrak{Y}_\alpha$ is strongly proper.

Proposition 3.1.7. *The condition that a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ be strongly proper is local with respect to the étale topology. That is, if we are given an étale covering $\{\mathfrak{Y}_\alpha \to \mathfrak{Y}\}$ such that each of the induced maps*

$$\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}_\alpha \to \mathfrak{Y}_\alpha$$

is strongly proper, then f is strongly proper.

Proof. Using Remark IX.4.17, Proposition VIII.1.4.11, and Proposition IX.8.24, we see that f is strongly separated, quasi-compact, and locally of finite presentation to order 0. It will therefore suffice to show that for every pullback diagram

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}^{\text{ét}} R & \longrightarrow & \mathfrak{Y}, \end{array}$$

the induced map $|\,\mathfrak{X}'\,| \to \operatorname{Spec}^{\mathrm{Z}} R$ is closed. Replacing $\mathfrak{Y}$ by $\operatorname{Spec}^{\text{ét}} R$, we are reduced to proving that the map $|\,\mathfrak{X}\,| \to |\,\mathfrak{Y}\,|$ is closed. We may assume that each $\mathfrak{Y}_\alpha$ is affine and that the collection of indices α is finite. Write $\coprod \mathfrak{Y}_\alpha = \operatorname{Spec}^{\text{ét}} R'$ and $\mathfrak{X}' = \mathfrak{X} \times_{\operatorname{Spec}^{\text{ét}} R} \operatorname{Spec}^{\text{ét}} R'$, so that R' is faithfully flat over R and the induced map $f' : \mathfrak{X}' \to \operatorname{Spec}^{\text{ét}} R'$ is proper. We have a commutative diagram of topological spaces

$$\begin{array}{ccc} |\,\mathfrak{X}'\,| & \xrightarrow{\psi} & |\,\mathfrak{X}\,| \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ \operatorname{Spec}^{\mathrm{Z}} R' & \xrightarrow{\psi'} & \operatorname{Spec}^{\mathrm{Z}} R. \end{array}$$

Fix a closed subset $K \subseteq |\,\mathfrak{X}\,|$; we wish to show that $\phi(K) \subseteq \operatorname{Spec}^{\mathrm{Z}} R$ is closed. Since $\psi^{-1}K$ is a closed subset of $|\,\mathfrak{X}'\,|$, the properness of f' implies that $\phi'(\psi^{-1}K)$ is a closed subset of $\operatorname{Spec}^{\mathrm{Z}} R'$. Corollary 1.4.11, gives $\psi'^{-1}(\phi(K)) = \phi'(\psi^{-1}K)$, so that $\psi'^{-1}(\phi(K))$ is a closed subset of $\operatorname{Spec}^{\mathrm{Z}} R'$. Since ψ' is a quotient map (Proposition VII.5.9), we conclude that $\phi(K)$ is a closed subset of $\operatorname{Spec}^{\mathrm{Z}} R$. □

Proposition 3.1.8. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a strongly proper map between quasi-separated spectral algebraic spaces. Then the induced map $|\,\mathfrak{X}\,| \to |\,\mathfrak{Y}\,|$ is closed.*

Proof. Writing $\mathfrak{Y}$ as a union of its quasi-compact open substacks, we can reduce to the case where $\mathfrak{Y}$ is quasi-compact. Choose an étale surjection $\operatorname{Spec}^{\text{ét}} R \to \mathfrak{Y}$, and form a pullback diagram

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec}^{\text{ét}} R & \longrightarrow & \mathfrak{Y}. \end{array}$$

We then obtain a diagram of topological spaces

$$\begin{array}{ccc} |\,\mathfrak{X}'\,| & \xrightarrow{\psi} & |\,\mathfrak{X}\,| \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ \operatorname{Spec}^{\mathrm{Z}} R & \xrightarrow{\psi'} & |\,\mathfrak{Y}\,|. \end{array}$$

Let $K \subseteq |\,\mathfrak{X}\,|$ be closed; we wish to show that $\phi(K)$ is closed. Since $\psi^{-1}K$ is a closed subset of $|\,\mathfrak{X}'\,|$, the properness of f implies that $\phi'(\psi^{-1}K)$ is a closed subset of $\operatorname{Spec}^{\mathrm{Z}} R'$. Corollary 1.4.11, gives $\psi'^{-1}(\phi(K)) = \phi'(\psi^{-1}K)$, so that $\psi'^{-1}(\phi(K))$ is a closed subset of $\operatorname{Spec}^{\mathrm{Z}} R$. Since ψ' is a quotient map (Proposition 1.4.14), we conclude that $\phi(K)$ is a closed subset of $|\,\mathfrak{Y}\,|$. □

Proposition 3.1.9. *Suppose we are given maps of spectral Deligne-Mumford stacks*

$$\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}.$$

(1) *If f and g are strongly proper, then $g \circ f$ is strongly proper.*

(2) *If $g \circ f$ is strongly proper and g is strongly separated, then f is strongly proper.*

Proof. We first prove (1). Assume that f and g are strongly proper. Using Remark IX.4.19, Proposition VIII.1.4.15, and Proposition IX.8.10, we see that $g \circ f$ is strongly separated, quasi-compact, and locally of finite presentation to order 0. To show that $g \circ f$ is proper, it will suffice to verify condition (*iv*) of Definition 3.1.1. Fix a map $\operatorname{Spec}^{\text{ét}} R \to \mathfrak{Z}$; we wish to show that the composite map

$$|\,\mathfrak{X} \times_{\mathfrak{Z}} \operatorname{Spec}^{\text{ét}} R| \xrightarrow{\phi} |\,\mathfrak{Y} \times_{\mathfrak{Z}} \operatorname{Spec}^{\text{ét}} R| \xrightarrow{\psi} \operatorname{Spec}^{\mathrm{Z}} R$$

is closed. The map ψ is closed by virtue of our assumption that g is strongly proper, and the map ϕ is closed by Proposition 3.1.8.

We now prove (2). Assume that g is strongly separated and that $g \circ f$ is strongly proper. The map f factors as a composition

$$\mathfrak{X} \xrightarrow{f'} \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \xrightarrow{f''} \mathfrak{Y}.$$

The map f'' is a pullback of $g \circ f$ and therefore strongly proper. The map f' is a pullback of the diagonal map $\mathfrak{Y} \to \mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{Y}$, and therefore a closed immersion. Example 3.1.3 shows that f' is strongly proper, so that $f = f'' \circ f'$ is strongly proper by assertion (1). □

Proposition 3.1.10. *Let A_0 be a connective $\mathbb{E}_\infty$-ring. Suppose we are given a filtered diagram of connective A_0-algebras $\{A_\alpha\}$ having colimit A. Let $n \geq 0$ be an integer, let $\mathfrak{X}_0 \in \mathrm{DM}_n^{\mathrm{fp}}(A_0)$. For each index α, let $\mathfrak{X}_\alpha$ denote the image of $\mathfrak{X}_0$ in $\mathrm{DM}_n^{\mathrm{fp}}(A_\alpha)$, and let $\mathfrak{X}$ denote the image of $\mathfrak{X}_0$ in $\mathrm{DM}_n^{\mathrm{fp}}(A)$. If $\mathfrak{X}$ is strongly proper over $\operatorname{Spec}^{\text{ét}} A$, then there exists an index α such that $\mathfrak{X}_\alpha$ is strongly proper over $\operatorname{Spec}^{\text{ét}} A_\alpha$.*

The proof of Proposition 3.1.10 will require some preliminaries.

Remark 3.1.11. Let $f : \mathfrak{X} \to \operatorname{Spec}^{\text{ét}} R$ be a map of spectral Deligne-Mumford stacks, and assume that $\mathfrak{X}$ is a quasi-compact quasi-separated algebraic space. Let $\phi : |\,\mathfrak{X}\,| \to \operatorname{Spec}^{\mathrm{Z}} R$ be the underlying map of topological spaces. Then the fibers of ϕ are quasi-compact. It follows that for any filtered collection of closed subsets $\{K_\alpha \subseteq |\,\mathfrak{X}\,|\}$, we have $\phi(\bigcap K_\alpha) = \bigcap \phi K(\alpha)$. Since $|\,\mathfrak{X}\,|$ has a basis of quasi-compact open subsets, every closed set $K \subseteq |\,\mathfrak{X}\,|$ can be obtained as a (filtered) intersection of closed subsets with quasi-compact complements. Consequently, to prove that ϕ is closed, it will suffice to show that $\phi(K) \subseteq \operatorname{Spec}^{\mathrm{Z}} R$ is closed whenever $K \subseteq |\,\mathfrak{X}\,|$ is a closed subset with quasi-compact complement.

Remark 3.1.12. Let $f : \mathfrak{X} \to \operatorname{Spec}^{\text{ét}} R$ be a map of spectral Deligne-Mumford stacks, and assume that $\mathfrak{X}$ is a quasi-compact quasi-separated algebraic space. Suppose we wish to verify that f is *universally closed*: that is, that f satisfies condition (*iv*) of Definition 3.1.1. Let $R \to R'$ be an arbitrary map of connective $\mathbb{E}_\infty$-rings and set $\mathfrak{X}' = \mathfrak{X} \times_{\operatorname{Spec}^{\text{ét}} R} \operatorname{Spec}^{\text{ét}} R'$; we wish to prove that the map $\phi : |\,\mathfrak{X}'\,| \to \operatorname{Spec}^{\mathrm{Z}} R'$ is closed. According to Remark 3.1.11, it suffices to show that $\phi(K) \subseteq \operatorname{Spec}^{\mathrm{Z}} R'$ is closed whenever $K \subseteq |\,\mathfrak{X}'\,|$ is the complement of a quasi-compact open subset of $|\,\mathfrak{X}'\,|$. Write $R' = \varinjlim R_\alpha$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$, where each R_α is of finite presentation over R, and set $\mathfrak{X}_\alpha = \mathfrak{X} \times_{\operatorname{Spec}^{\text{ét}} R} \operatorname{Spec}^{\text{ét}} R_\alpha$. According to Proposition 1.4.15, every quasi-compact open subset of $|\,\mathfrak{X}'\,|$ is the inverse image of a quasi-compact open subset of some $|\,\mathfrak{X}_\alpha\,|$. It will therefore suffice to show that the map $\phi_\alpha : |\,\mathfrak{X}_\alpha\,| \to \operatorname{Spec}^{\mathrm{Z}} R_\alpha$ is closed. In other words, to verify condition (*iv*) of Definition 3.1.1, it suffices to treat the case where R' is finitely presented over R. In particular, if R is Noetherian, we may assume that R' is also Noetherian (Proposition A.7.2.5.31).

Lemma 3.1.13. *Suppose we are given a commutative diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ & \searrow \quad \swarrow_{g} & \\ & \mathfrak{Z}. & \end{array}$$

Assume that $g \circ f$ is strongly proper, that g is strongly separated and locally of finite presentation to order 0*, and that f is surjective. Then g is strongly proper.*

Proof. Without loss of generality, we may assume that $\mathfrak{Z} = \operatorname{Spec}^{\text{ét}} R$ is affine. Then $\mathfrak{X}$ is quasi-compact. Since f is surjective, we deduce that $\mathfrak{Y}$ is quasi-compact, so that g is quasi-compact. To complete the proof that g is strongly proper, it will suffice to verify condition (iv) of Definition 3.1.1. Let R' be a connective R-algebra; we wish to show that the map $|\operatorname{Spec}^{\text{ét}} R' \times_{\operatorname{Spec}^{\text{ét}} R} \mathfrak{Y}| \to \operatorname{Spec}^{\mathrm{Z}} R'$ is closed. Replacing $\mathfrak{Z}$ by $\operatorname{Spec}^{\text{ét}} R'$, we are reduced to proving that the map $|\mathfrak{Y}| \to |\mathfrak{Z}|$ is closed. Let $K \subseteq |\mathfrak{Y}|$ be a closed subset. Since f is surjective, and $g \circ f$ is proper, we deduce that

$$g(K) = g(f(f^{-1}(K))) = (g \circ f)(f^{-1}K)$$

is a closed subset of $|\mathfrak{Z}|$, as desired. □

Proof of Proposition 3.1.10. Using Remark 3.1.2, we may reduce to the case where $n = 0$. Using Corollary 2.5.8, we may assume without loss of generality that $\mathfrak{X}_0$ is a separated spectral algebraic space. Replacing A_0 by $\pi_0 A_0$ and each A_α by $\pi_0 A_\alpha$, we may suppose that A_0 is discrete. Write A_0 as a filtered colimit of finitely generated subrings $B_\beta \subseteq A_0$. Using Theorem 2.3.2, we can reduce to the case where $\mathfrak{X}_0$ is the image of an object $\mathfrak{X}_\beta \in \mathrm{DM}_0^{\mathrm{fp}}(B_\beta)$. Replacing $\mathfrak{X}_0$ by $\mathfrak{X}_\beta$, we can reduce to the case where A_0 is a finitely generated discrete ring (and in particular Noetherian).

We now invoke Chow's lemma (see [31]) to obtain a diagram

$$\mathfrak{X}_0 \leftarrow \mathfrak{X}_0 \times_{\operatorname{Spec}^{\text{ét}} A_0} \mathbf{P}^m_{A_0} \xleftarrow{i'} \mathfrak{X}'_0 \xrightarrow{j_0} \overline{\mathfrak{X}}'_0 \xrightarrow{i} \mathbf{P}^n_{A_0}$$

of 0-truncated separated spectral algebraic spaces, where $\mathbf{P}^m_{A_0}$ and $\mathbf{P}^n_{A_0}$ denote projective spaces over $\operatorname{Spec}^{\text{ét}} A_0$, i and i' are closed immersions, j_0 is an open immersion, and the map $\mathfrak{X}'_0 \to \mathfrak{X}_0$ is surjective. Let $\mathfrak{X}'$ and $\overline{\mathfrak{X}}'$ denote the images of $\mathfrak{X}'_0$ and $\overline{\mathfrak{X}}'_0$ in $\mathrm{DM}_0^{\mathrm{fp}}(A)$. Since $\mathfrak{X}$ is proper over A and i' is a closed immersion, we deduce that $\mathfrak{X}'$ is proper over A. The map $j : \mathfrak{X}' \to \overline{\mathfrak{X}}'$ factors as a composition

$$\mathfrak{X}' \xrightarrow{j'} \mathfrak{X}' \times_{\operatorname{Spec}^{\text{ét}} A} \overline{\mathfrak{X}}' \xrightarrow{j''} \overline{\mathfrak{X}}'$$

where j' is a closed immersion (since $\overline{\mathfrak{X}}'$ is separated) and j'' has closed image (since $\mathfrak{X}'$ is proper over A). It follows that j_0 is an open immersion with closed image and therefore also a closed immersion. For each index α, let $\mathfrak{X}_\alpha$ be the image of $\mathfrak{X}_0$ in $\mathrm{DM}_0^{\mathrm{fp}}(A_\alpha)$, and define $\mathfrak{X}'_\alpha$ and $\overline{\mathfrak{X}'}_\alpha$ similarly. Using Proposition 2.5.7, we deduce that there exists an index α such that the induced map $\mathfrak{X}'_\alpha \to \overline{\mathfrak{X}}'_\alpha$ is a closed immersion, so that there exists a closed immersion $\mathfrak{X}'_\alpha \to \mathbf{P}^n_{A_\alpha}$. Using Example 3.1.4, we deduce that $\mathfrak{X}'_\alpha$ is proper over A_α. Since we have a surjection $\mathfrak{X}'_\alpha \to \mathfrak{X}_\alpha$, Lemma 3.1.13 implies that $\mathfrak{X}_\alpha$ is proper over A_α, as desired. □

3.2 The Direct Image Theorem

Recall the direct image theorem for proper maps of algebraic spaces:

Theorem 3.2.1. *Let $f : X \to Y$ be a proper map between locally Noetherian algebraic spaces, and let $\mathcal{F}$ be an object in the abelian category of coherent sheaves on X. Then, for each $i \geq 0$, the higher direct image $R^i f_* \mathcal{F}$ is a coherent sheaf on Y.*

For a proof, we refer the reader to [31]. In this section, we will use Theorem 3.2.1 to deduce an analogous direct image theorem in the setting of spectral algebraic geometry. Here, we do not need any Noetherian hypotheses:

Theorem 3.2.2. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of spectral Deligne-Mumford stacks which is strongly proper and locally almost of finite presentation. Then the pushforward functor $f_* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{Y})$ carries almost perfect objects of $\mathrm{QCoh}(\mathfrak{X})$ to almost perfect objects of $\mathrm{QCoh}(\mathfrak{Y})$.*

Example 3.2.3. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map between spectral algebraic spaces which is strongly proper and locally almost of finite presentation, and let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})^{\heartsuit}$. Assume that $\mathfrak{Y}$ is quasi-compact locally Noetherian, so that $\mathfrak{X}$ is also locally Noetherian. Let X and Y denote the underlying algebraic spaces of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, so that f induces a map of algebraic spaces $\pi : X \to Y$. Then the heart $\mathrm{QCoh}(\mathfrak{X})^{\heartsuit}$ is equivalent to the abelian category of quasi-coherent sheaves on X; let $\mathcal{F}_0$ denote the image of $\mathcal{F}$ under this equivalence. Using Proposition A.7.2.5.17, we see that $\mathcal{F}_0$ is coherent if and only if $\mathcal{F}$ is almost perfect. If these conditions are satisfied, then Theorem 3.2.2 implies that $f_* \mathcal{F} \in \mathrm{QCoh}(\mathfrak{Y})$ is almost perfect. Using Proposition A.7.2.5.17 again (and the quasi-compactness of $\mathrm{QCoh}(\mathfrak{Y})$), we see that this is equivalent to the following pair of conditions:

(*i*) For each $i \geq 0$, the sheaf $R^i \pi_* \mathcal{F}_0 \simeq \pi_{-i}(f_* \mathcal{F})$ is coherent.

(*ii*) The higher direct images $R^i \pi_* \mathcal{F}_0 \simeq \pi_{-i}(f_* \mathcal{F})$ vanish for $i \gg 0$.

In particular, we can regard Theorem 3.2.2 as a generalization of Theorem 3.2.1.

Before giving the proof of Theorem 3.2.2, let us describe an application.

Definition 3.2.4. Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a map of spectral Deligne-Mumford stacks. We will say that f is *finite* if it satisfies the following pair of conditions:

(1) The map f is affine.

(2) The pushforward $f_* \mathcal{O}_{\mathcal{X}}$ is perfect to order 0 (as a quasi-coherent sheaf on $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$).

Proposition 3.2.5. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. The following conditions are equivalent:*

(1) *The map f is finite.*

(2) *The map f is strongly proper and locally quasi-finite.*

Proof. We may assume without loss of generality that $\mathfrak{Y} = \mathrm{Spec}^{\text{ét}} R$ is affine. We first prove that (1) $\Rightarrow$ (2). Assume that f is finite; then $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} A$ for some connective $\mathbb{E}_\infty$-ring A for which $\pi_0 A$ is finitely generated as a discrete module over $\pi_0 R$. Then $\mathfrak{X}$ is obviously a quasi-compact separated spectral algebraic space, which is locally of finite presentation to order 0 over R. To prove (2), it will suffice to show that for every map of $\mathbb{E}_\infty$-rings $R \to R'$, the induced map of topological spaces $\mathrm{Spec}^{\mathrm{Z}}(R' \otimes_R A) \to \mathrm{Spec}^{\mathrm{Z}} R'$ is closed. Let $I \subseteq \pi_0(R' \otimes_R A)$ be an ideal and set $B = (\pi_0 R' \otimes_R A)/I$. We wish to show that the map $\mathrm{Spec}^{\mathrm{Z}} B \to \mathrm{Spec}^{\mathrm{Z}} \pi_0 R'$ has closed image. This follows from the observation that B is finitely generated as a (discrete) module over $\pi_0 R'$.

We now prove that (2) $\Rightarrow$ (1). According to Theorem 1.2.1, the map f is quasi-affine. We may therefore choose a quasi-compact open immersion $j : \mathfrak{X} \to \mathrm{Spec}^{\text{ét}} A$ for some connective $\mathbb{E}_\infty$-algebra A over R. The projection map $\mathrm{Spec}^{\text{ét}} A \to \mathrm{Spec}^{\text{ét}} R$ is strongly separated, so that j induces a closed immersion $\gamma : \mathfrak{X} \to \mathfrak{X} \times_{\mathrm{Spec}^{\text{ét}} R} \mathrm{Spec}^{\text{ét}} A$. Since f is strongly proper, the canonical map $|\mathfrak{X} \times_{\mathrm{Spec}^{\text{ét}} R} \mathrm{Spec}^{\text{ét}} A| \to |\mathrm{Spec}^{\text{ét}} A|$ is closed. It follows that j has closed image in $|\mathrm{Spec}^{\text{ét}} A| \simeq \mathrm{Spec}^{\mathrm{Z}} A$, so that j is a clopen immersion and therefore $\mathfrak{X}$ is affine. Write $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} B$. We wish to show that $\pi_0 B$ is finitely generated as a module over $\pi_0 R$. Since f is locally of finite presentation to order 0, we are given that $\pi_0 B$ is finitely generated as a commutative ring over $\pi_0 R$. It will therefore suffice to show that every element $x \in \pi_0 B$ is integral over

$\pi_0 R$. Let R' denote the commutative ring $(\pi_0 R)[y]$. Let $I \subseteq (\pi_0 B)[y]$ be the ideal generated by the element $1 - xy$. Since f is strongly proper, the map

$$\operatorname{Spec}^{\mathrm{Z}}(\pi_0 B)[\frac{1}{x}] \to \operatorname{Spec}^{\mathrm{Z}}(\pi_0 B)[y]/I \to \operatorname{Spec}^{\mathrm{Z}}(\pi_0 B)[y] \to \operatorname{Spec}^{\mathrm{Z}}(\pi_0 R)[y]$$

has closed image $K \subseteq \operatorname{Spec}^{\mathrm{Z}}(\pi_0 R)[y]$, determined by some ideal $J \subseteq (\pi_0 R)[y]$. Note that the closed set K does not intersect the closed subset defined by the ideal $(y) \subseteq (\pi_0 R)[y]$, so that J and (y) generate the unit ideal in $(\pi_0 R)[y]$. It follows that J contains an element of the form $1 + a_1 y + \cdots + a_n y^n$, where the coefficients a_i belong to $\pi_0 R$. Replacing $1 + p(y)$ with a suitable power, we may assume that the image of $1 + a_1 y + \cdots + a_n y^n$ in $(\pi_0 B)[\frac{1}{x}]$ vanishes. It follows that $x^{m+n} + a_1 x^{m+n-1} + \cdots + a_n x^m$ vanishes in $\pi_0 B$ for $m \gg 0$, so that x is integral over $\pi_0 R$ as desired. □

We now turn to the proof of Theorem 3.2.2. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks which is strongly proper and locally almost of finite presentation; we wish to show that the pushforward functor f_* carries almost perfect objects of $\operatorname{QCoh}(\mathfrak{X})$ to almost perfect objects of $\operatorname{QCoh}(\mathfrak{Y})$. This assertion is local on $\mathfrak{Y}$, so we may assume without loss of generality that $\mathfrak{Y}$ is affine. It will therefore suffice to prove the following more precise result:

Proposition 3.2.6. *Let $f : \mathfrak{X} \to \operatorname{Spec}^{ét} R$ be a map of spectral Deligne-Mumford stacks which is strongly proper and locally almost of finite presentation, and let*

$$f_* : \operatorname{QCoh}(\mathfrak{X}) \to \operatorname{QCoh}(\operatorname{Spec}^{ét} R) \simeq \operatorname{Mod}_R$$

be the direct image functor. Then there exists an integer $m \gg 0$ with the following property: for every object $\mathcal{F} \in \operatorname{QCoh}(\mathfrak{X})$ which is perfect to order $n+1$, the direct image $f_* \mathcal{F} \in \operatorname{QCoh}(\operatorname{Spec}^{ét} R) \simeq \operatorname{Mod}_R$ is perfect to order $(n - m)$.

The proof of Proposition 3.2.6 will require some preliminaries. First, we need a slight refinement of Proposition VIII.2.5.13.

Lemma 3.2.7. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a separated spectral algebraic space. Suppose that there exists a finite collection of objects $U_0, \ldots, U_n \in \mathcal{X}$ satisfying the following conditions:*

(1) *Each U_i is (-1)-truncated.*

(2) *Each U_i is affine.*

(3) *The objects U_i cover $\mathcal{X}$: that is, the coproduct $\coprod U_i$ is a 0-connective object of $\mathcal{X}$.*

Then the global sections functor $\Gamma : \operatorname{QCoh}(\mathfrak{X}) \to \operatorname{Sp}$ carries $\operatorname{QCoh}(\mathfrak{X})_{\geq 0}$ into $\operatorname{Sp}_{\geq -n}$.

Proof. We proceed by induction on n. If $n = -1$, then $\mathfrak{X}$ is empty and the result is obvious. Assume therefore that $n \geq 0$. For every object $U \in \mathcal{X}$, let $\Gamma_U : \operatorname{QCoh}(\mathfrak{X}) \to \operatorname{Sp}$ be the functor given by evaluation at U. Let $V = \tau_{\leq -1}(\coprod_{1 \leq i \leq n} U_i)$ and let $V' = U_0 \times V$. It follows from assumption (3) that the pushout $U_0 \coprod_{V'} V$ is a final object of $\mathcal{X}$. Let $\mathcal{F} \in \operatorname{QCoh}(\mathfrak{X})_{\geq 0}$, so that we have a fiber sequence

$$\Gamma(\mathcal{F}) \to \Gamma_{U_0}(\mathcal{F}) \oplus \Gamma_V(\mathcal{F}) \to \Gamma_{V'}(\mathcal{F})$$

and therefore an exact sequence of abelian groups

$$\pi_{i+1}\Gamma_{V'}(\mathcal{F}) \to \pi_i \Gamma(\mathcal{F}) \to \pi_i \Gamma_{U_0}(\mathcal{F}) \oplus \pi_i \Gamma_V(\mathcal{F}).$$

Since $\mathfrak{X}$ is a separated spectral algebraic space, the products $U_0 \times U_i$ are affine. It follows from the inductive hypothesis that $\pi_{i+1}\Gamma_{V'}(\mathcal{F})$ vanishes for $i < -n$ and that $\pi_i \Gamma_V(\mathcal{F})$ vanishes for $i \leq -n$. Moreover, since U_0 is affine, the functor Γ_{U_0} is t-exact, so that $\pi_i \Gamma_{U_0}(\mathcal{F}) \simeq 0$ for $i < 0$. It follows that $\pi_i \Gamma(\mathcal{F}) \simeq 0$ for $i < -n$. □

Lemma 3.2.8. *Let $f : \mathfrak{X} \to \operatorname{Spec}^{ét} R$ be a map of spectral Deligne-Mumford stacks. Assume that $\mathfrak{X}$ is a quasi-compact, quasi-separated spectral algebraic space and that f is locally almost of finite presentation. Then there exists an integer $m \gg 0$ with the following property:*

$(*)$ *Let $n \geq 0$ be an integer. Then there is a connective Noetherian $\mathbb{E}_\infty$-ring R_0, a quasi-compact, quasi-separated spectral algebraic space $\mathfrak{X}_0$ which is finitely n-presented over $\operatorname{Spec} R_0$, and a commutative diagram*

$$\begin{array}{ccc} \tau_{\leq n}\,\mathfrak{X} & \longrightarrow & \mathfrak{X}_0 \\ \downarrow & & \downarrow \\ \operatorname{Spec}^{ét} R & \longrightarrow & \operatorname{Spec}^{ét} R_0 \end{array}$$

for which the induced map $\tau_{\leq n}\,\mathfrak{X} \to \tau_{\leq n}(\mathfrak{X}_0 \times_{\operatorname{Spec}^{ét} R_0} \operatorname{Spec}^{ét} R)$ induces an equivalence of n-truncations, and the global sections functor $\Gamma : \operatorname{QCoh}(\mathfrak{X}_0) \to \operatorname{Sp}$ carries $\operatorname{QCoh}(\mathfrak{X}_0)_{\geq 0}$ into $\operatorname{Sp}_{\geq -m}$.

Proof. Using Theorem 1.3.8, we can choose a scallop decomposition

$$\emptyset = \mathfrak{U}_0 \to \mathfrak{U}_1 \to \cdots \to \mathfrak{U}_k = \mathfrak{X}$$

for $\mathfrak{X}$. For $0 < i \leq k$, choose an excision square σ_i:

$$\begin{array}{ccc} \mathfrak{V}_i & \xrightarrow{\phi_i} & \operatorname{Spec} A_i \\ \downarrow & & \downarrow \\ \mathfrak{U}_{i-1} & \longrightarrow & \mathfrak{U}_i \end{array}$$

where ϕ_i is a quasi-compact open immersion. Then $\mathfrak{V}_i$ determines a quasi-compact open subset $V_i \subseteq \operatorname{Spec}^{\mathrm{Z}}(\pi_0 A_i)$, which can be written as the union of m_i affine open subsets of $\operatorname{Spec}^{\mathrm{Z}}(\pi_0 A_i)$ for some integer m_i. Choose any integer m which is strictly larger than each m_i. We will show that m has the desired properties.

Fix an integer $n \geq 0$; we wish to verify property $(*)$. Let $\mathfrak{X}' = \tau_{\leq n}\,\mathfrak{X}$, so that the n-truncation of the above data yields a scallop decomposition

$$\emptyset = \mathfrak{U}'_0 \to \mathfrak{U}'_1 \to \cdots \to \mathfrak{U}'_k = \mathfrak{X}'$$

for $\mathfrak{X}'$ and excision squares σ'_i :

$$\begin{array}{ccc} \mathfrak{V}'_i & \xrightarrow{\phi_i} & \operatorname{Spec}^{\text{ét}} A'_i \\ \downarrow & & \downarrow \\ \mathfrak{U}'_{i-1} & \longrightarrow & \mathfrak{U}'_i, \end{array}$$

where $A'_i = \tau_{\leq n} A_i$. Since the ∞-category $\operatorname{CAlg}^{\mathrm{cn}}$ is compactly generated, we can write R as the colimit of a filtered diagram $\{R_\alpha\}$ of connective $\mathbb{E}_\infty$-rings, where each R_α is finitely presented over the sphere spectrum and therefore Noetherian (Proposition A.7.2.5.31). Without loss of generality, we may assume that this diagram is indexed by a filtered partially ordered set P (Proposition T.5.3.1.16). Theorem 2.3.2 yields an equivalence of ∞-categories $\operatorname{DM}^{\mathrm{fp}}_n(R) \simeq \varinjlim_\alpha \operatorname{DM}^{\mathrm{fp}}_n(R_\alpha)$. We may therefore choose an index $\alpha \in P$ such that the diagrams σ'_i lift to diagrams σ^α_i :

$$\begin{array}{ccc} \mathfrak{V}^\alpha_i & \xrightarrow{\phi^\alpha_i} & \mathfrak{W}^\alpha_i \\ \downarrow & & \downarrow \\ \mathfrak{U}^\alpha_{i-1} & \longrightarrow & \mathfrak{U}^\alpha_i\,. \end{array}$$

Enlarging α if necessary, we may assume that each ϕ_i^α is an open immersion (Corollary 2.5.3), that each of the maps $\mathfrak{V}_i^\alpha \to \mathfrak{U}_{i-1}^\alpha$ is étale (Proposition 2.5.2), that each $\mathfrak{W}_i^\alpha$ is affine (Proposition 2.5.1), that each σ_i^α is a pushout square, and that $\mathfrak{U}_0^\alpha$ is empty. Using Proposition VIII.2.5.3 we deduce that each σ_i^α is an excision square, so that the induced maps $\mathfrak{U}_{i-1}^\alpha \to \mathfrak{U}_i^\alpha$ are open immersions, and therefore the sequence

$$\emptyset = \mathfrak{U}_0^\alpha \to \cdots \to \mathfrak{U}_m^\alpha$$

is a scallop decomposition of $\mathfrak{X}_0 = \mathfrak{U}_m^\alpha$. For each $0 < i \leq k$, choose a collection of open immersions $\{\operatorname{Spec} B_{i,j} \to \mathfrak{V}_i'\}_{1 \leq j \leq m_i}$ which are jointly surjective. Enlarging α if necessary, we can assume that each of these maps lifts to a morphism $\xi_{i,j} : \mathfrak{Y}_{i,j} \to \mathfrak{V}_i^\alpha$ in $\mathrm{DM}_n^{\mathrm{fp}}(R_\alpha)$. Enlarging α further if necessary, we may assume that each $\mathfrak{Y}_{i,j}$ is affine (Proposition 2.5.1), that each $\xi_{i,j}$ is an open immersion (Corollary 2.5.3), and that the maps $\xi_{i,j}$ are jointly surjective (Proposition 2.5.9). Write $\mathfrak{W}_i^\alpha = \operatorname{Spec} A_i^\alpha$, so that $\mathfrak{V}_i^\alpha$ corresponds to the open substack of $\operatorname{Spec} A_i^\alpha$ classified by a union of m_i open subsets of $\operatorname{Spec}^{\mathrm{Z}}(\pi_0 A_i^\alpha)$.

We now set $R_0 = R_\alpha$. By construction, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X}_0 \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} R_0 \end{array}$$

which induces an equivalence $\mathfrak{X}' \simeq \tau_{\leq n}(\mathfrak{X}_0 \times_{\operatorname{Spec} R_0} \operatorname{Spec} R)$. The spectral Deligne-Mumford stack $\mathfrak{X}_0$ admits a scallop decomposition, and is therefore a quasi-compact, quasi-separated spectral algebraic space (Theorem 1.3.8).

To complete the proof, we need to show that the global sections functor $\Gamma : \mathrm{QCoh}(\mathfrak{X}_0) \to \mathrm{Sp}$ carries $\mathrm{QCoh}(\mathfrak{X}_0)_{\geq 0}$ into $\mathrm{Sp}_{\geq -m}$. Let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}_0)_{\geq 0}$. We prove by induction on $0 \leq i \leq k$ that each of the spectra $\Gamma(\mathfrak{U}_i^\alpha; \mathcal{F} \,|\, \mathfrak{U}_i^\alpha)$ is $(-m)$-connective. When $i = 0$, this is obvious (since $\mathfrak{U}_0^\alpha$ is empty). To carry out the inductive step, we note that for $i > 0$ we have a pullback diagram of spectra

$$\begin{array}{ccc} \Gamma(\mathfrak{U}_i^\alpha; \mathcal{F} \,|\, \mathfrak{U}_i^\alpha) & \longrightarrow & \Gamma(\mathfrak{U}_{i-1}^\alpha; \mathcal{F} \,|\, \mathfrak{U}_i^\alpha) \\ \downarrow & & \downarrow \\ \Gamma(\operatorname{Spec}^{\text{ét}} A_i^\alpha; \mathcal{F} \,|\, \operatorname{Spec}^{\text{ét}} A_i^\alpha) & \longrightarrow & \Gamma(\mathfrak{V}_i^\alpha; \mathcal{F} \,|\, \mathfrak{V}_i^\alpha). \end{array}$$

The inductive hypothesis implies that $\Gamma(\mathfrak{U}_{i-1}^\alpha; \mathcal{F} \,|\, \mathfrak{U}_{i-1}^\alpha)$ is $(-m)$-connective, and the spectrum

$$\Gamma(\operatorname{Spec} A_i^\alpha; \mathcal{F} \,|\, \operatorname{Spec} A_i^\alpha)$$

is connective by virtue of our assumption that $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}_0)_{\geq 0}$. Since $m > m_i$, it will suffice to show that $\Gamma(\mathfrak{V}_i^\alpha; \mathcal{F} \,|\, \mathfrak{V}_i^\alpha)$ is $(-m_i)$-connective. This follows from Lemma 3.2.7. □

Remark 3.2.9. In the situation of Lemma 3.2.8, we can assume that the integer m has the following additional property:

($*'$) The global sections functor $\Gamma : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{Sp}$ carries $\mathrm{QCoh}(\mathfrak{X})_{\geq 0}$ to $\mathrm{Sp}_{\geq -m}$.

This property follows from the construction given in the proof of Lemma 3.2.8. It can also be ensured by enlarging m, using Proposition VIII.2.5.13.

Remark 3.2.10. In the situation of Lemma 3.2.8, assume that the map $f : \mathfrak{X} \to \operatorname{Spec}^{\text{ét}} R$ is strongly proper. Then we can assume that the maps $\mathfrak{X}_0 \to \operatorname{Spec}^{\text{ét}} R_0$ appearing in ($*$) is strongly proper: this follows from Proposition 3.1.10.

Proof of Proposition 3.2.6. Assume that $f : \mathfrak{X} \to \operatorname{Spec}^{\text{ét}} R$ is strongly proper and locally almost of finite presentation. Choose an integer $m \gg 0$ satisfying the conclusion of Lemma 3.2.8 and Remark 3.2.9. Let $\mathcal{F} \in \operatorname{QCoh}(\mathfrak{X})$ be perfect to order $n+1$. We will show that $f_* \mathcal{F} \in \operatorname{QCoh}(\operatorname{Spec} R) \simeq \operatorname{Mod}_R$ is perfect to order $n-m$. Since $\mathfrak{X}$ is quasi-compact, the condition that $\mathcal{F}$ is perfect to order $n+1$ implies that $\mathcal{F} \in \operatorname{QCoh}(\mathfrak{X})_{\geq -k}$ for some integer $k \gg 0$. Replacing $\mathcal{F}$ by $\mathcal{F}[k]$ and n by $n+k$, we may suppose that $\mathcal{F}$ is connective.

Choose a fiber sequence

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'',$$

where $\mathcal{F}'' \in \operatorname{QCoh}(\mathfrak{X})_{\leq n}$ and $\mathcal{F}' \in \operatorname{QCoh}(\mathfrak{X})_{\geq n+1}$. We then obtain a fiber sequence of R-modules

$$f_* \mathcal{F}' \to f_* \mathcal{F} \to f_* \mathcal{F}'',$$

where $f_* \mathcal{F}'$ is $(n+1-m)$-connective. It follows that we have an equivalence of truncations $\tau_{\leq n-m} f_* \mathcal{F} \to \tau_{\leq n-m} f_* \mathcal{F}''$. It will therefore suffice to show that $f_* \mathcal{F}''$ is perfect to order $n-m$. Note that $\mathcal{F}''$ is perfect to order $n+1$ by Remark VIII.2.6.6. We may therefore replace $\mathcal{F}$ by $\mathcal{F}''$ and thereby reduce to the case where $\mathcal{F}$ is n-truncated. Let $\mathfrak{X}'$ denote the n-truncation of $\mathfrak{X}$, so that $\mathcal{F}$ is the direct image of an (essentially unique) quasi-coherent sheaf $\mathcal{F}_0 \in \operatorname{QCoh}^{n-fp}(\mathfrak{X}')$ (Corollary VIII.2.5.24).

Choose a commutative diagram

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{f} & \mathfrak{X}_0 \\ \downarrow & & \downarrow{\scriptstyle f_0} \\ \operatorname{Spec}^{\text{ét}} R & \longrightarrow & \operatorname{Spec}^{\text{ét}} R_0 \end{array}$$

as in Lemma 3.2.8. According to Remark 3.2.10, we may assume that f_0 is strongly proper. Write R as the colimit of a filtered diagram $\{R_\alpha\}$ of $\mathbb{E}_\infty$-algebras of finite presentation over R_0. Since R_0 is Noetherian, each R_α is Noetherian (Proposition A.7.2.5.31). Using Theorem 2.4.4 and Corollary VIII.2.5.24, we deduce that $\operatorname{QCoh}^{n-fp}(\mathfrak{X}')$ is equivalent to the filtered colimit of the ∞-categories $\operatorname{QCoh}^{n-fp}(\mathfrak{X}' \times_{\operatorname{Spec} R_0} \operatorname{Spec} R_\alpha)$. We may therefore assume that there exists an index α, an object $\mathcal{G} \in \operatorname{QCoh}^{n-fp}(\mathfrak{X}_0 \times_{\operatorname{Spec}^{\text{ét}} R_0} \operatorname{Spec}^{\text{ét}} R_\alpha)$, and an equivalence $\mathcal{F}_0 \simeq \tau_{\leq n} g^* \mathcal{G}$, where $g : \mathfrak{X}' \to \mathfrak{X}_0 \times_{\operatorname{Spec}^{\text{ét}} R_0} \operatorname{Spec}^{\text{ét}} R_\alpha$ is the canonical map. We have a fiber sequence of quasi-coherent sheaves

$$\mathcal{K} \to g^* \mathcal{G} \to \mathcal{F}_0$$

on $\mathfrak{X}'$, where $\mathcal{K}$ is $(n+1)$-connective. Let $i : \mathfrak{X}' \to \mathfrak{X}$ be the canonical map and $i_* : \operatorname{QCoh}(\mathfrak{X}') \to \operatorname{QCoh}(\mathfrak{X})$ the associated pushforward functor. Since i is affine, i_* is right exact. We therefore have a fiber sequence

$$i_* \mathcal{K} \to i_* g^* \mathcal{G} \to \mathcal{F}$$

in $\operatorname{QCoh}(\mathfrak{X})$, where $i_* \mathcal{K}$ is $(n+1)$-connective. It follows that $f_* i_* \mathcal{K}$ is $(n+1-m)$-connective, so that $\tau_{\leq n-m} f_* \mathcal{F} \simeq \tau_{\leq n-m} f_* i_* g^* \mathcal{G}$. It will therefore suffice to show that $f_* i_* g^* \mathcal{G}$ is perfect to order $(n-m)$.

Write $\mathfrak{X}_\alpha = \mathfrak{X}_0 \times_{\operatorname{Spec}^{\text{ét}} R_0} \operatorname{Spec}^{\text{ét}} R_\alpha$. Let $h : \mathfrak{X}_\alpha \to \operatorname{Spec}^{\text{ét}} R_\alpha$ be the projection map, and let

$$h_* : \operatorname{QCoh}(\mathfrak{X}_0 \times_{\operatorname{Spec}^{\text{ét}} R_0} \operatorname{Spec}^{\text{ét}} R_\alpha) \to \operatorname{Mod}_{R_\alpha}$$

denote the associated pushforward functor. Using Corollary 1.3.9, we obtain an equivalence of R-modules

$$f_* i_* g^* \mathcal{G} \simeq R \otimes_{R_\alpha} h_* \mathcal{G}.$$

It will therefore suffice to show that $h_* \mathcal{G}$ is perfect to order $(n-m)$ (Proposition VIII.2.6.13). Note that $\mathfrak{X}_\alpha$ is locally Noetherian (Proposition 2.2.4) and that $\mathcal{G} \in \operatorname{QCoh}(\mathfrak{X}_\alpha)$ is coherent (Proposition VIII.2.6.24). It will therefore suffice to prove the following:

(∗) Let $\mathcal{G} \in \operatorname{QCoh}(\mathfrak{X}_\alpha)$ be coherent. Then $h_*(\mathcal{G}) \in \operatorname{Mod}_{R_\alpha}$ is almost perfect. That is, the homotopy groups $\pi_i(h_*(\mathcal{G}))$ are finitely generated as modules over the commutative Noetherian ring $\pi_0 R_\alpha$ (see Proposition A.7.2.5.17).

The collection of those objects $\mathcal{G} \in \mathrm{QCoh}(\mathfrak{X}_\alpha)$ for which $h_*(\mathcal{G})$ is almost perfect is closed under extensions. Consequently, to show that this collection contains all coherent objects of $\mathrm{QCoh}(\mathfrak{X}_\alpha)$, it suffices to show that it contains every coherent object of the heart $\mathrm{QCoh}(\mathfrak{X}_\alpha)^\heartsuit$. Let $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ denote the 0-truncation of $\mathfrak{X}_\alpha$, and let $j : \mathfrak{Y} \to \mathfrak{X}_\alpha$ denote the canonical map. If $\mathcal{G} \in \mathrm{QCoh}(\mathfrak{X}_\alpha)^\heartsuit$, then we can write $\mathcal{G} = j_* \overline{\mathcal{G}}$ for some coherent sheaf $\overline{\mathcal{G}} \in \mathrm{QCoh}(\mathfrak{Y})^\heartsuit$ (Corollary VIII.2.5.24). It will therefore suffice to show that the pushforward of $\overline{\mathcal{G}}$ along the composite map

$$\mathfrak{Y} \xrightarrow{h'} \mathrm{Spec}(\pi_0 R) \to \mathrm{Spec}\, R$$

is almost perfect. Unwinding the definitions (and using Proposition A.7.2.5.17), we must show that each of the cohomology groups $\mathrm{H}^i(\mathcal{Y}; \overline{\mathcal{G}})$ is finitely generated as a module over the commutative ring $\pi_0 R$. This is an immediate consequence of Theorem 3.2.1 (see Example 3.2.3). □

3.3 Proper Linear ∞-Categories

Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack. In this section, we will introduce the notion of a *locally proper* quasi-coherent stack on $\mathfrak{X}$ (Definition 3.3.6). Our main result is a categorified version of the proper direct image theorem: if $f : \mathfrak{X} \to \mathfrak{Y}$ is a strongly proper morphism which is locally almost of finite presentation and has finite Tor-amplitude, then the pushforward functor $f_* : \mathrm{QStk}(\mathfrak{X}) \to \mathrm{QStk}(\mathfrak{Y})$ carries locally proper quasi-coherent stacks on $\mathfrak{X}$ to locally proper quasi-coherent stacks on $\mathfrak{Y}$ (Theorem 3.3.11).

We begin with a few simple observations about linear ∞-categories.

Remark 3.3.1. Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ and $\mathcal{C}'$ be R-linear ∞-categories, and let $F : \mathcal{C} \to \mathcal{C}'$ be an R-linear functor. Then F is a colimit-preserving functor between stable ∞-categories, and therefore admits a right adjoint $G : \mathcal{C}' \to \mathcal{C}$. Suppose that $\mathcal{C}$ is compactly generated and that F carries compact objects of $\mathcal{C}$ to compact objects of $\mathcal{C}'$. It follows from Proposition T.5.5.7.2 that G preserves small filtered colimits, so that Remark VII.6.6 allows us to regard G as an R-linear functor from $\mathcal{C}'$ to $\mathcal{C}$.

Construction 3.3.2. Let R be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be an R-linear ∞-category, and let $C \in \mathcal{C}$ be a compact object. The construction $M \mapsto M \otimes C$ determines an R-linear functor $\mathrm{LMod}_R \to \mathcal{C}$, which carries compact objects of LMod_R to compact objects of $\mathcal{C}$. Invoking Remark 3.3.1, we see that this functor admits a right adjoint $\mathcal{C} \to \mathrm{LMod}_R$. We will denote this functor by $D \mapsto \mathrm{Mor}_\mathcal{C}(C, D)$. Here $\mathrm{Mor}_\mathcal{C}(C, D) \in \mathrm{LMod}_R$ is a classifying object for morphisms from C to D: that is, it is characterized by the existence of canonical homotopy equivalences

$$\mathrm{Map}_{\mathrm{LMod}_R}(M, \mathrm{Mor}_\mathcal{C}(C, D)) \to \mathrm{Map}_\mathcal{C}(M \otimes C, D).$$

Definition 3.3.3. Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be an R-linear ∞-category. We will say that $\mathcal{C}$ is *proper* if the following conditions are satisfied:

(1) The ∞-category $\mathcal{C}$ is compactly generated.

(2) For every pair of compact objects $M, N \in \mathcal{C}$, the R-module $\mathrm{Mor}_\mathcal{C}(M, N)$ is perfect.

Remark 3.3.4. In the setting of differential graded categories, the condition of properness has been studied by a number of authors: see for example [34], [64], and [67].

We now record some basic stability properties enjoyed by the class of proper linear ∞-categories.

Proposition 3.3.5. *Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be an R-linear ∞-category. Then:*

(1) *If $R \to R'$ is a map of $\mathbb{E}_2$-rings, then the R'-linear ∞-category $\mathrm{LMod}_{R'}(\mathcal{C})$ is proper.*

(2) *Suppose there exists a finite collection of étale morphisms $\{R \to R_\alpha\}$ such that the induced map $R \to R_\alpha$ is faithfully flat. If each $\mathrm{LMod}_{R_\alpha}(\mathcal{C})$ is a proper R_α-linear ∞-category, then $\mathcal{C}$ is proper as an R-linear ∞-category.*

Proof. We first prove (1). If $\mathcal{C}$ is proper, then it is compactly generated and therefore $\mathrm{LMod}_{R'}(\mathcal{C})$ is compactly generated (Example XI.6.3). We must show that for every pair of compact objects $X, Y \in \mathrm{LMod}_{R'}(\mathcal{C})$, the R'-module $\mathrm{Mor}_{\mathrm{LMod}_{R'}(\mathcal{C})}(X, Y)$ is perfect. Let us first regard Y as fixed, and let $\mathcal{X} \subseteq \mathrm{LMod}_{R'}(\mathcal{C})$ be the full subcategory spanned by those compact objects X for which $\mathrm{Mor}_{\mathrm{LMod}_{R'}(\mathcal{C})}(X, Y)$ is perfect. Then $\mathcal{X}$ is an idempotent complete, stable subcategory of $\mathrm{LMod}_{R'}(\mathcal{C})$. To show that it contains every compact object of $\mathrm{LMod}_{R'}(\mathcal{C})$, it will suffice to show that it contains every object of the form $R' \otimes X_0$, where X_0 is a compact object of $\mathcal{C}$. Let us now regard X_0 as fixed; we wish to show that $\mathrm{Mor}_{\mathrm{LMod}_{R'}(\mathcal{C})}(R' \otimes X_0, Y) \simeq \mathrm{Mor}_{\mathcal{C}}(X_0, Y)$ is a perfect R'-module for every compact object $Y \in \mathrm{LMod}_{R'}(\mathcal{C})$. Arguing as above, we may suppose that $Y \simeq R' \otimes Y_0$ for some compact object $Y_0 \in \mathcal{C}$. We then have an equivalence of R'-modules $R' \otimes_R \mathrm{Mor}_{\mathcal{C}}(X_0, Y_0) \to \mathrm{Mor}_{\mathrm{LMod}_{R'}(\mathcal{C})}(X, Y)$, and are therefore reduced to proving that $\mathrm{Mor}_{\mathcal{C}}(X_0, Y_0)$ is a perfect R-module. This follows from our assumption that $\mathcal{C}$ is proper.

We now prove (2). Using Theorem XI.6.1 we conclude that $\mathcal{C}$ is a compactly generated ∞-category. Fix compact objects $X, Y \in \mathcal{C}$. For every index α, the module

$$R_\alpha \otimes_R \mathrm{Mor}_{\mathcal{C}}(X, Y) \simeq \mathrm{Mod}_{\mathrm{LMod}_{R_\alpha}(\mathcal{C})}(R_\alpha \otimes X, R_\alpha \otimes Y)$$

is perfect. Using Proposition XI.6.21, we deduce that $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is a perfect left R-module. □

Proposition 3.3.5 asserts that the condition of properness can be tested locally for the étale topology. This motivates the following:

Definition 3.3.6. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack and let $\mathcal{C} \in \mathrm{QStk}(\mathfrak{X})$. We will say that $\mathcal{C}$ is *locally proper* if, for every map $\eta : \mathrm{Spec}^{\text{ét}} R \to \mathfrak{X}$, the pullback $\eta^* \mathcal{C}$ is a proper R-linear ∞-category.

Remark 3.3.7. Let $\mathfrak{X} \simeq \mathrm{Spec}^{\text{ét}} R$ be an affine spectral Deligne-Mumford stack and let $\mathcal{C}$ be a quasi-coherent stack on $\mathfrak{X}$. Then $\mathcal{C}$ is locally proper (as a quasi-coherent stack on $\mathfrak{X}$) if and only if it is proper as an R-linear ∞-category: this follows immediately from the first assertion of Proposition 3.3.5.

Using Proposition 3.3.5, we deduce the following:

Proposition 3.3.8. *Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack and let $\mathcal{C}$ be a quasi-coherent stack on $\mathfrak{X}$. Then:*

(1) *Let $f : \mathfrak{Y} \to \mathfrak{X}$ be any map of spectral Deligne-Mumford stacks. If $\mathcal{C}$ is locally proper, then $f^* \mathcal{C} \in \mathrm{QStk}(\mathfrak{Y})$ is locally proper.*

(2) *Suppose we are given a collection of étale maps $\{f_\alpha : \mathfrak{X}_\alpha \to \mathfrak{X}\}$ which induce an étale surjection $\coprod_\alpha \mathfrak{X}_\alpha \to \mathfrak{X}$. If each pullback $f_\alpha^* \mathcal{C} \in \mathrm{QStk}(\mathfrak{X}_\alpha)$ is locally proper, then $\mathcal{C}$ is locally proper.*

Our primary goal is to study the stability of the class of proper linear ∞-categories under pushforwards. We first need a little bit more terminology.

Definition 3.3.9. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks and let $n \geq 0$ be an integer. We will say that f is *has* Tor-*amplitude* $\leq n$ if, for every commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}^{\text{ét}} B & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}^{\text{ét}} A & \longrightarrow & \mathfrak{Y} \end{array}$$

where the horizontal maps are étale , the $\mathbb{E}_\infty$-ring B has Tor-amplitude $\leq n$ as an A-module (see Definition A.7.2.5.21).

Example 3.3.10. A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is flat if and only if it has Tor-amplitude ≤ 0.

We can now formulate our main result.

Theorem 3.3.11. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. Assume that:*

(1) *The map f is strongly proper.*

(2) *There exists an integer n such that the map f has* Tor*-amplitude $\leq n$.*

(3) *The map f is locally almost of finite presentation.*

Let $\mathcal{C} \in \mathrm{QStk}(\mathfrak{X})$ be locally proper. Then $f_* \mathcal{C} \in \mathrm{QStk}(\mathfrak{Y})$ is locally proper.

Corollary 3.3.12. *Let $\mathfrak{X}$ be a spectral algebraic space which is strongly proper, locally almost of finite presentation, and of finite* Tor*-amplitude over some connective $\mathbb{E}_\infty$-ring R. Then $\mathrm{QCoh}(\mathfrak{X})$ is a proper R-linear ∞-category.*

Before giving the proof of Theorem 3.3.11, let us collect a few basic facts about morphisms of finite Tor-amplitude.

Lemma 3.3.13. *Let $f : \mathfrak{X} \to \mathrm{Spec}^{\acute{e}t} R$ be a map of spectral Deligne-Mumford stacks, and let $n \geq 0$. The following conditions are equivalent:*

(1) *The map f has* Tor*-amplitude $\leq n$.*

(2) *For every étale map $\mathrm{Spec}^{\acute{e}t} A \to \mathfrak{X}$, A has* Tor*-amplitude $\leq n$ as an R-module.*

Proof. It is clear that (1) $\Rightarrow$ (2). Conversely, suppose that (2) is satisfied, and suppose we are given an étale map $g : \mathrm{Spec}^{\text{ét}} A \to \mathfrak{X}$ such that $f \circ g$ factors as a composition $\mathrm{Spec}^{\text{ét}} A \to \mathrm{Spec}^{\text{ét}} R' \to \mathrm{Spec}^{\text{ét}} R$, for some R' which is étale over R. We wish to show that A has Tor-amplitude $\leq n$ as an R'-module. Since R' is étale over R, A is a retract (as an R'-module) of $R' \otimes_R A$, which is of Tor-amplitude $\leq n$ over R'. $\square$

Remark 3.3.14. A map $f : \mathrm{Spec}^{\text{ét}} B \to \mathrm{Spec}^{\text{ét}} A$ of affine spectral Deligne-Mumford stacks has Tor-amplitude $\leq n$ if and only if B has Tor-amplitude $\leq n$ as an A-module.

Proposition 3.3.15. *The condition that a map of spectral Deligne-Mumford stacks $f : \mathfrak{X} \to \mathfrak{Y}$ be of* Tor*-amplitude $\leq n$ is local on the source with respect to the fpqc topology (see Definition VIII.1.5.26).*

Proof. First suppose that f has Tor-amplitude $\leq n$, and that we are given a flat map $g : \mathfrak{X}' \to \mathfrak{X}$. We wish to show that $g \circ f$ has Tor-amplitude $\leq n$. Consider a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}^{\text{ét}} C & \longrightarrow & \mathfrak{X}' \\ \downarrow & & \downarrow \\ \mathrm{Spec}^{\text{ét}} A & \longrightarrow & \mathfrak{Y}; \end{array}$$

we wish to show that B has Tor-amplitude $\leq n$ as an A-module. In other words, we wish to show that if M is a discrete A-module, then $C \otimes_A M$ is n-truncated. This assertion is local on C with respect to the étale topology. We may therefore suppose that the map $\mathrm{Spec}^{\text{ét}} C \to \mathrm{Spec}^{\text{ét}} A \times_{\mathfrak{Y}} \mathfrak{X}$ factors as a composition

$$\mathrm{Spec}^{\text{ét}} C \to \mathrm{Spec}^{\text{ét}} B \xrightarrow{u} \mathrm{Spec}^{\text{ét}} A \times_{\mathfrak{Y}} \mathfrak{X},$$

where u is étale . Since f has Tor-amplitude $\leq n$, we see that $B \otimes_A M$ is n-truncated. Then $C \otimes_A M \simeq C \otimes_B (B \otimes_A M)$ is n-truncated by virtue of the fact that C is flat over B.

Now suppose that we are given a flat covering $\{g_\alpha : \mathfrak{X}_\alpha \to \mathfrak{X}\}$ such that each $g_\alpha \circ f$ has Tor-amplitude $\leq n$; we wish to show that f has Tor-amplitude $\leq n$. We may assume without loss of generality that $\mathfrak{Y} = \mathrm{Spec}^{\text{ét}} A$ is affine. Choose an étale map $\mathrm{Spec}^{\text{ét}} B \to \mathfrak{X}$; we wish to show that B has Tor-amplitude $\leq n$ over A (see Lemma 3.3.13). Since the g_α form a flat covering, we can find finitely many étale maps $\mathrm{Spec}^{\text{ét}} C_\alpha \to \mathfrak{X}_\alpha \times_{\mathfrak{X}} \mathrm{Spec}^{\text{ét}} B$ such that $C = \prod C_\alpha$ is faithfully flat over B. If M is a discrete A-module, then $C \otimes_A M \simeq C \otimes_B (B \otimes_A M)$ is n-truncated; it follows that $B \otimes_A M$ is n-truncated so that B has Tor-amplitude $\leq n$ over A. $\square$

Proposition 3.3.16. *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}. \end{array}$$

If f has Tor*-amplitude $\leq n$, so does f'. The converse holds if g is a flat covering.*

Proof. Suppose first that f has Tor-amplitude $\leq n$. To prove that f' has Tor-amplitude $\leq n$, we may assume without loss of generality that $\mathfrak{Y}' = \operatorname{Spec}^{\text{ét}} A'$ is affine. Choose a faithfully flat étale morphism $A' \to A''$ such that the composite map

$$\operatorname{Spec}^{\text{ét}} A'' \to \operatorname{Spec}^{\text{ét}} A' \to \mathfrak{Y}$$

factors through an étale map $\operatorname{Spec}^{\text{ét}} A \to \mathfrak{Y}$. Using Proposition 3.3.15, we are reduced to proving that for every étale map $\operatorname{Spec}^{\text{ét}} B' \to \mathfrak{X}_{\operatorname{Spec}^{\text{ét}} A'} \operatorname{Spec}^{\text{ét}} A''$, B' has Tor-amplitude $\leq n$ over A'. Using Proposition 3.3.15 we may further reduce to the case where the map

$$\operatorname{Spec}^{\text{ét}} B' \to \operatorname{Spec}^{\text{ét}} A \times_{\mathfrak{Y}} \mathfrak{X}$$

factors through some étale map $\operatorname{Spec}^{\text{ét}} B \to \operatorname{Spec}^{\text{ét}} A \times_{\mathfrak{Y}} \mathfrak{X}$. Then the map $A' \to B'$ factors as a composition

$$A' \to A'' \to A'' \otimes_A B \to B',$$

where the first and third map are étale , and the middle map has Tor-amplitude $\leq n$. It follows that B' has Tor-amplitude $\leq n$ over A', as desired.

Now suppose that g is a flat covering and that f' has Tor-amplitude $\leq n$; we wish to show that f has the same property. We may assume without loss of generality that $\mathfrak{Y} = \operatorname{Spec}^{\text{ét}} A$ is affine. Using Proposition 3.3.15 we can further reduce to the case where $\mathfrak{X} = \operatorname{Spec}^{\text{ét}} B$ is affine. Since g is a flat covering, we can choose an étale map $\operatorname{Spec}^{\text{ét}} A' \to \mathfrak{Y}'$ such that A' is faithfully flat over B. Because f' has Tor-amplitude $\leq n$, we deduce that $A' \otimes_A B$ has Tor-amplitude $\leq n$ over A'. It then follows from Lemma VIII.2.6.16 that B has Tor-amplitude $\leq n$ over A. □

Lemma 3.3.17. *Let $f : R \to A$ be a map of connective $\mathbb{E}_1$-rings, and let M be a left A-module. Suppose that A has* Tor*-amplitude $\leq m$ as a left R-module, and the M has* Tor*-amplitude $\leq n$ as a left A-module. Then M has* Tor*-amplitude $\leq m + n$ as a left R-module.*

Proof. Let $N \in (\operatorname{LMod}_R)_{\leq p}$; we wish to show that $N \otimes_R M$ is $p + m + n$-truncated. We have $N \otimes_R M \simeq (N \otimes_R A) \otimes_A M$. The desired result now follows from the observation that $N \otimes_R A$ is $(p+m)$-truncated. □

Proposition 3.3.18. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{Z}$ be maps of spectral Deligne-Mumford stacks. If f has* Tor*-amplitude $\leq m$ and g has* Tor*-amplitude $\leq n$, then $g \circ f$ has* Tor*-amplitude $\leq m + n$.*

Proof. Using Propositions 3.3.15 and 3.3.16, we can reduce to the case where $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ are affine. In this case, the desired result follows from Lemma 3.3.17 and Remark 3.3.14. □

Proposition 3.3.19. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks which is of* Tor*-amplitude $\leq n$. Assume that f is quasi-compact, quasi-separated, and exhibits $\mathfrak{X}$ are a relative spectral algebraic space over $\mathfrak{Y}$. Let $\mathcal{F} \in \operatorname{QCoh}(\mathfrak{X})$ be a quasi-coherent sheaf which is locally of* Tor*-amplitude $\leq k$. Then the pushforward $f_* \mathcal{F} \in \operatorname{QCoh}(\mathfrak{Y})$ has* Tor*-amplitude $\leq n + k$.*

Proof. The assertion is local on $\mathfrak{Y}$; we may therefore suppose that $\mathfrak{Y} \simeq \operatorname{Spec}^{\text{ét}} R$ is affine. Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Let us say that an object $U \in \mathcal{X}$ is *good* if $\mathcal{F}(U)$ is of Tor-amplitude $\leq n + k$ over R. It follows from Lemma 3.3.17 that every affine object of $\mathcal{X}$ is good, and Proposition A.7.2.5.23 implies that the collection of good objects of $\mathcal{X}$ is closed under pushouts. Using Theorem 1.3.8 and Corollary VIII.2.5.9, we conclude that the final object of $\mathcal{X}$ is good, so that $f_* \mathcal{F}$ has Tor-amplitude $\leq n + k$. □

Proposition 3.3.20. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. Assume that:*

(1) *The map f is strongly proper.*

(2) *The map f has* Tor*-amplitude $\leq n$ for some n.*

(3) *The map f is locally almost of finite presentation.*

Then the pushforward functor f_* carries perfect objects of $\mathrm{QCoh}(\mathfrak{X})$ *to perfect objects of* $\mathrm{QCoh}(\mathfrak{Y})$.

Proof. Combine Proposition 3.3.19, Theorem 3.2.2, and Proposition A.7.2.5.23. □

Remark 3.3.21. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack and let $\mathcal{C} \in \mathrm{QStk}(\mathfrak{X})$ be a quasi-coherent stack on $\mathfrak{X}$ which is locally compactly generated. Let $\mathcal{Q}_{\mathfrak{X}} \in \mathrm{QStk}(\mathfrak{X})$ be the quasi-coherent stack which assigns to each map $\eta : \mathrm{Spec}^{\text{ét}} R \to \mathfrak{X}$ the R-linear ∞-category $\mathcal{Q}_{\mathfrak{X}}(\eta) = \mathrm{Mod}_R$. Every object $C \in \mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ induces a map of quasi-coherent stacks $F : \mathcal{Q}_{\mathfrak{X}} \to \mathcal{C}$. If C is locally compact, then we can apply Construction 3.3.2 pointwise to obtain an R-linear functor $\mathcal{C}(\eta) \to \mathcal{Q}_{\mathfrak{X}}(\eta) \simeq \mathrm{Spec}^{\text{ét}} R$ for each point $\eta : \mathrm{Spec}^{\text{ét}} R \to \mathfrak{X}$. These functors amalgamate to a map of quasi-coherent stacks $e_C : \mathcal{C} \to \mathcal{Q}_{\mathfrak{X}}$. The composite map

$$\mathrm{QCoh}(\mathfrak{X}; \mathcal{C}) \xrightarrow{e_C} \mathrm{QCoh}(\mathfrak{X}; \mathcal{Q}) = \mathrm{QCoh}(\mathfrak{X}) \xrightarrow{\Gamma} \mathrm{Sp},$$

can be identified with the spectrum-valued functor $D \mapsto \mathrm{Mor}_{\mathrm{QCoh}(\mathfrak{X};\mathcal{C})}(C, D)$ corepresented by C.

We are now ready to give the proof of Theorem 3.3.11.

Proof of Theorem 3.3.11. It follows from Theorem 1.5.10 that $\mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ is compactly generated, and an object of $\mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ is compact if and only if it is locally compact. It will therefore suffice to prove that if $M, N \in \mathrm{QCoh}(\mathfrak{X}; \mathcal{C})$ are locally compact, then the R-module $\mathrm{Mor}_{\mathrm{QCoh}(\mathfrak{X};\mathcal{C})}(M, N)$ is perfect. Let $e_M : \mathcal{C} \to \mathcal{Q}_{\mathfrak{X}}$ be defined as in Remark 3.3.21, so that $\mathrm{Mor}_{\mathrm{QCoh}(\mathfrak{X};\mathcal{C})}(M, N)$ is given by applying the pushforward functor f_* to $e_M(N) \in \mathrm{QCoh}(\mathfrak{X}; \mathcal{Q}) \simeq \mathrm{QCoh}(\mathfrak{X})$. Using Proposition 3.3.20, we are reduced to proving that $e_M(N) \in \mathrm{QCoh}(\mathfrak{X})$ is perfect. This follows immediately from our assumption that $\mathcal{C}$ is locally proper. □

We close this section by collecting a few other consequences of Proposition 3.3.20. Recall that if $\mathfrak{X}$ is a spectral Deligne-Mumford stacks, an object $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ is perfect if and only if it is dualizable (Proposition VIII.2.7.28). If these conditions are satisfied, we denote a dual of $\mathcal{F}$ by $\mathcal{F}^\vee$.

Proposition 3.3.22. *Let $f : \mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks which is strongly proper, has* Tor*-amplitude $\leq n$ for some n, and is locally almost of finite presentation. Suppose that $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ is perfect. Then:*

(1) *The pushforward $f_* \mathcal{F}$ is a perfect object of* $\mathrm{QCoh}(\mathfrak{Y})$. *Denote its dual by $(f_* \mathcal{F})^\vee$, so that $f^* f_* \mathcal{F}$ is a perfect object of* $\mathrm{QCoh}(\mathfrak{X})$ *with dual $f^*(f_* \mathcal{F})^\vee$.*

(2) *Let $\phi_0 : f^* f_* \mathcal{F} \otimes \mathcal{F}^\vee \to \mathcal{O}_{\mathcal{X}}$ be the morphism obtained by composing the counit map $f^* f_* \mathcal{F} \to \mathcal{F}$ with the evaluation $\mathcal{F} \otimes \mathcal{F}^\vee \to \mathcal{O}_{\mathcal{X}}$, and let $\phi : \mathcal{F}^\vee \to (f^* f_* \mathcal{F})^\vee \simeq f^*(f_* \mathcal{F})^\vee$ be the morphism determined by ϕ_0. For any quasi-coherent sheaf $\mathcal{G}$ on $\mathfrak{Y}$, the composition with ϕ induces a homotopy equivalence*

$$\theta : \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{Y})}((f_* \mathcal{F})^\vee, \mathcal{G}) \to \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X})}(f^*(f_* \mathcal{F})^\vee, f^* \mathcal{G}) \to \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{F}^\vee, f^* \mathcal{G}).$$

Proof. Assertion (1) follows from Proposition 3.3.20. We now prove (2). Let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ be perfect and let $\mathcal{G} \in \mathrm{QCoh}(\mathfrak{Y})$ be arbitrary. Unwinding the definitions, we can identify θ with the canonical map

$$\mathrm{Map}_{\mathrm{QCoh}(\mathfrak{Y})}(\mathcal{O}_{\mathcal{Y}}, f_* \mathcal{F} \otimes \mathcal{G}) \to \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{Y})}(\mathcal{O}_{\mathcal{Y}}, f_*(\mathcal{F} \otimes f^* \mathcal{G})) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{O}_{\mathcal{X}}, \mathcal{F} \otimes f^* \mathcal{G}).$$

It follows from Remark 1.3.14 that θ is a homotopy equivalence. □

Proposition 3.3.23. (1) *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map between quasi-compact, quasi-separated spectral algebraic spaces. Assume that f is locally almost of finite presentation, strongly proper, and has* Tor*-amplitude $\leq n$ for some n. Then the pullback functor $f^* : \mathrm{QCoh}(\mathfrak{Y}) \to \mathrm{QCoh}(\mathfrak{X})$ admits a left adjoint $f_+ : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{Y})$.*

(2) *Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{g'} & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \xrightarrow{g} & \mathfrak{Y}, \end{array}$$

where f and f' satisfy the assumptions of (1)*. Then the diagram of ∞-categories*

$$\begin{array}{ccc} \mathrm{QCoh}(\mathfrak{Y}) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathfrak{X}) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\ \mathrm{QCoh}(\mathfrak{Y}') & \xrightarrow{f'^*} & \mathrm{QCoh}(\mathfrak{X}') \end{array}$$

is left adjointable. In other words, the canonical natural transformation $f'_+ \circ g'^* \to g^* f_+$ is an equivalence of functors from $\mathrm{QCoh}(\mathfrak{X})$ *to* $\mathrm{QCoh}(\mathfrak{Y}')$.

Proof. We first prove (1). For every object $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, let $e(\mathcal{F}) : \mathrm{QCoh}(\mathfrak{Y}) \to \mathcal{S}$ denote the functor given by $e(\mathcal{F})(\mathcal{G}) = \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{F}, f^* \mathcal{G})$. Let $\mathcal{C} \subseteq \mathrm{QCoh}(\mathfrak{X})$ denote the full subcategory spanned by those objects $\mathcal{F}$ for which the functor $e(\mathcal{F})$ is corepresented by an object of $\mathrm{QCoh}(\mathfrak{Y})$. To prove the existence of the left adjoint f_+, it will suffice to show that $\mathcal{C} = \mathrm{QCoh}(\mathfrak{X})$ (see Proposition T.5.2.4.2). Because the ∞-category $\mathrm{QCoh}(\mathfrak{Y})$ admits small colimits, the ∞-category $\mathcal{C}$ is closed under small colimits in $\mathrm{QCoh}(\mathfrak{X})$. Corollary 1.5.12 implies that $\mathrm{QCoh}(\mathfrak{X})$ is generated by perfect objects under filtered colimits. It will therefore suffice to show that $e(\mathcal{F})$ is corepresentable in the special case where $\mathcal{F}$ is perfect. This follows from Proposition 3.3.22, which shows that $e(\mathcal{F})$ is corepresented by the object $(f_* \mathcal{F}^\vee)^\vee$.

We now prove (2). We wish to show that for every object $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, the canonical map $\lambda : f'_+ g'^* \mathcal{F} \to g^* f_+ \mathcal{F}$ is an equivalence in $\mathrm{QCoh}(\mathfrak{Y}')$. Note that both sides are compatible with the formation of colimits in $\mathcal{F}$. Since $\mathrm{QCoh}(\mathfrak{X})$ is generated under filtered colimits by perfect objects, we may assume without loss of generality that $\mathcal{F}$ is perfect. Unwinding the descriptions of the functors f_+ and f'_+ supplied above, we see that λ can be identified with the dual of the canonical map $g^* f_* \mathcal{F}^\vee \to f'_* g'^* \mathcal{F}^\vee$, which is an equivalence by Corollary 1.3.9. □

Remark 3.3.24. In the situation of Proposition 3.3.23, suppose that $\mathfrak{X}$ is locally of Tor-amplitude $\leq n$ over $\mathfrak{Y}$. Then the pullback functor f^* carries $\mathrm{QCoh}(\mathfrak{Y})_{\leq m}$ to $\mathrm{QCoh}(\mathfrak{X})_{\leq m+n}$ for every integer m. It follows that the left adjoint f_+ carries $\mathrm{QCoh}(\mathfrak{X})_{\geq m}$ into $\mathrm{QCoh}(\mathfrak{Y})_{\geq m-n}$. In particular, f_+ carries almost connective objects of $\mathrm{QCoh}(\mathfrak{X})$ to almost connective objects of $\mathrm{QCoh}(\mathfrak{Y})$.

Remark 3.3.25. In the situation of Proposition 3.3.23, the functor f_+ carries perfect objects of $\mathrm{QCoh}(\mathfrak{X})$ to perfect objects of $\mathrm{QCoh}(\mathfrak{Y})$ (this follows immediately from the proof). The functor f_+ also carries almost perfect objects of $\mathrm{QCoh}(\mathfrak{X})$ to almost perfect objects of $\mathrm{QCoh}(\mathfrak{Y})$. To prove this, we can work locally on $\mathfrak{Y}$ and thereby reduce to the case where $\mathfrak{Y} = \mathrm{Spec}^{\text{ét}} R$ is affine. Suppose that $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ is almost perfect, and suppose we are given a filtered diagram $\{M_\alpha\}$ of m-truncated R-modules having colimit M. If f is locally of Tor-amplitude $\leq n$, then $f^* M_\alpha$ is a filtered diagram in $\mathrm{QCoh}(\mathfrak{X})_{\leq m+n}$ having colimit $f^* M$. Using the fact that $\mathcal{F}$ is almost perfect, we deduce that the canonical map $\theta : \varinjlim \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{F}, f^* M_\alpha) \to \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{F}, f^* M)$ is an equivalence. Identifying θ with the canonical map $\varinjlim \mathrm{Map}_{\mathrm{Mod}_R}(f_+ \mathcal{F}, M_\alpha) \to \mathrm{Map}_{\mathrm{Mod}_R}(f_+ \mathcal{F}, M)$, we deduce that $f_+ \mathcal{F}$ is almost perfect, as desired.

3.4 Valuative Criteria

Let $f : X \to Y$ be a map of schemes which is quasi-compact, separated, and locally of finite type. According to the *valuative criterion of properness*, the map f is proper if and only if it satisfies the following condition:

$(*)$ For every valuation ring V with residue field K and every commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \operatorname{Spec} V & \longrightarrow & Y, \end{array}$$

there exists a dotted arrow as indicated, rendering the diagram commutative (since f is separated, the dotted arrow is essentially unique).

Our goal in this section is to establish a similar valuative criterion in the setting of spectral algebraic geometry. Since the condition that a map $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral algebraic spaces be strongly proper depends only on the underlying ordinary algebraic spaces of $\mathfrak{X}$ and $\mathfrak{Y}$, our result can be formally deduced from the usual valuative criterion (for maps between algebraic spaces). We will reproduce a proof here for the sake of completeness.

Theorem 3.4.1 (Valuative Criterion for Properness)**.** *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a quasi-compact, strongly separated map of spectral Deligne-Mumford stacks which is locally of finite presentation to order 0. Then f is strongly proper if and only if the following condition is satisfied:*

$(*)$ *For every valuation ring V with fraction field K and every commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec}^{\acute{e}t} K & \longrightarrow & \mathfrak{X} \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec}^{\acute{e}t} V & \longrightarrow & \mathfrak{Y}, \end{array}$$

there exists a dotted arrow as indicated, rendering the diagram commutative.

Moreover, if $\mathfrak{Y}$ is locally Noetherian, then it suffices to verify condition $(*)$ in the special case where V is a discrete valuation ring.

Before giving the proof, let us deduce some consequences.

Corollary 3.4.2 (Valuative Criterion for Separatedness)**.** *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a quasi-separated map of spectral Deligne-Mumford stacks which represent functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that f is a relative spectral algebraic space (that is, that the induced map $X(R) \to Y(R)$ has discrete homotopy fibers, for every commutative ring R). The following conditions are equivalent:*

(1) *The map f is strongly separated.*

(2) *The diagonal map $\delta : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is proper.*

(3) *For every valuation ring A with residue field K, the canonical map $X(V) \to X(K) \times_{Y(K)} Y(V)$ is (-1)-truncated (that is, it is the inclusion of a summand).*

Moreover, if f is locally of finite presentation to order 0 and $\mathfrak{Y}$ is locally Noetherian, then it suffices to verify condition (3) in the special case where V is a discrete valuation ring.

Proof. The implication (1) $\Rightarrow$ (2) is immediate (since any closed immersion is proper). Let $\mathfrak{Z} = \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$, and let $\delta' : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{X}$ be the diagonal of the map δ. Since f is a relative spectral algebraic space, the map δ' induces an equivalence between the underlying 0-truncated spectral Deligne-Mumford stacks, and is therefore a closed immersion. It follows that δ is strongly separated. Since $\mathfrak{X}$ is quasi-separated, the map δ is quasi-compact. Since δ admits a left homotopy inverse, it is locally of finite presentation to order 0. Using Theorem 3.4.1, we see that δ is proper if and only if the following condition is satisfied:

- $(*)$ Let $Z : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by $\mathfrak{Z}$. Then, for every valuation ring A with residue field K, the canonical map $X(V) \to X(K) \times_{Z(K)} Z(V)$ is surjective on connected components.

Unwinding the definitions, we see that (2) $\Leftrightarrow$ $(*)$ $\Leftrightarrow$ (3).

Write $\mathfrak{Z} = (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ and $\mathfrak{Z}_0 = (\mathcal{Z}, \pi_0 \, \mathcal{O}_{\mathcal{Z}})$. Note that δ is proper if and only if the induced map $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Z}_0 \to \mathfrak{Z}_0$ is proper. If f is locally of finite presentation to order 0 and $\mathfrak{Y}$ is locally Noetherian, then $\mathfrak{Z}_0$ is locally Noetherian. Using Theorem 3.4.1, we deduce that δ is proper if and only if condition $(*)$ is satisfied whenever V is a discrete valuation ring (which is equivalent to the requirement that (3) is satisfied whenever V is a discrete valuation ring).

To complete the proof, it will suffice to show that (2) $\Rightarrow$ (1). Assume that δ is proper. Since δ is locally quasi-finite, we conclude that δ is finite (Proposition 3.2.5). Choose a map $\mathrm{Spec}^{\text{ét}} R \to \mathfrak{Z}$, so that $\mathfrak{X} \times_{\mathfrak{Z}} \mathrm{Spec}^{\text{ét}} R \simeq \mathrm{Spec}^{\text{ét}} R'$ for some R-algebra R'. We wish to prove that the underlying map of commutative rings $\pi_0 R \to \pi_0 R'$ is surjective. Replacing R by $\pi_0 R$, we may assume that R is a commutative ring. Since a map of discrete R-modules $M \to N$ is surjective if and only if it surjective after localization at any prime ideal $\mathfrak{p}$ of R, we may replace R by $R_{\mathfrak{p}}$ and thereby reduce to the case where R is local. Since $\pi_0 R'$ is finitely generated as a module over R, we may use Nakayama's lemma to replace R by its residue field and thereby reduce to the case where R is a field k. Then $\pi_0 R'$ is a finite dimensional algebra over k. We will complete the proof by showing that the dimension of $\pi_0 R'$ is ≤ 1. For this, it suffices to show that the inclusion of the first factor induces an isomorphism

$$\pi_0 R' \to (\pi_0 R') \otimes_k (\pi_0 R') \simeq \pi_0 (R' \otimes_k R').$$

This follows immediately from our observation that δ' induces an equivalence on the underlying 0-truncated spectral Deligne-Mumford stacks. □

Corollary 3.4.3. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a quasi-compact, quasi-separated morphism of spectral Deligne-Mumford stacks which is locally of finite presentation to order* 0*. Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functors represented by $\mathfrak{X}$ and $\mathfrak{Y}$, and suppose that f is a relative spectral algebraic space. Then f is strongly proper if and only if, for every valuation ring V with residue field K, the induced map*

$$X(V) \to X(K) \times_{Y(K)} Y(V)$$

is a homotopy equivalence. Moreover, if $\mathfrak{Y}$ is locally Noetherian, then it suffices to verify this condition in the special case where V is a discrete valuation ring.

Proof. Combine Theorem 3.4.1 with Corollary 3.4.2. □

We now turn to the proof of Theorem 3.4.1. We will need a few preliminaries.

Lemma 3.4.4. *Let R be a commutative ring, let K be a field, and let $\phi : R \to K$ be a ring homomorphism. Let $\mathfrak{p} \subseteq R$ be a prime ideal containing* $\ker(\phi)$*. Then there exists a valuation subring $V \subseteq K$ (with fraction field K) such that $\phi(R) \subseteq V$ and $\mathfrak{p} = \phi^{-1}\mathfrak{m}$, where $\mathfrak{m}$ denotes the maximal ideal of V. Moreover, if R is Noetherian, $\mathfrak{p} \neq \ker(\phi)$, and K is finitely generated over R, then we can arrange that V is a discrete valuation ring.*

Proof. We first treat the general case (where R is not assumed to be Noetherian). Replacing R by the localization $R_{\mathfrak{p}}$, we may assume that R is a local ring with maximal ideal $\mathfrak{p}$. Let P denote the partially ordered consisting of subrings $A \subseteq K$ which contain $\phi(R)$ and satisfy $\mathfrak{p}A \neq A$. Using Zorn's lemma, we deduce that P has a maximal element, which we will denote by V. We will show that V has the desired properties.

We first claim that V is a local ring. Choose elements $x, y \in V$ with $x + y = 1$; we must show that either x or y is invertible in V. Since $V/\mathfrak{p}V \neq 0$, one of the localizations $(V/\mathfrak{p}V)[\frac{1}{x}]$ and $(V/\mathfrak{q}V)[\frac{1}{y}]$ must be nonzero. Without loss of generality, we may assume that $(V/\mathfrak{q}V)[\frac{1}{x}] \neq 0$, so that $V[\frac{1}{x}] \neq \mathfrak{q}V[\frac{1}{x}]$. The maximality of V then implies that $V = V[\frac{1}{x}]$, so that x is invertible in V.

Let $\mathfrak{m}$ denote the maximal ideal of V. Since $\mathfrak{p}V$ is a proper ideal of V, we have $\mathfrak{p}V \subseteq \mathfrak{m}$ and therefore $\mathfrak{p} \subseteq \phi^{-1}\mathfrak{m}$. Since $\mathfrak{p}$ is a maximal ideal of R, we conclude that $\mathfrak{p} = \phi^{-1}\mathfrak{m}$.

We now complete the proof by showing that V is a valuation ring with fraction field K. Let x be a nonzero element of K; we wish to show that either x or x^{-1} belongs to V. If x^{-1} does not belong to V, then the subring $V' \subseteq K$ generated by V and x^{-1} is strictly larger than V and therefore satisfies $V' = \mathfrak{p}V'$. In particular, we can write $1 = \sum_{0 \leq i \leq n} c_i x^{-i}$ for some coefficients $c_i \in \mathfrak{p}V \subseteq \mathfrak{m}$. Then $x^n = \sum_{1 \leq i \leq n} \frac{c_i}{1-c_0} x^{n-i}$ so that x is integral over V. If x does not belong to V, then the subring $V'' \subseteq K$ generated by V and x properly contains V and is finitely generated as an V-module. The maximality of V implies that $V'' = \mathfrak{p}V''$. Using Nakayama's lemma, we deduce that $V'' = 0$ and obtain a contradiction. This completes the proof of the first assertion.

Now suppose that R is Noetherian, $\mathfrak{p} \neq \ker(\phi)$, and that K is finitely generated over R. Replacing R by its image in K, we may suppose that R is a subring of K. Let $x_1, \ldots, x_n \in K$ be a transcendence basis for K over the fraction field of R. Replacing R by $R[x_1, \ldots, x_n]$ and $\mathfrak{p}$ by $\mathfrak{p}[x_1, \ldots, x_n]$, we may reduce to the case where K is a finite algebraic extension of the fraction field of R. Replacing R by the localization $R_{\mathfrak{p}}$, we may assume that R is a local ring with maximal ideal $\mathfrak{p}$. Since R is Noetherian, we can choose a finite set of generators $y_1, \ldots, y_m \in \mathfrak{p}$ for the ideal $\mathfrak{p}$. For $1 \leq i \leq m$, let R_i denote the subring of K generated by R together with the elements $\frac{y_j}{y_i}$. We now claim:

($*$) There exists $1 \leq i \leq m$ such that y_i is not invertible in R_i.

Suppose that ($*$) is not satisfied: that is, $\frac{1}{y_i} \in R_i$ for every index i. Then each $\frac{1}{y_i}$ can be written as a polynomial (with coefficients in R) in the variables $\frac{y_j}{y_i}$. Clearing denominators, we deduce that there exists an integer a such that $y_i^a \in \mathfrak{p}^{a+1}$ for every index i. It follows that $\mathfrak{p}^b \subseteq \mathfrak{p}^{b+1}$ for $b > a(m-1)$. Since R is Noetherian with maximal ideal $\mathfrak{p}$, the Krull intersection theorem implies that $\bigcap_{b \geq 0} \mathfrak{p}^b = 0$, so that $\mathfrak{p}^b = 0$ for $b > a(m-1)$. In particular, $\mathfrak{p}$ consists of nilpotent elements of R. Since $R \subseteq K$ is an integral domain, we deduce that $\mathfrak{p} = 0$, contradicting our assumption that $\mathfrak{p} \neq \ker(\phi)$. This completes the proof of ($*$).

Using ($*$), let us choose an index i such that y_i is not invertible in R_i. Let $\mathfrak{q}$ be minimal among prime ideals of R_i which contain y_i. Then $\mathfrak{q}$ contains each y_j, so that $\mathfrak{q} \cap R$ contains $\mathfrak{p}$. Since $\mathfrak{p}$ is a maximal ideal of R, we deduce that $\mathfrak{q} \cap R = \mathfrak{p}$. We may therefore replace R by $(R_i)_{\mathfrak{q}}$ (which is Noetherian, since it is finitely generated over R) and thereby reduce to the case where the prime ideal $\mathfrak{p}$ of R is minimal among prime ideals containing some element $x \in R$. By Krull's Hauptidealsatz, we deduce that R has Krull dimension 1. Let R' be the integral closure of R in K. The Krull-Akizuki theorem guarantees that R' is a Dedekind domain. Since R' is integral over R, the maximal ideal $\mathfrak{p}$ of R can be lifted to a maximal ideal $\mathfrak{p}'$ of R'. Then $V = R'_{\mathfrak{p}'}$ is a discrete valuation ring with the desired properties. □

Lemma 3.4.5. *Let V be a valuation ring with maximal ideal $\mathfrak{m}$ and residue field K, let K' be an extension field of K having degree n, and let R be a subring of K' containing V. Then there are at most finitely many prime ideals $\mathfrak{q} \subseteq R$ such that $\mathfrak{q} \cap V = \mathfrak{m}$.*

Proof. For every prime ideal $\mathfrak{q} \subseteq R$, there exists a valuation ring $V' \subseteq K'$ with fraction field K' and maximal ideal $\mathfrak{m}'$ such that $R \subseteq V'$ and $\mathfrak{q} = R \cap \mathfrak{m}'$ (Lemma 3.4.4). In particular, it follows that $\mathfrak{m} = \mathfrak{m}' \cap V$. Consequently, V' determines a valuation on K' extending the valuation on K determined by V. According to Exercise 12.1 of [51], there are at most finitely many possibilities for the valuation ring V', hence at most finitely many possibilities for the prime ideal $\mathfrak{q} \subseteq R$. □

Proof of Theorem 3.4.1. We first prove that condition $(*)$ is necessary. Without loss of generality, we may replace $\mathfrak{Y}$ by $\operatorname{Spec}^{\text{ét}} V$. Let K denote the fraction field of V; we wish to show that if $f : \mathfrak{X} \to \operatorname{Spec}^{\text{ét}} V$ is proper, then every map $\phi : \operatorname{Spec}^{\text{ét}} K \to \mathfrak{X}$ (of spectral algebraic spaces over V) extends to a map $\operatorname{Spec}^{\text{ét}} A \to \mathfrak{X}$. The map ϕ determines a point $\eta \in |\mathfrak{X}|$. Let Z denote the smallest closed subset of $|\mathfrak{X}|$ containing the point η. Let $\mathfrak{Z}$ denote the reduced closed substack of $\mathfrak{X}$ corresponding to the subset Z. We claim that composite map $\mathfrak{Z} \hookrightarrow \mathfrak{X} \xrightarrow{f} \operatorname{Spec}^{\text{ét}} V$ is locally quasi-finite. This assertion is local on $\mathfrak{X}$. Let us therefore choose an étale map $g : \operatorname{Spec}^{\text{ét}} A \to \mathfrak{X}$, and set

$$\operatorname{Spec}^{\text{ét}} K \times_{\mathfrak{X}} \operatorname{Spec}^{\text{ét}} A \simeq \operatorname{Spec}^{\text{ét}} K' \qquad \mathfrak{Z} \times_{\mathfrak{X}} \operatorname{Spec}^{\text{ét}} A \simeq \operatorname{Spec}^{\text{ét}} B.$$

Since g is étale, K' is a product of separable algebraic extension fields of K. Similarly, $\operatorname{Spec}^{\text{ét}} B \to \mathfrak{Z}$ is étale, so that B is a reduced commutative ring. By construction, the map $\operatorname{Spec}^{\text{ét}} K \to \mathfrak{Z}$ induces an injective map of commutative rings $B \hookrightarrow K'$. Since f is locally of finite presentation to order 0, B is finitely generated as a commutative ring over V. To show that $\mathfrak{Z} \to \operatorname{Spec}^{\text{ét}} V$ is locally quasi-finite, we wish to that for every prime ideal $\mathfrak{p} \subseteq V$, there are only finitely many prime ideals of B lying over $\mathfrak{p}$. Replacing V by $V_{\mathfrak{p}}$, we may reduce to the case where $\mathfrak{p}$ is the maximal ideal of V. In this case, the desired result follows from Lemma 3.4.5.

The map $\mathfrak{Z} \to \operatorname{Spec}^{\text{ét}} V$ is strongly proper and locally quasi-finite, and therefore finite (Proposition 3.2.5). It follows that we can write $\mathfrak{Y} \simeq \operatorname{Spec}^{\text{ét}} R$ for some commutative ring R which is finitely generated as an V-module. Moreover, the map $\operatorname{Spec}^{\text{ét}} K \to \mathfrak{Y}$ induces an injection $R \to K$. We may therefore identify R with a subalgebra of K which is finitely generated as a module over V. Since V is a valuation ring of K, it is integrally closed in K. It follows that $R \simeq V$, so that the inclusion $\mathfrak{Z} \hookrightarrow \mathfrak{X}$ gives the desired extension of ϕ.

Suppose now that $(*)$ is satisfied and that we are given a pullback diagram

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec}^{\text{ét}} R & \longrightarrow & \mathfrak{Y}; \end{array}$$

we wish to prove that the induced map of topological spaces $|\mathfrak{X}'| \to \operatorname{Spec}^{\mathrm{Z}} R$ is closed. Let Z be a closed subset of $|\mathfrak{X}'|$ and let $\mathfrak{Z}$ be the corresponding reduced closed substack of $\mathfrak{X}'$. Choose an étale surjection $\operatorname{Spec}^{\text{ét}} B \to \mathfrak{Z}$ (so that B is a reduced commutative ring) and let I denote the kernel of the induced map of commutative rings $\pi_0 R \to B$. We will prove that $\psi(Z) \subseteq \operatorname{Spec}^{\mathrm{Z}} R$ agrees with the image of the closed embedding $\operatorname{Spec}^{\mathrm{Z}}(\pi_0 R)/I \hookrightarrow \operatorname{Spec}^{\mathrm{Z}} R$. To this end, let $\mathfrak{q}$ be a prime ideal of $\pi_0 R$ containing the ideal I; we wish to show that $\mathfrak{q}$ belongs to $\psi(Z)$. Using Zorn's lemma, we see that there is a prime ideal $\mathfrak{p} \subseteq \mathfrak{q}$ of $\pi_0 R$ which is minimal among prime ideals which contain I. The injection of commutative rings $(\pi_0 R)/I \hookrightarrow B$ induces an injection $((\pi_0 R)/I)_{\mathfrak{p}} \to B_{\mathfrak{p}}$, so that the localization $B_{\mathfrak{p}}$ is nonzero. It follows that $B_{\mathfrak{p}}$ contains a prime ideal, which is the localization of a prime ideal $\mathfrak{p}' \subseteq B$. Note that the image of $\mathfrak{p}'$ in $\operatorname{Spec}^{\mathrm{Z}} R$ belongs to the image of the inclusion $\operatorname{Spec}^{\mathrm{Z}}((\pi_0 R)/I)_{\mathfrak{p}} \hookrightarrow \operatorname{Spec}^{\mathrm{Z}} R$. By construction, the ring $((\pi_0 R)/I)_{\mathfrak{p}}$ contains a unique prime ideal, whose image in $\operatorname{Spec}^{\mathrm{Z}} R$ coincides with $\mathfrak{p}$. It follows that the map $\operatorname{Spec}^{\mathrm{Z}} B \to \operatorname{Spec}^{\mathrm{Z}} R$ carries $\mathfrak{p}'$ to $\mathfrak{p}$.

Let K denote the fraction field of $B/\mathfrak{p}'$ and let $\psi : \pi_0 R \to K$ be the induced map. Using Lemma 3.4.4, we can choose a valuation ring $V \subseteq K$ with residue field K and maximal ideal $\mathfrak{m}$, such that $\psi^{-1}\mathfrak{m} = \mathfrak{q}$. This determines a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}^{\text{ét}} K & \longrightarrow & \mathfrak{X}' \\ \downarrow & \overset{i}{\nearrow} & \downarrow \\ \operatorname{Spec}^{\text{ét}} V & \longrightarrow & \operatorname{Spec}^{\text{ét}} R. \end{array}$$

Applying condition $(*)$, we deduce the existence of a dotted arrow as indicated in the diagram. Since the map $\operatorname{Spec}^{\text{ét}} K \to \mathfrak{X}'$ factors through the closed immersion $\mathfrak{Z} \hookrightarrow \mathfrak{X}'$, the map i also factors through $\mathfrak{Z}$. It follows that $\psi(Y)$ contains the image of the map $\operatorname{Spec}^{\mathrm{Z}} V \to \operatorname{Spec}^{\mathrm{Z}} R$, which includes the point $\mathfrak{q} \in \operatorname{Spec}^{\mathrm{Z}} R$. This completes the proof that f is strongly proper.

Now let us assume that $\mathfrak{Y}$ is locally Noetherian, and that condition $(*)$ is satisfied whenever V is a discrete valuation ring. We wish to show that f is strongly proper. The assertion is local on $\mathfrak{Y}$; we may therefore assume that $\mathfrak{Y} = \operatorname{Spec}^{\text{ét}} R$ for some Noetherian $\mathbb{E}_\infty$-ring R. We wish to show that for every pullback diagram

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec}^{\text{ét}} R' & \longrightarrow & \operatorname{Spec}^{\text{ét}} R, \end{array}$$

the induced map of topological spaces $|\mathfrak{X}'| \to \operatorname{Spec}^{\mathrm{Z}} R'$ is closed. Using Remark 3.1.12, we assume without loss of generality that R' is Noetherian. Replacing R by R', we are reduced to proving that the map $|\mathfrak{X}| \to \operatorname{Spec}^{\mathrm{Z}} R$ is closed. The proof now proceeds as in the previous case, using the second part of Lemma 3.4.4 to arrange that the valuation ring V is actually discrete. □

4 Completions of Modules

One of the most basic constructions in commutative algebra is that of *completion*. If R is a commutative ring, $I \subseteq R$ is an ideal, and M is an R-module, then the I-adic completion $Cpl(M; I)$ of M is defined to be the inverse limit

$$\varprojlim M/I^n M.$$

This construction behaves exceptionally well if R is a Noetherian ring and we restrict our attention to finitely generated R-modules. In this case, the construction $M \mapsto Cpl(M; I)$ is an exact functor and there is a canonical isomorphism

$$Cpl(M; I) \simeq Cpl(R; I) \otimes_R M.$$

If R is not Noetherian (or if R is Noetherian, and we wish to consider R-modules which are not finitely generated), then the situation is more complicated. In this setting, the construction $M \mapsto Cpl(M; I)$ is usually ill-behaved. Nevertheless, there is an analogous construction in the derived category of R-modules which enjoys good formal properties in general (at least when the ideal I is finitely generated). We say that an R-module spectrum M is *I-complete* if, for every element $x \in I$, the (homotopy) inverse limit of the tower

$$\cdots M \xrightarrow{x} M \xrightarrow{x} M$$

vanishes. The collection of I-complete R-module spectra form a localization of the ∞-category of Mod_R. That is, for any $M \in \operatorname{Mod}_R$, there exists a morphism $M \to M^{\wedge}_I$ which is universal among maps from M to I-complete R-modules. We refer to $M^{\wedge}_I$ as the *I-completion* of M.

Our goal in this section is to study the ∞-category of I-complete R-modules in the setting of an arbitrary $\mathbb{E}_\infty$-ring R (where $I \subseteq \pi_0 R$ is an arbitrary finitely generated ideal). We begin in §4.1 by studying the related notions of I-nilpotent and I-local R-modules (and the corresponding localization and colocalization constructions, which are closely related to Grothendieck's theory of local cohomology). In §4.3 we introduce the ∞-category of I-complete R-modules, prove the existence of the I-completion functor $M \mapsto M^{\wedge}_I$, and study its properties. In §4.3, we specialize to the case where R is Noetherian, and show that the I-completion functor is closely related to the classical I-adic completion (at least for R-modules which are almost perfect; see Proposition 4.3.6).

4.1 I-Nilpotent and I-Local Modules

Let A be an $\mathbb{E}_\infty$-ring and let $U \subseteq \operatorname{Spec}^{\mathrm{Z}} A$ be a quasi-compact open subset. Then U determines a quasi-compact open immersion $j : \mathfrak{U} \subseteq \operatorname{Spec}^{\text{ét}} A$. The pushforward functor $j_* : \operatorname{QCoh}(\mathfrak{U}) \to \operatorname{QCoh}(\operatorname{Spec}^{\text{ét}} A) \simeq$

Mod_A is a fully faithful embedding, whose essential image is a localization of Mod_A. Our goal in this section is to give a purely algebraic description of this localization. We begin by reviewing some commutative algebra.

Definition 4.1.1. Let R be a commutative ring and let M be a discrete R-module. For each element $x \in M$, we let $\mathrm{Supp}(x)$ denote the set $\{a \in R : (\exists n) a^n x = 0\}$. We will refer to $\mathrm{Supp}(m)$ as the *support of* x.

Remark 4.1.2. Let R and M be as in Definition 4.1.1. If $a^n x = 0$ and $b^{n'} x = 0$, then the binomial formula implies that $(a+b)^{n+n'} x = 0$. It follows immediately that $\mathrm{Supp}(x)$ is an ideal of R. Moreover, this ideal is *radical*: that is, if $a^k \in \mathrm{Supp}(x)$, then $a \in \mathrm{Supp}(x)$.

Definition 4.1.3. Let A be an $\mathbb{E}_2$-ring, and let $I \subseteq \pi_0 A$ be an ideal. We will say an object $M \in \mathrm{LMod}_A$ is *I-nilpotent* if, for every element $x \in \pi_k M$, we have $I \subseteq \mathrm{Supp}(x) \subseteq \pi_0 A$. We let LMod_A^{I-nil} denote the full subcategory of LMod_A spanned by the I-nilpotent objects.

Remark 4.1.4. Since the support of any element $m \in \pi_k M$ is a radical ideal in $\pi_0 A$, we see that Definition 4.1.3 depends only on the radical of the ideal I (or, equivalently, the closed subset of the Zariski spectrum of $\pi_0 A$ determined by I).

Remark 4.1.5. Let A be an $\mathbb{E}_2$-ring, and let $I \subseteq \pi_0 A$ be the sum of a collection of ideals $I_\alpha \subseteq \pi_0 A$. Then an object $M \in \mathrm{LMod}_A$ is I-nilpotent if and only if M is I_α-nilpotent for every index α.

Proposition 4.1.6. *Let A be an $\mathbb{E}_2$-ring and $I \subseteq \pi_0 A$ an ideal. Then LMod_A^{I-nil} is closed under desuspension and small colimits, and is therefore a stable subcategory of LMod_A.*

Proof. It is obvious that LMod_A^{I-nil} is closed under desuspension. To prove that it is closed under small colimits, it will suffice to show that it is closed under coequalizers and small coproducts. We first treat the case of coproducts. Assume that $M \in \mathrm{LMod}_A$ is given as the coproduct of a collection of I-nilpotent objects $M_\alpha \in \mathrm{LMod}_A$. Let $x \in \pi_k M$, so that x is given by a family of elements $x_\alpha \in \pi_k M_\alpha$. Fix $a \in I$; we wish to show that $a^n x = 0 \in \pi_k M$ for $n \gg 0$. Since each M_α is I-nilpotent, we can choose n_α such that $a^{n_\alpha} x_\alpha = 0 \in \pi_k M_\alpha$. Moreover, we have $x_\alpha = 0$ for almost all α; we may therefore assume that $n_\alpha = 0$ for almost all α. Taking n to be the supremum of the set $\{n_\alpha\}$, we deduce that $a^n x = 0$ as desired.

We now show that the collection of I-nilpotent objects of LMod_A is closed under coequalizers. Let $M, N \in \mathrm{LMod}_A$ be I-nilpotent and suppose we are given a pair of maps $f, g : M \to N$. Then the coequalizer of f and g can be identified with the cofiber P of $f - g$. We have an exact triangle $M \to N \to P$, whence an exact sequence of homotopy groups

$$\pi_k N \xrightarrow{\phi} \pi_k P \xrightarrow{\psi} \pi_{k-1} M.$$

Fix $x \in \pi_k P$ and $a \in I$. Since M is I-nilpotent, we have $\psi(a^n x) = a^n \psi(x) = 0$ for n sufficiently large. It follows from exactness that $a^n x = \phi(y)$ for some $y \in \pi_k N$. Since N is I-nilpotent, we have $a^{n'} y = 0$ for n' sufficiently large, so that $0 = \phi(a^{n'} y) = a^{n'} \phi(y) = a^{n+n'} x$. □

Corollary 4.1.7. *Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be an ideal. Then LMod_A^{I-nil} is bitensored over LMod_A. More precisely, if $M \in \mathrm{LMod}_A$ is I-nilpotent and $N \in \mathrm{LMod}_A$ is arbitrary, then $M \otimes_A N$ and $N \otimes_A M$ are I-nilpotent.*

Proof. Fix an I-nilpotent object $M \in \mathrm{LMod}_A$. Let $\mathcal{C} \subseteq \mathrm{LMod}_A$ be the full subcategory of LMod_A spanned by those objects N such that $M \otimes_A N$ and $N \otimes_A M$ are I-nilpotent. It follows from Proposition 4.1.6 that $\mathcal{C}$ is closed under desuspension and small colimits in LMod_A. Since $A \in \mathcal{C}$, we conclude that $\mathcal{C} = \mathrm{LMod}_A$. □

Remark 4.1.8. Let A be a connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be an ideal. It follows immediately from the definition that if $M \in \mathrm{LMod}_A$ is I-nilpotent, then $\tau_{\geq 0} M$ and $\tau_{\leq 0} M$ are I-nilpotent. It follows that the t-structure on LMod_A induces a t-structure on $\mathrm{LMod}_A^{(-nilI)}$, for which the inclusion $\mathrm{LMod}_A^{(-nilI)} \hookrightarrow \mathrm{LMod}_A$ is t-exact.

Definition 4.1.9. Let A be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 A$ be an ideal, and let $M \in \mathrm{LMod}_A$. We will say that M is *I-local* if $\mathrm{Map}_{\mathrm{LMod}_A}(N, M)$ is contractible for every I-nilpotent A-module N. We let LMod_A^{I-loc} denote the full subcategory of LMod_A spanned by the I-local objects.

Notation 4.1.10. Let A be an $\mathbb{E}_2$-ring and let $a \in \pi_0 A$. We let $A[a^{-1}]$ denote the $\mathbb{E}_2$-ring introduced in Example A.7.5.0.7. For every object $M \in \mathrm{LMod}_A$, we let $M[\frac{1}{a}] = A[\frac{1}{a}] \otimes_A M$. Proposition A.7.2.2.13 supplies a canonical isomorphism of graded abelian groups

$$\pi_*(M[\frac{1}{a}]) \simeq (\pi_* M)[\frac{1}{a}].$$

It follows that if M is I-nilpotent for some ideal $I \subseteq \pi_0 A$ which contains a, then $M[\frac{1}{a}] \simeq 0$.

Remark 4.1.11. Let A be an $\mathbb{E}_2$-ring and let $a \in \pi_0 A$, so that right multiplication by x induces a morphism of left A-modules $r_a : A \to A$. We observe that r_a induces an equivalence after tensoring with $A[\frac{1}{a}]$. It follows that precomposition by r_a induces a homotopy equivalence from the mapping space $\mathrm{Map}_{\mathrm{LMod}_A}(A, A[\frac{1}{a}])$ to itself, so that the tautological map $i : A \to A[\frac{1}{a}]$ determines a map from the colimit of the diagram

$$A \xrightarrow{r_a} A \xrightarrow{r_a} A \to \cdots$$

into $A[\frac{1}{a}]$. By computing the homotopy groups on each side, we see that this map is an equivalence. It follows that for any left A-module M, we can identify $M[\frac{1}{a}] \simeq A[\frac{1}{a}] \otimes_A M$ with the colimit of the diagram

$$M \xrightarrow{r_a} M \xrightarrow{r_a} M \to \cdots,$$

where $r_a : M \to M$ denotes the map given by

$$M \simeq A \otimes_A M \xrightarrow{r_a \otimes \mathrm{id}} A \otimes_A M \simeq M.$$

Proposition 4.1.12. *Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. Then there exists a left A-module V such that the functor $M \mapsto V \otimes_A M$ is right adjoint to the inclusion $\mathrm{LMod}_A^{I-nil} \to \mathrm{LMod}_A$.*

Proof. Choose generators $x_1, \ldots, x_n$ for the ideal I. For $1 \leq i \leq n$, let V_i be the fiber of the localization map $A \to A[\frac{1}{x_i}]$. Let $V = \bigotimes_{1 \leq i \leq n} V_i$ and let $\alpha : V \to A$ be the evident map. We claim that α exhibits the functor $M \mapsto V \otimes_A M$ as a right adjoint to the inclusion $\mathrm{LMod}_A^{I-nil} \to \mathrm{LMod}_A$. To prove this, it suffices to show the following:

- (a) For each $M \in \mathrm{LMod}_A$, the tensor product $V \otimes_A M$ is I-nilpotent. To prove this, it suffices to show that $V \otimes_A M$ is (x_i)-torsion for $1 \leq i \leq n$ (Remark 4.1.5). Using Corollary 4.1.7, we are reduced to the problem of showing that V_i is (x_i)-nilpotent. For this, we observe that V_i is the colimit of A-modules $V(m)_i$, where $V(m)_i$ is the cofiber of the map $A \xrightarrow{x_i^m} A$ given by right multiplication by x_i^m (Remark 4.1.11). According to Proposition 4.1.6, it will suffice to show that each $V(m)_i$ is (x_i)-nilpotent. We now observe that the exact sequence

$$\pi_{k+1} A \xrightarrow{x_i^m} \pi_{k+1} A \to \pi_k V(m)_i \to \pi_k A \xrightarrow{x_i^m} \pi_k M$$

  guarantees that $\pi_k V(m)_i$ is annihilated by x_i^{2m}.

- (b) Let $N \in \mathrm{LMod}_A^{I-nil}$; we must show that α induces a homotopy equivalence $\mathrm{Map}_{\mathrm{LMod}_A}(N, V \otimes_A M) \to \mathrm{Map}_{\mathrm{LMod}_A}(N, M)$. This map is given as a composition of maps

$$\theta_i : \mathrm{Map}_{\mathrm{LMod}_A}(N, V_i \otimes_A V_{i+1} \otimes_A \cdots \otimes_A V_n \otimes_A M) \to \mathrm{Map}_{\mathrm{LMod}_A}(N, V_{i+1} \otimes_A \cdots \otimes_A V_n \otimes_A M).$$

  It will therefore suffice to show that each θ_i is a homotopy equivalence. For this, it suffices to show that each of the mapping spaces

$$\mathrm{Map}_{\mathrm{LMod}_A}(N, A[x_i^{-1}] \otimes_A V_{i+1} \otimes_A \cdots \otimes_A V_n \otimes_A M)$$

is contractible. This mapping space can be identified with

$$\mathrm{Map}_{\mathrm{LMod}_{A[\frac{1}{x_i}]}}(N[\frac{1}{x_i}], A[\frac{1}{x_i}] \otimes_A V_{i+1} \otimes_A \cdots \otimes_A V_n \otimes_A M).$$

This space is contractible, since our assumption that N is I-nilpotent implies that $N[\frac{1}{x_i}] \simeq 0$.

□

Notation 4.1.13. Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. We let $\Gamma_I : \mathrm{LMod}_A \to \mathrm{LMod}_A^{I-nil}$ denote a right adjoint to the inclusion functor $\mathrm{LMod}_A^{I-nil} \hookrightarrow \mathrm{LMod}_A$, whose existence is guaranteed by Proposition 4.1.12.

Example 4.1.14. In the situation of Proposition 4.1.12, suppose that the ideal I is generated by a single element x. For any $M \in \mathrm{LMod}_A$, we have a canonical fiber sequence

$$\Gamma_{(x)} M \to M \to M[\frac{1}{x}].$$

Proposition 4.1.15. *Let A be an $\mathbb{E}_2$-ring and $I \subseteq \pi_0 A$ a finitely generated ideal. Then the ∞-category LMod_A^{I-nil} is compactly generated. Moreover, the inclusion $\mathrm{LMod}_A^{I-nil} \hookrightarrow \mathrm{LMod}_A$ carries compact objects to compact objects.*

Proof. Choose a collection of elements $x_1, \ldots, x_n \in \pi_0 A$ which generate the ideal I. For $1 \leq i \leq n$, let Q_i denote the cofiber of the map $r_{x_i} : A \to A$ given by left multiplication with x_i. Each Q_i is (x_i)-nilpotent. It follows from Corollary 4.1.7 that the tensor product $Q = \bigotimes_{1 \leq i \leq n} Q_i$ is (x_i)-nilpotent for each i, so that Q is I-nilpotent by Remark 4.1.5. By construction, Q is a perfect object of LMod_A, and in particular a compact object of LMod_A^{I-nil}. We will complete the proof by showing that the collection of shifts $\{Q[k]\}_{k \in \mathbf{Z}}$ generates LMod_A^{I-nil} under small colimits. To prove this, let $\mathcal{C} \subseteq \mathrm{LMod}_A$ be the smallest full subcategory which contains each $Q[n]$ and is closed under small colimits. Then $\mathcal{C}$ is presentable, and the inclusion $F : \mathcal{C} \to \mathrm{LMod}_A^{I-nil}$ preserves small colimits. It follows from Corollary T.5.5.2.9 that the functor F admits a right adjoint G. To prove that $\mathcal{C} = \mathrm{LMod}_A^{I-nil}$, it will suffice to prove that G is conservative. Since G is an exact functor between stable ∞-categories, it will suffice to show that if $M \in \mathrm{LMod}_A^{I-nil}$ satistifes $G(M) \simeq 0$, then $M \simeq 0$.

We will prove that for $0 \leq i \leq n$, the tensor product

$$M(i) = Q_i \otimes_A Q_{i-1} \otimes_A \cdots \otimes_A Q_1 \otimes_A M$$

is zero. The proof proceeds by descending induction on i. We first treat the case $i = n$. Observe that each Q_i is a dualizable object of the monoidal ∞-category LMod_A, whose dual is given by $\ker r_{x_i} \simeq Q_i[-1]$. We conclude that for each $k \in \mathbf{Z}$, the space

$$\mathrm{Map}_{\mathrm{LMod}_A}(A[k], Q_n \otimes_A \cdots \otimes_A Q_1 \otimes_A M) \simeq \mathrm{Map}_{\mathrm{LMod}_A}(Q[k-n], M) \simeq \mathrm{Map}_{\mathcal{C}}(Q[k-n], G(M))$$

is contractible.

We now carry out the inductive step. Assume that $M(i+1) \simeq 0$; we will prove that $M(i) \simeq 0$. There is an evident cofiber sequence

$$M(i) \stackrel{r_{x_i}}{\to} M(i) \to M(i+1).$$

Since $M(i+1) \simeq 0$, we conclude that r_{x_i} induces an equivalence from $M(i)$ to itself. Combining this observation with Remark 4.1.11, we conclude that the map $\beta : M(i) \simeq M(i)[\frac{1}{x_i}]$ is an equivalence. However, since $M(i)$ is (x_i)-torsion (Corollary 4.1.7), Remark 4.1.11 also implies that β is zero, so that $M(i) \simeq 0$ as desired. □

Corollary 4.1.16. *Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. Then the functor $\Gamma_I : \mathrm{LMod}_A \to \mathrm{LMod}_A^{I-nil}$ preserves small colimits.*

Proof. Since Γ_I is evidently an exact functor, it will suffice to show that Γ_I commutes with small filtered colimits. This follows from Propositions 4.1.15 and T.5.5.7.2. Alternatively, it can be deduced immediately from the description given in Proposition 4.1.12. ☐

Remark 4.1.17. Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be an arbitrary ideal. It is easy to see that the ∞-category LMod_A^{I-nil} is accessible. Proposition 4.1.6 implies that LMod_A^{I-nil} is presentable and that the inclusion $\mathrm{LMod}_A^{I-nil} \subseteq \mathrm{LMod}_A$ preserves small colimits. It follows from Corollary T.5.5.2.9 that this inclusion admits a right adjoint $\Gamma_I : \mathrm{LMod}_A \to \mathrm{LMod}_A^{I-nil}$. However, the functor Γ_I is difficult to describe in the case where I is not finitely generated.

Proposition 4.1.18. *Let A be a connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be an ideal. Then:*

(1) *The functor $\Gamma_I : \mathrm{LMod}_A \to \mathrm{LMod}_A^{I-nil}$ is left t-exact.*

(2) *Let $M \in (\mathrm{LMod}_A)_{\leq 0}$. Then the canonical map $\Gamma_I M \to M$ induces an injection*

$$\theta : \pi_0 \Gamma_I M \to \pi_0 M,$$

whose image is the collection of I-nilpotent elements of $\pi_0 M$.

(3) *Let L_I denote the cofiber of the natural transformation $\Gamma_I \to \mathrm{id}$¿ Then L_I is a left t-exact functor from LMod_A to itself.*

Proof. Assertion (1) follows from the observation that the inclusion functor $\mathrm{LMod}_A^{I-nil} \hookrightarrow \mathrm{LMod}_A$ is right t-exact, and assertion (3) follows immediately from (2). We will prove (2). Let K denote the kernel of the map θ. Since $\pi_0 \Gamma_I M$ is I-nilpotent, the module K is I-nilpotent. It follows that the canonical map

$$\mathrm{Ext}^0_A(K, \Gamma_I M) \to \mathrm{Ext}^0_A(K, M)$$

is bijective. Since the composite map $K \to \Gamma_I M \to M$ vanishes, we conclude that the map $K \to \Gamma_I M$ vanishes, so that $K \simeq 0$. This proves that θ is injective. The image of θ is a quotient of $\pi_0 \Gamma_I M$, and therefore consists of I-nilpotent elements. Conversely, suppose $x \in \pi_0 M$ is I-nilpotent. Then the submodule $Ax \subseteq \pi_0 \Gamma_I M$ is I-nilpotent, so that the canonical map $\mathrm{Ext}^0_A(Ax, \Gamma_I M) \to \mathrm{Ext}^0_A(Ax, M)$ is bijective. It follows that the map $Ax \to M$ factors through $\Gamma_I M$, so that $x \in \pi_0 M$ belongs to the image of θ. ☐

Remark 4.1.19. In the situation of Proposition 4.1.18, suppose that $I \subseteq \pi_0 A$ is finitely generated. Then we can identify the image of θ with the submodule of $\pi_0 M$ spanned by those elements which are annihilated by I^n for $n \gg 0$.

Remark 4.1.20. Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. Proposition A.1.4.5.11 implies that there exists an accessible t-structure (with trivial heart) on LMod_A with $(\mathrm{LMod}_A)_{\geq 0} = \mathrm{LMod}_A^{I-nil}$ and $(\mathrm{LMod}_A)_{\leq 0} = \mathrm{LMod}_A^{I-loc}$. In particular, the inclusion $\mathrm{LMod}_A^{I-loc} \hookrightarrow \mathrm{LMod}_A$ admits a left adjoint, which we will denote by $L_I : \mathrm{LMod}_A \to \mathrm{LMod}_A^{I-loc}$. We have a fiber sequence of functors

$$\Gamma_I \to \mathrm{id} \to L_I$$

from LMod_A to itself. It follows from Corollary 4.1.16 that the composite functor

$$\mathrm{LMod}_A \xrightarrow{L_I} \mathrm{LMod}_A^{I-loc} \hookrightarrow \mathrm{LMod}_A$$

preserves small colimits. In particular, LMod_A^{I-loc} is closed under small colimits in LMod_A.

Remark 4.1.21. Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. Since the inclusion functor $\mathrm{LMod}_A^{I-loc} \to \mathrm{LMod}_A$ preserves small colimits, Proposition T.5.5.7.2 implies that the localization functor L_I carries compact objects of LMod_A to compact objects of LMod_A^{I-loc}. It follows that the ∞-category LMod_A^{I-loc} is compactly generated.

Remark 4.1.22. Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. It follows from Corollary 4.1.7 that the localization functor L_I is compatible with the monoidal structure on LMod_A, in the sense of Definition A.2.2.1.6. It follows that LMod_A^{I-loc} inherits the structure of a monoidal ∞-category, and that the localization $L_I : \mathrm{LMod}_A \to \mathrm{LMod}_A^{I-loc}$ has the structure of a monoidal functor (Proposition A.2.2.1.9). The same reasoning shows that if A is an $\mathbb{E}_n$-ring for $2 \leq n \leq \infty$, then L_I has the structure of an $\mathbb{E}_{n-1}$-monoidal functor.

Remark 4.1.23. Let A be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and let M be an I-local left A-module. Then for any left A-module N, the tensor products $M \otimes_A N$ and $N \otimes_A M$ are I-local. To prove this, we can use the fact that the full subcategory $\mathrm{LMod}_A^{I-loc} \subseteq \mathrm{LMod}_A$ is closed under small colimits to reduce to the case where $N \simeq A[n]$ for some integer n, in which case the result is obvious.

Remark 4.1.24. Let A be a connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. Then the ∞-category LMod_A^{I-loc} inherits a t-structure, where $(\mathrm{LMod}_A^{I-loc})_{\leq 0} = \mathrm{LMod}_A^{I-loc} \cap (\mathrm{LMod}_A)_{\leq 0}$. To prove this, we let $\mathcal{C}$ be the smallest full subcategory of LMod_A^{I-loc} which is closed under colimits and extensions and contains $L_I(A)$. It follows from Proposition A.1.4.5.11 that there exists an accessible t-structure on LMod_A^{I-loc} with $(\mathrm{LMod}_A^{I-loc})_{\geq 0} = \mathcal{C}$, and it follows immediately that $(\mathrm{LMod}_A^{I-loc})_{\leq 0} = \mathrm{LMod}_A^{I-loc} \cap (\mathrm{LMod}_A)_{\leq 0}$.

By construction, the inclusion functor $\mathrm{LMod}_A^{I-loc} \hookrightarrow \mathrm{LMod}_A$ is left t-exact and its left adjoint L_I is right t-exact. Note that for $M \in (\mathrm{LMod}_A)_{\leq 0}$, we have $\Gamma_I M \in (\mathrm{LMod}_A)_{\leq 0}$, and the canonical map $\pi_0 \Gamma_I M \to \pi_0 M$ is injective. Using the fiber sequence

$$\Gamma_I \to \mathrm{id} \to L_I,$$

we conclude that L_I is also left t-exact.

Proposition 4.1.25. *Let A be an $\mathbb{E}_2$-ring and let $I, J \subseteq \pi_0 A$ be finitely generated ideals. Then $\mathrm{LMod}_A^{I+J-loc}$ is generated under extensions by the full subcategories*

$$\mathrm{LMod}_A^{I-loc}, \mathrm{LMod}_A^{J-loc} \subseteq \mathrm{LMod}_A^{I+J-loc}.$$

Proof. It is clear from the definitions that $\mathrm{LMod}_A^{I-loc}, \mathrm{LMod}_A^{J-loc} \subseteq \mathrm{LMod}_A^{I+J-loc}$. Let $\mathcal{C}$ be the smallest full subcategory of LMod_A which contains LMod_A^{I-loc} and LMod_A^{J-loc} and is closed under extensions; we wish to show that the inclusion $\mathcal{C} \subseteq \mathrm{LMod}_A^{I+J-loc}$ is an equivalence. Let $M \in \mathrm{LMod}_A$ be $(I+J)$-local; we wish to show that $M \in \mathcal{C}$. Consider the fiber sequence

$$\Gamma_I M \to M \to L_I M.$$

Since $L_I M \in \mathcal{C}$, we may replace M by $\Gamma_I M$ and thereby assume that M is I-nilpotent. Similarly, we can replace M by $\Gamma_J M$ and thereby assume that M is J-nilpotent (Proposition 4.1.12 and Corollary 4.1.7 show that this replacement does not injure our assumption that M is I-nilpotent). Then M is $(I+J)$-nilpotent (Remark 4.1.5). Since M is $(I+J)$-local, we conclude that $M \simeq 0$ and therefore $M \in \mathcal{C}$ as desired. □

4.2 Completion of Modules

Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be an ideal. In §4.1, we saw that the ∞-category LMod_A admits a "semi-orthogonal decomposition" into subcategories LMod_A^{I-loc} and LMod_A^{I-nil}, where LMod_A^{I-nil} denotes the full subcategory spanned by the I-nilpotent objects and LMod_A^{I-loc} the full subcategory spanned by the I-local objects. In particular, we can characterize LMod_A^{I-nil} as the *right orthogonal* to the subcategory LMod_A^{I-loc}: that is, the full subcategory spanned by those objects $M \in \mathrm{LMod}_A$ such that $\mathrm{Ext}_A^n(M, N) \simeq 0$ for every $N \in \mathrm{LMod}_A^{I-loc}$ and every integer n. If I is finitely generated, then the subcategory LMod_A^{I-loc} is closed under small colimits. It is therefore also sensible to consider the *left orthogonal* to the subcategory LMod_A^{I-loc}.

Definition 4.2.1. Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. We will say that a left A-module M is *I-complete* if, for every I-local object $N \in \mathrm{LMod}_A$, the groups $\mathrm{Ext}^n_A(N, M)$ vanish for each $n \in \mathbf{Z}$. We let LMod_A^{I-comp} denote the full subcategory of LMod_A spanned by the I-complete objects.

Lemma 4.2.2. *Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. Then the inclusion functor $\mathrm{LMod}_A^{I-comp} \hookrightarrow \mathrm{LMod}_A$ admits an accessible left adjoint. In particular, LMod_A^{I-comp} is a presentable ∞-category.*

Proof. Proposition A.1.4.5.11 implies that LMod_A admits an accessible t-structure with $(\mathrm{LMod}_A)_{\geq 0} = \mathrm{LMod}_A^{I-loc}$. It follows immediately that $(\mathrm{LMod}_A)_{\leq 0} = \mathrm{LMod}_A^{I-comp}$, so that the associated truncation functor is left adjoint to the inclusion $\mathrm{LMod}_A^{I-comp} \hookrightarrow \mathrm{LMod}_A$. □

Notation 4.2.3. Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. We will indicate the left adjoint to the inclusion $\mathrm{LMod}_A^{I-comp} \hookrightarrow \mathrm{LMod}_A$ by $M \mapsto M^\wedge_I$. We will refer to $M^\wedge_I$ as the *I-completion of M*.

Remark 4.2.4. Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. Let $\alpha : M \to M'$ be a morphism of left A-modules which induces an equivalence of completions $M^\vee_I \to M'^\vee_I$, and let N be an arbitrary left A-module. It follows from Remark 4.1.23 that the induced maps

$$(M \otimes_A N)^\wedge_I \to (M' \otimes_A N)^\wedge_I \qquad (N \otimes_A M)^\wedge_I \to (N \otimes_A M')^\wedge_I$$

are equivalences.

Since the subcategories LMod_A^{I-comp} and LMod_A^{I-nil} can be described as the left and right orthogonals of the ∞-category LMod_A^{I-loc}, they are canonically equivalent to one another:

Proposition 4.2.5. *Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. Then the I-completion functor induces an equivalence of ∞-categories*

$$F : \mathrm{LMod}_A^{I-nil} \to \mathrm{LMod}_A^{I-comp}.$$

Proof. The functor F admits a right adjoint G, given by the restriction $\Gamma_I | \mathrm{LMod}_A^{I-comp}$. We claim that G is a homotopy inverse to F. We first show that the unit map $u : \mathrm{id} \to G \circ F$ is an equivalence. In other words, we claim that if $M \in \mathrm{LMod}_A$ is I-nilpotent, then the canonical map $\alpha : M \to \Gamma_I M^\wedge_I$ is an equivalence. We can factor α as a composition

$$M \xrightarrow{\alpha'} \Gamma_I M \xrightarrow{\alpha''} \Gamma_I M^\wedge_I,$$

where α' is an equivalence since M is assumed to be I-nilpotent. The fiber of α'' is given by $\Gamma_I K$, where K is the fiber of the map $M \to M^\wedge_I$. Since K is I-local, we have $\Gamma_I K \simeq 0$ so that α'' is an equivalence. It follows that α is an equivalence as desired.

We now show that the counit map $v : F \circ G \to \mathrm{id}$ is an equivalence. In other words, we show that if $N \in \mathrm{LMod}_A$ is I-complete, then the canonical map $\beta : (\Gamma_I N)^\wedge_I \to N$ is an equivalence. The map β factors as a composition

$$(\Gamma_I N)^\wedge_I \xrightarrow{\beta'} N^\wedge_I \xrightarrow{\beta''} N$$

where β'' is an equivalence by virtue of our assumption that N is I-complete. It will therefore suffice to show that β' is an equivalence. We now observe that the cofiber of β' is given by $(L_I N)^\wedge_I$, which is zero since $L_I N$ is I-local. □

Remark 4.2.6. Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. The proof of Corollary 4.1.7 shows that if $M \in \mathrm{LMod}_A$ is I-local and $N \in \mathrm{LMod}_A$ is arbitrary, then $M \otimes_A N$ and $N \otimes_A M$ are I-local. It follows that the I-completion functor $M \mapsto M^\vee_I$ is compatible with the monoidal structure on LMod_A, in the sense of Definition A.2.2.1.6. Applying Proposition A.2.2.1.9, we conclude that LMod_A^{I-comp}

inherits the structure of a monoidal ∞-category, and that the completion $M \mapsto M^{\wedge}_{I}$ can be promoted to a monoidal functor. The same reasoning shows that if A is an E_n-ring for $2 \leq n \leq \infty$, then the I-completion has the structure of an $\mathbb{E}_{n-1}$-monoidal functor.

If A is an $\mathbb{E}_\infty$-ring, we will denote LMod_A^{I-comp} simply by Mod_A^{I-comp}. Then Mod_A^{I-comp} inherits the structure of a symmetric monoidal ∞-category with unit object $A^{\wedge}_{I}$. In particular, the completion $A^{\wedge}_{I}$ inherits the structure of an $\mathbb{E}_\infty$-algebra over A, and every I-complete A-module admits an essentially unique structure of $A^{\wedge}_{I}$-module.

Our next goal is to understand the I-completion functor $M \mapsto M^{\wedge}_{I}$ more explicitly. We begin by studying the case where I is a principal ideal.

Proposition 4.2.7. *Let A be an $\mathbb{E}_2$-ring and let $x \in \pi_0 A$. For any A-module M, let $T(M)$ denote the limit of the tower*

$$\cdots \to M \xrightarrow{r_x} M \xrightarrow{r_x} M,$$

where r_x is induced by multiplication by x (see Remark 4.1.11). Then the (x)-completion of M is given by the cofiber of the canonical map $\theta : T(M) \to M$.

Proof. For every integer n, let $\{\pi_n M\}$ denote the tower of abelian groups

$$\cdots \to \pi_n M \xrightarrow{x} \pi_n M \xrightarrow{x} \pi_n M.$$

Multiplication by x induces a map of towers $\{\pi_n M\} \to \{\pi_n M\}$, which is an isomorphism of the underlying pro-objects in the category of abelian groups. It follows that multiplication by x is bijective on $\varprojlim\{\pi_n M\}$ and $\varprojlim^1\{\pi_n M\}$. For every integer n, we have a Milnor exact sequence

$$0 \to \varprojlim{}^1\{\pi_{n+1} M\} \to \pi_n T(M) \to \varprojlim\{\pi_n M\} \to 0.$$

It follows that multiplication by x is bijective on $\pi_n T(M)$, so that $T(M) \simeq T(M)[\frac{1}{x}]$ and therefore $T(M)$ is (x)-local. It follows that $T(M)^{\wedge}_{(x)} \simeq 0$ so that the map $M^{\wedge}_{I} \to \mathrm{cofib}(\theta)^{\wedge}_{I}$ is an equivalence. To complete the proof, it will suffice to show that $\mathrm{cofib}(\theta)$ is (x)-complete. We note that $\mathrm{cofib}(\theta)$ can be identified with the limit of a tower $\{U_n(M)\}$ in LMod_A, where $U_n(M)$ is the cofiber of the map $r_{x^n} : M \to M$. It will therefore suffice to show that each $U_n(M)$ is (x)-complete. For this, we note that

$$\mathrm{Map}_{\mathrm{LMod}_A}(N, U_n(M)) \simeq \mathrm{Map}_{\mathrm{LMod}_A}(U_n(N)[1], M) \simeq *$$

if N is (x)-local. □

Corollary 4.2.8. *Let A be an $\mathbb{E}_2$-ring, let $M \in \mathrm{LMod}_A$, and let $x \in \pi_0 A$. The following conditions are equivalent:*

(1) *The module M is (x)-complete.*

(2) *The limit of the tower*

$$\cdots \to M \xrightarrow{r_x} M \xrightarrow{r_x} M$$

vanishes.

Corollary 4.2.9. *Let A be an $\mathbb{E}_2$-ring, let $x \in \pi_0 A$, and let $M \in \mathrm{LMod}_A$.*

(1) *If $\pi_k M \simeq 0$ for $k < 0$, then $\pi_k M^{\wedge}_{(x)} \simeq 0$ for $k < 0$.*

(2) *If $\pi_k M \simeq 0$ for $k > 0$, then $\pi_k M^{\wedge}_{(x)} \simeq 0$ for $k > 1$.*

Proof. Let $T(M)$ be as in the statement of Proposition 4.2.7, so that we have an exact sequence

$$\pi_k M \to \pi_k M^{\wedge}_{(x)} \to \pi_{k-1} T(M).$$

In case (1), the desired result follows from the observation that $\pi_{k-1} T(M) \simeq 0$ for $k < 0$. In case (2), we observe instead that $\pi_{k-1} T(M) \simeq 0$ for $k > 1$. □

Corollary 4.2.10. *Let A be an $\mathbb{E}_2$-ring and let $x \in \pi_0 A$. For any finitely generated ideal $I \subseteq \pi_0 A$, the (x)-completion functor $M \to M^{\vee}_{(x)}$ carries I-complete objects to I-complete objects.*

Proof. Since the collection of I-complete objects of LMod_A is closed under limits, it is clear that if $M \in \mathrm{LMod}_A$ is I-complete then $T(M)$ is I-complete. The desired result now follows from the description of (x)-completion provided by Proposition 4.2.7. □

The following observation allows us to reduce the general study of completions to the case of completions along principal ideals:

Proposition 4.2.11. *Let A be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and let $I' \subseteq \pi_0 A$ be the ideal generated by I together with an element $x \in \pi_0 A$. For any A-module M, the composite map*

$$M \xrightarrow{\alpha} M^{\wedge}_I \xrightarrow{\beta} (M^{\wedge}_I)^{\wedge}_{(x)}$$

exhibits $(M^{\wedge}_I)^{\wedge}_{(x)}$ as an I'-completion of M.

Proof. It is clear that $(M^{\wedge}_I)^{\wedge}_{(x)}$ is (x)-complete, and Corollary 4.2.10 shows that it is also I-complete. Using Proposition 4.1.25 we deduce that $(M^{\wedge}_I)^{\wedge}_{(x)}$ is I'-complete. It will therefore suffice to show that the fiber of $\beta \circ \alpha$ is I'-local. We argue that the fibers of α and β are both I'-local. This is clear, since the fiber of α is I-local and the fiber of β is (x)-local. □

Corollary 4.2.12. *Let A be an $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and let $M \in \mathrm{LMod}_A$. The following conditions are equivalent:*

(1) *The module M is I-complete.*

(2) *For each $x \in I$, the module M is (x)-complete.*

(3) *There exists a set of generators $x_1, \ldots, x_n$ for the ideal I such that M is (x_i)-complete for $1 \leq i \leq n$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious. We prove that (3) ⇒ (1). For $0 \leq i \leq n$, let $I(i)$ be the ideal generated by $x_1, \ldots, x_i$. We prove that M is $I(i)$-complete by induction on i, the case $i = 0$ being trivial. Assume that $i < n$ and that M is $I(i)$-complete. Then the map $\alpha : M \to M^{\wedge}_{I(i)}$ is an equivalence. Since M is x_{i+1}-complete, the map $\beta : M \to M^{\wedge}_{(x_{i+1})}$ is also an equivalence. Using Proposition 4.2.11, we deduce that the map $M \to M^{\wedge}_{I(i+1)}$ is an equivalence, so that M is $I(i+1)$-complete. □

We now study the behavior of completions with respect to truncation. Our main result can be stated as follows:

Theorem 4.2.13. *Let A be a connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. A module $M \in \mathrm{LMod}_A$ is I-complete if and only if each $\pi_k M$ is I-complete, when regarded as a discrete A-module.*

The proof of Theorem 4.2.13 will require some preliminaries.

Lemma 4.2.14. *Let A be a connective $\mathbb{E}_2$-ring, let $x \in \pi_0 A$, and let $M \in \mathrm{LMod}_A$. Assume that $M \in (\mathrm{LMod}_A)_{\leq 0}$ and that M is (x)-complete. Then $\tau_{\geq 0} M$ is also (x)-complete.*

Proof. Let $M' = \tau_{\geq 0} M$ and $M'' = \tau_{\leq -1} M$, so we have a diagram of cofiber sequences

$$\begin{array}{ccccc} M' & \longrightarrow & M & \longrightarrow & M'' \\ \downarrow & & \downarrow & & \downarrow \\ M'^{\wedge}_{(x)} & \longrightarrow & M^{\wedge}_{(x)} & \longrightarrow & M''^{\wedge}_{(x)}. \end{array}$$

Corollary 4.2.9 implies that the homotopy groups $\pi_k M''^{\wedge}_{(x)}$ vanish for $k > 0$ and that the homotopy groups $\pi_k M'^{\vee}_{(x)}$ vanish for $k \notin \{0, 1\}$. It follows that $\pi_1 M'^{\wedge}_{(x)} \simeq \pi_1 M^{\wedge}_{(x)}$ and that we have a short exact sequence

$$0 \to \pi_0 M'^{\wedge}_{(x)} \xrightarrow{\alpha} \pi_0 M^{\wedge}_{(x)} \to \pi_0 M''^{\wedge}_{(x)} \to 0.$$

Note that the composite map $\pi_0 M' \to \pi_0 M'^{\wedge}_{(x)} \to \pi_0 M^{\wedge}_{(x)}$ coincides with the composition $\pi_0 M' \to \pi_0 M \to \pi_0 M^{\wedge}_{(x)}$, which is an isomorphism (since M is assumed to be (x)-complete). It follows that α is surjective and therefore an isomorphism. Similarly, since M is (x)-complete and belongs to $(\mathrm{LMod}_A)_{\leq 0}$, we deduce that $\pi_1 M^{\wedge}_{(x)} \simeq 0$, so that $\pi_1 M'^{\wedge}_{(x)} \simeq 0$. It follows that the map $M' \to M'^{\wedge}_{(x)}$ induces an isomorphism on homotopy groups and is therefore an equivalence. □

Lemma 4.2.15. *Let A be a connective $\mathbb{E}_2$-ring, let $x \in \pi_0 A$, and let $M \in \mathrm{LMod}_A$. If M is (x)-complete, then $\tau_{\leq 0} M$ is (x)-complete.*

Proof. Let $M'' = \tau_{\leq 0} M$, so we have a fiber sequence

$$M' \to M \to M''$$

with $M' \in (\mathrm{LMod}_A)_{\geq 1}$. Corollary 4.2.9 shows that $M'^{\wedge}_{(x)} \in (\mathrm{LMod}_A)_{\geq 1}$, so that the map $M^{\wedge}_{(x)} \to M''^{\wedge}_{(x)}$ induces an isomorphism $\pi_k M \simeq \pi_k M^{\wedge}_{(x)} \to \pi_k M''^{\wedge}_{(x)}$ for $k \leq 0$. We may therefore replace M by $M''^{\wedge}_{(x)}$, so that $M \in (\mathrm{LMod}_A)_{\leq 1}$ by Corollary 4.2.9. We have a fiber sequence

$$\tau_{\geq 1} M \to M \to \tau_{\leq 0} M$$

where M is (x)-complete by assumption and $\tau_{\geq 1} M$ is (x)-complete by Lemma 4.2.14, so that $\tau_{\leq 0} M$ is (x)-complete as desired. □

Proposition 4.2.16. *Let A be a connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. If $M \in \mathrm{LMod}_A$ is I-complete, then the truncations $\tau_{\leq n} M$ and $\tau_{\geq n} M$ are I-complete for every integer n.*

Proof. In view of the fiber sequence

$$\tau_{\geq n+1} M \to M \to \tau_{\leq n} M,$$

it will suffice to show that $\tau_{\leq n} M$ is I-complete for each $n \in \mathbf{Z}$. Replacing M by $M[-n]$, we may reduce to the case $n = 0$, which follows from Lemma 4.2.15 and Corollary 4.2.12. □

Remark 4.2.17. It follows from Proposition 4.2.16 that the ∞-category LMod_A^{I-comp} inherits a t-structure, with

$$(\mathrm{LMod}_A^{I-comp})_{\leq 0} = (\mathrm{LMod}_A)_{\leq 0} \cap \mathrm{LMod}_A^{I-comp} \qquad (\mathrm{LMod}_A^{I-comp})_{\geq 0} = (\mathrm{LMod}_A)_{\geq 0} \cap \mathrm{LMod}_A^{I-comp}.$$

In particular, the inclusion functor $\mathrm{LMod}_A^{I-comp} \hookrightarrow \mathrm{LMod}_A$ is t-exact, so its left adjoint $M \mapsto M^{\wedge}_I$ is right t-exact (a fact we already observed in Corollary 4.2.9 in the special case where the ideal I is principal).

Proposition 4.2.18. *Let A be a connective $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. Then:*

(1) *The t-structure on LMod_A^{I-comp} (described in Remark 4.2.17) is both right and left complete.*

(2) *The completion functor $M \mapsto M^{\wedge}_{I}$ is of finite left t-amplitude. In other words, there exists an integer n such that if $M \in (\mathrm{LMod}_A)_{\leq 0}$, then $M^{\wedge}_{I} \in (\mathrm{LMod}_A)_{\leq n}$.*

Proof. We first prove (2). Choose generators $x_1, \ldots, x_n$ for the ideal I. Proposition 4.2.11 implies that the I-completion functor can be obtained by composing the (x_i)-completion functors for $1 \leq i \leq n$. It therefore suffices to treat the case where I is principal, which follows from Corollary 4.2.9.

We now prove (1). First we show that LMod_A^{I-comp} is left complete. SInce LMod_A is left complete, it will suffice to show that if $M \in \mathrm{LMod}_A$ is an object such that $\tau_{\leq k} M$ is I-complete for all $k \in \mathbf{Z}$, then M is I-complete. This is clear, since the collection of I-complete objects is stable under small limits, and $M \simeq \varprojlim \tau_{\leq k} M$.

The proof of right completeness is slightly more difficult. Arguing as above, we are reduced to showing that if $M \in \mathrm{LMod}_A$ is such that $\tau_{\geq k} M$ is I-complete for $k \in \mathbf{Z}$, then M is I-complete. Fix an integer m; we will prove that the completion map $M \to M^{\wedge}_{I}$ induces an isomorphism $\pi_{m'} M \to \pi_{m'} M^{\wedge}_{I}$ for $m' \geq m$. To prove this, we choose n as in (2) and set $k = m - n$. Assertion (2) guarantees that the cofiber of the map $\theta : (\tau_{\geq k} M)^{\wedge}_{I} \to M^{\vee}_{I}$ belongs to $(\mathrm{LMod}_A)_{\leq m-1}$. We have a commutative diagram

$$\begin{array}{ccc} \pi_{m'} \tau_{\geq k} M & \xrightarrow{\alpha} & \pi_{m'} M \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta'} \\ \pi_{m'} (\tau_{\geq k} M)^{\wedge}_{I} & \xrightarrow{\alpha'} & \pi_{m'} M^{\wedge}_{I}. \end{array}$$

Here the maps α and α' are isomorphisms for $m' \geq m$, and the map β is an isomorphism for all m' (since $\tau_{\geq k} M$ is I-complete), so that β' is an isomorphism as desired. □

Remark 4.2.19. In the situation of Proposition 4.2.18, assertion (2) can be made more specific: the proof shows that we can take n to be the minimal number of generators for the ideal I (or any other ideal having the same radical as I).

Proof of Theorem 4.2.13. Let M be a left A-module. If M is I-complete, then Proposition 4.2.16 implies that each homotopy group $\pi_k M$ is I-complete. Conversely, suppose that each $\pi_k M$ is I-complete. We will prove that the completion map $\alpha : M \to M^{\wedge}_{I}$ induces an isomorphism $\pi_m M \to \pi_m M^{\wedge}_{I}$ for each $m \in \mathbf{Z}$. To prove this, choose n as in Proposition 4.2.18, and consider the map of fiber sequences

$$\begin{array}{ccccc} \tau_{\geq m-n} M & \xrightarrow{\alpha} & M & \longrightarrow & \tau_{\leq m-n-1} M \\ \downarrow & & \downarrow & & \downarrow \\ (\tau_{\geq m-n} M)^{\wedge}_{I} & \xrightarrow{\alpha^{\vee}} & M^{\wedge}_{I} & \longrightarrow & (\tau_{\leq m-n-1} M)^{\wedge}_{I}. \end{array}$$

The associated long exact sequence shows that α and $\alpha^{\vee}$ induce isomorphisms on π_m. It will therefore suffice to show that $M' = \tau_{\geq m-n} M$ is I-complete. We prove by induction on m' that $\tau_{\leq m'} M'$ is I-complete; it will then follow that $M' \simeq \varprojlim \tau_{\leq m'} M'$ is I-complete. For $m' < m$ we have $\tau_{\leq m'} M' \simeq 0$ and the result is obvious. The inductive step follows from the existence of a fiber sequence

$$(\pi_{m'} M')[m'] \to \tau_{\leq m'} M' \to \tau_{\leq m'-1} M',$$

since for $m' \geq m$ the module $\pi_{m'} M' \simeq \pi_{m'} M$ is I-complete by assumption. □

4.3 Completion in the Noetherian Case

Let A be an $\mathbb{E}_2$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. In §4.1 and §4.2, we discussed the functors

$$M \mapsto M^{\wedge}_{I} \qquad M \mapsto \Gamma_I(M),$$

given by localization and colocalization with respect to the subcategories

$$\mathrm{LMod}_A^{I-comp} \subseteq \mathrm{LMod}_A \supseteq \mathrm{LMod}_A^{I-nil}$$

of I-complete and I-nilpotent modules, respectively. In this section, we will study the behavior of these functors in the special case where A is Noetherian. In particular, we will show that when A is a Noetherian commutative ring (regarded as a discrete $\mathbb{E}_2$-ring), then these functors reduce to familiar constructions in commutative algebra (see Theorem 4.3.1 and Corollary 4.3.10).

We begin by analyzing the functor $M \mapsto \Gamma_I(M)$.

Theorem 4.3.1. *Let A be a Noetherian commutative ring, let $\mathcal{A}$ denote the abelian category of discrete A-modules, so that we have a canonical equivalence of ∞-categories*

$$\bigcup_k (\mathrm{Mod}_A)_{\leq k} \simeq \mathcal{D}^+(\mathcal{A}).$$

Let $I \subseteq A$ be an ideal, let $F : \mathcal{A} \to \mathcal{A}$ denote the functor given by $F(M) = \{x \in M : (\exists n \geq 0) I^n x = 0\}$. Then F is a left exact functor, and the diagram

$$\begin{array}{ccc} \mathcal{D}^+(\mathcal{A}) & \xrightarrow{RF} & \mathcal{D}^+(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathrm{Mod}_A & \xrightarrow{\Gamma_I} & \mathrm{Mod}_A \end{array}$$

commutes up to canonical homotopy (here RF denotes the right derived functor of F; see Example A.1.3.2.3).

The proof of Theorem 4.3.1 will require a bit of commutative algebra.

Lemma 4.3.2. *Let A be a Noetherian commutative ring containing an ideal I. Let M be an injective object in the abelian category $\mathcal{A}$ of discrete A-modules, and let $M' \subseteq M$ be the submodule consisting of those elements which are annihilated by I^n for $n \gg 0$. Then M' is also an injective object of $\mathcal{A}$.*

Proof. Suppose we are given an inclusion $P \subseteq Q$ of discrete A-modules; we wish to show that every map $f : P \to M'$ can be extended to a map $Q \to M'$. Let S be the partially ordered set of all pairs (P', f'), where P' is a submodule of Q containing P and $f' : P' \to M'$ is a morphism extending f. Then S satisfies the hypotheses of Zorn's lemma, and therefore contains a maximal element. Replacing P by this maximal element, we may assume that $f : P \to M'$ does not admit an extension to any larger submodule of Q. If $P = Q$, there is nothing to prove; otherwise, we can choose an element $y \in Q - P$. Let $J = \{a \in A : ay \in P\}$ and let $g : J \to M'$ be the map given by $g_0(a) = f(ay)$. Since J is finitely generated, there exists $n \gg 0$ such that I^n annihilates $g(J)$. According to the Artin-Rees lemma (see [1]), there exists n' such that $I^{n'} \cap J \subseteq I^n J$, so that g vanishes on $I^{n'} \cap J$. Then g induces a map $J/(I^{n'} \cap J) \to M$. Since M is injective, we can extend this to a map $A/I^{n'} \to M$. This map automatically factors through M', and therefore determines a map $g' : A \to M'$ extending f_0. By construction, we have a pushout diagram of discrete A-modules

$$\begin{array}{ccc} J & \longrightarrow & A \\ \downarrow & & \downarrow \\ P & \longrightarrow & P + Ay. \end{array}$$

Thus f and g' can be amalgamated to a map $f' : (P + Ay) \to M'$, contrary to our assumption. □

Proof of Theorem 4.3.1. We first show that F is left exact. Suppose we are given a short exact sequence of discrete A-modules

$$0 \to M' \xrightarrow{\iota} M \xrightarrow{F(\eta)} M''.$$

We wish to show that the induced sequence

$$0 \to F(M') \overset{F(\iota)}{\to} F(M) \to F(\eta) \to F(M'')$$

is exact. The injectivity of $F(\iota)$ is clear. If $x \in F(M)$ belongs to the kernel of $F(\eta)$, then x is an I-nilpotent element of M belonging to $\ker(\eta)$. We can then write $x = \iota(y)$ for some $y \in M'$. Since ι is injective, the I-nilpotence of x implies the I-nilpotence of y, so that $y \in F(M')$ and $x = F(\iota)(y)$.

We next observe that the functor F is given by $F(M) = \pi_0(\Gamma_I M)$ (see Proposition 4.1.18 and Remark 4.1.19; note that since Γ_I is left t-exact, this gives another proof of the left exactness of F). According to Theorem A.1.3.2.2, the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{D}^+(\mathcal{A}) & \xrightarrow{RF} & \mathcal{D}^+(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathrm{Mod}_A & \xrightarrow{\Gamma_I} & \mathrm{Mod}_A \end{array}$$

is equivalent to the assertion that the functor Γ_I carries injective objects of $\mathcal{A}$ to discrete A-modules. Since A is Noetherian, the ideal I is generated by finitely many elements $x_1, \ldots, x_n \in A$. The proof of Proposition 4.1.12 shows that $\Gamma_I = \Gamma_{(x_1)} \circ \cdots \circ \Gamma_{(x_n)}$. We will show that each $\Gamma_{(x_i)}$ carries injective objects of $\mathcal{A}$ to discrete A-modules. Lemma 4.3.2 shows that $\Gamma_{(x_i)}$ carries injective objects of $\mathcal{A}$ to injective objects of $\mathcal{A}$, so that Γ_I has the same property.

Fix $1 \leq i \leq n$, let $x = x_i$, and let M be an injective object of $\mathcal{A}$; we wish to show that $\Gamma_{(x)} M$ is discrete. Using Example 4.1.14, we see that $\Gamma_{(x)} M$ can be identified with the fiber of the map $\lambda : M \to M[\frac{1}{x}]$. It will therefore suffice to show that λ is an epimorphism in $\mathcal{A}$. Fix $y \in M$ and $k \geq 0$; we wish to show that $\frac{\lambda(y)}{x^k} \in M[\frac{1}{x}]$ belongs to the image of λ. For each $m \geq 0$, let $J(m) = \{a \in A : a x^m = 0\}$. We have an ascending chain

$$0 = J(0) \subseteq J(1) \subseteq J(2) \subseteq \cdots$$

of ideals in A. Since A is Noetherian, this chain is eventually constant. We can therefore choose $J(m) = J(m+k)$. Define $f : x^{m+k} A \to M$ by the formula $f(x^{m+k} a) = x^m a y \in M$ (this does not depend on the choice of a, since $J(m) = J(m+k)$). Since M is injective, we can extend f to a map $f' : A \to M$. Let $y' = f'(1)$. Then $x^{k+m} y' = f'(x^{k+m}) = f(x^{k+m}) = x^m y$. It follows that $x^{k+m} \lambda(y') = x^m \lambda(y)$, so that $\frac{\lambda(y)}{x^k} \simeq \lambda(y')$ belongs to the image of λ as desired. □

Corollary 4.3.3. *Let A be a Noetherian commutative ring, let $\mathcal{A}$ denote the abelian category of discrete A-modules, and let $I \subseteq A$ be an ideal. Let $L_I : \mathrm{LMod}_A \to \mathrm{LMod}_A^{I-nil}$ denote a left adjoint to the inclusion, and let $G : \mathcal{A} \to \mathcal{A}$ be the functor given by $G(M) = \pi_0 L_I M$. Then G is a left exact functor from $\mathcal{A}$ to itself. Moreover, the diagram*

$$\begin{array}{ccc} \mathcal{D}^+(\mathcal{A}) & \xrightarrow{RG} & \mathcal{D}^+(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathrm{Mod}_A & \xrightarrow{L_I} & \mathrm{Mod}_A \end{array}$$

commutes up to canonical homotopy, where RG denotes the right derived functor of G.

Proof. Since L_I is left t-exact (Proposition 4.1.18), the functor G is left exact. If $M \in \mathcal{A}$ is injective, the proof of Theorem 4.3.1 shows that $\Gamma_I M$ is a discrete A-module. Using the fiber sequence

$$\Gamma_I M \to M \to L_I M$$

(and the injectivity of the map $\pi_0 \Gamma_I M \to \pi_0 M$), we deduce that $L_I M$ is discrete. This proves that the functor L_I carries injective objects of $\mathcal{A}$ to discrete objects of Mod_A, so that the commutativity of the diagram follows from Theorem A.1.3.2.2. □

Remark 4.3.4. In the situation of Corollary 4.3.3, the functor $G : \mathcal{A} \to \mathcal{A}$ carries a discrete A-module M to the set of all global sections of the quasi-coherent sheaf associated to M over the open subscheme $U \subseteq \operatorname{Spec} A$ determined by the ideal I. Using Theorem 4.3.1 and Corollary 4.3.3, we see that the fiber sequence of functors

$$\Gamma_I \to \mathrm{id} \to L_I$$

reproduces Grothendieck's theory of *local cohomology.*

We now turn to the study of completions. We begin by recalling the completion construction in classical commutative algebra.

Definition 4.3.5. Let A be a commutative ring containing an ideal I and let M be a discrete A-module. The *I-adic completion* of M to be the discrete A-module given by the inverse limit $\varprojlim M/I^n M$. We will denote the I-adic completion of M by $Cpl(M; I)$.

The I-adic completion of Definition 4.3.5 and the I-completion of Notation 4.2.3 are closely related:

Proposition 4.3.6. *Let A be a connective $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and let M be a discrete A-module. Assume that M is Noetherian: that is, that every submodule of M is finitely generated. For every set S, there is a canonical equivalence $M'^{\wedge}_I \simeq Cpl(M'; I)$, where $M' = \bigoplus_{\beta \in S} M$. In particular, we have $M^{\wedge}_I \simeq Cpl(M; I)$*

Proof. We work by induction on the minimal number of generators of I. If $I = (0)$ there is nothing to prove. Otherwise, we may assume that $I = J + (x)$ for some $x \in \pi_0 A$ and that $M'^{\wedge}_J = Cpl(M'; J)$. Using Proposition 4.2.11, we deduce that $M'^{\wedge}_I \simeq (Cpl(M'; J))^{\wedge}_{(x)}$. For $m, n \geq 0$, we let $X_{m,n}$ denote the cofiber of the map $M'/J^m M' \to M'/J^m M'$ given by multiplication by x^n. Then $\pi_i X_{m,n}$ vanishes for $i \notin \{0, 1\}$, and Proposition 4.2.7 implies that $M'^{\wedge}_I \simeq \varprojlim\{X_{m,n}\}$. It follows that there is a canonical isomorphism $\pi_1 M'^{\wedge}_I \simeq \varprojlim\{\pi_1 X_{m,n}\}$ and a short exact sequence

$$0 \to \varprojlim{}^1\{\pi_1 X_{m,n}\} \to \pi_0 M'^{\wedge}_I \to \varprojlim\{\pi_0 X_{m,n}\} \to 0.$$

To complete the proof, it will suffice to show that $\varprojlim\{\pi_1 X_{m,n}\} \simeq \varprojlim{}^1\{\pi_1 X_{m,n}\} \simeq 0$. In fact, we claim that $\{\pi_1 X_{m,n}\}_{m,n \geq 0}$ is trivial as a pro-object in the category of abelian groups. To prove this, it suffices to show that for each $m, n \geq 0$, there exists $n' \geq n$ such that the induced map $\pi_1 X_{m,n'} \to \pi_1 X_{m,n}$ is zero. For each $k \geq 0$, let $Y(k) = \{y \in M/I^m M : x^k y = 0\}$. Since M is Noetherian, the quotient $M/I^m M$ is also Noetherian, so the ascending chain of submodules

$$0 = Y(0) \subseteq Y(1) \subseteq Y(2) \subseteq \cdots$$

must eventually stabilizer. It follows that there exists $k \geq 0$ such that if $y \in M/I^m M$ is annihilated by x^{k+1}, then it is annihilated by x^k. It follows immediately that the map $\pi_1 X_{m,n+k} \to \pi_1 X_{m,n}$ is zero as desired. □

Corollary 4.3.7. *Let A be a connective $\mathbb{E}_2$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and let $M \in$ LMod_A. Assume that $\pi_n M$ is Noetherian when regarded as a discrete $\pi_0 A$-module, for every integer n. Then, for each integer n, there is a canonical isomorphism $\pi_n M^{\wedge}_I \simeq Cpl(\pi_n M; I)$.*

Proof. In view of Proposition 4.3.6, it will suffice to show that we have an equivalence of R-modules

$$\pi_n M^{\wedge}_I \simeq (\pi_n M)^{\wedge}_I.$$

We have a fiber sequence

$$\tau_{\geq n+1} M \to M \to \tau_{\leq n} M,$$

hence a fiber sequence

$$(\tau_{\geq n+1} M)^{\wedge}_I \to M^{\wedge}_I \to (\tau_{\leq n} M)^{\wedge}_I.$$

Since the functor of I-completion is right t-exact, the associated long exact sequence of homotopy groups gives an isomorphism $\pi_n M_I^\vee \simeq \pi_n(\tau_{\leq n} M)_I^\vee$. Replacing M by $\tau_{\leq n} M$, we may reduce to the case where M is n-truncated. Let $N = \tau_{\leq n-1} M$. We have a fiber sequence

$$(\pi_n M)[n] \to M \to N,$$

hence a fiber sequence

$$(\pi_n M)_I^\wedge[n] \to M_I^\wedge \to N_I^\wedge.$$

Using the associated long exact sequence, we are reduced to proving that $N_I^\wedge$ is $(n-1)$-truncated. We first prove by descending induction on k that $(\tau_{\geq k} N)_I^\wedge$ is $(n-1)$-truncated. For $k \geq n$, there is nothing to prove. Assume therefore that $k < n$.

$$(\tau_{\geq k+1} N)_I^\wedge \to (\tau_{\leq k} N)_I^\wedge \to (\pi_k N)_I^\wedge[k].$$

The inductive hypothesis implies that $(\tau_{\geq k+1} N)_I^\wedge$ is $(n-1)$-truncated, and Proposition 4.3.6 implies that $(\pi_k N)_I^\wedge$ is discrete. It follows that $(\tau_{\geq k} N)_I^\wedge$ is $(n-1)$-truncated. We have a fiber sequence

$$(\tau_{\geq k} N)_I^\wedge \to N_I^\wedge \to (\tau_{\leq k-1} N)_I^\wedge.$$

For $k \ll 0$, Proposition 4.2.18 implies that $(\tau_{\leq k-1} N)_I^\wedge$ is $(n-1)$-truncated, so that $N_I^\wedge$ is $(n-1)$-truncated as desired. □

Let A be an $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ a finitely generated ideal, and let M be an R-module. Using Remark 4.2.6, we see that $R_I^\wedge$ inherits the structure of an $\mathbb{E}_\infty$-algebra over R, and that $M_I^\wedge$ has the structure of an $R_I^\wedge$-module. We therefore obtain a canonical map $R_I^\wedge \otimes_R M \to M_I^\wedge$.

Proposition 4.3.8. *Let R be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let M be an almost perfect R-module. Then the canonical map*

$$R_I^\wedge \otimes_R M \to M_I^\wedge$$

is an equivalence. In particular, if R is I-complete, then M is I-complete.

Proof. Fix an integer n; we will show that the map $\phi_M : R_I^\wedge \otimes_R M \to M_I^\wedge$ is n-connective. Since M is almost perfect, there exists a perfect R-module N and an n-connective map $N \to M$. We have a commutative diagram

$$\begin{array}{ccc} R_I^\wedge \otimes_R N & \xrightarrow{\phi_N} & N_I^\wedge \\ \downarrow & & \downarrow \\ R_I^\wedge \otimes_R M & \xrightarrow{\phi_M} & M_I^\wedge. \end{array}$$

Since the I-completion functor is left t-exact (and therefore $R_I^\wedge$ is connective), the vertical maps in this diagram are n-connective. It will therefore suffice to show that the map ϕ_N is n-connective. Let $\mathcal{C} \subseteq \mathrm{Mod}_R$ be the full subcategory spanned by those objects N for which ϕ_N is an equivalence. Then $\mathcal{C}$ is a stable subcategory which is closed under the formation of retracts. Consequently, to show that $\mathcal{C}$ contains all perfect R-modules, it suffices to show that $\mathcal{C}$ contains R, which is clear. □

Corollary 4.3.9. *Let A be a Noetherian $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 A$ be an ideal. Then the completion $A_I^\wedge$ is flat over A.*

Proof. Let M be a discrete A-module; we wish to show that $A_I^\wedge \otimes_A M$ is discrete. Since the construction $M \mapsto A_I^\wedge \otimes_A M$ commutes with filtered colimits, we can assume that M is finitely presented (when regarded as a module over the commutative ring $\pi_0 A$). In this case, M is almost perfect as an A-module (Proposition A.7.2.5.17), so that $A_I^\wedge \otimes_A M$ can be identified with the I-completion $M_I^\wedge$ (Proposition 4.3.8). The discreteness of $M_I^\wedge$ now follows from Proposition 4.3.6. □

Corollary 4.3.10. *Let A be a Noetherian commutative ring and let $\mathcal{A}$ denote the abelian category of discrete A-modules, so that we have a canonical equivalence of ∞-categories*

$$\bigcup_k (\mathrm{Mod}_A)_{\leq k} \simeq \mathcal{D}^+(\mathcal{A}).$$

(see Proposition A.7.1.1.15). Let $I \subseteq A$ be an ideal, and let $F : \mathcal{A} \to \mathcal{A}$ be the functor given by $M \mapsto \pi_0 M_I^\wedge$. Then:

(1) *The functor F is right exact.*

(2) *If M is a free A-module, there is a canonical isomorphism $F(M) \simeq Cpl(M; I)$.*

(3) *The diagram of ∞-categories*

$$\begin{array}{ccc} \mathcal{D}^-(\mathcal{A}) & \xrightarrow{LF} & \mathcal{D}^-(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathrm{Mod}_A & \xrightarrow{M \mapsto M_I^\wedge} & \mathrm{Mod}_A \end{array}$$

commutes up to canonical homotopy, where LF denotes the left derived functor of F.

Proof. Assertion (1) follows from the right t-exactness of the functor $M \mapsto M_I^\wedge$ (Remark 4.2.17), and assertion (2) follows from Proposition 4.3.6. Assertion (3) follows from (2) and Theorem A.1.3.2.2. □

Remark 4.3.11. Let A be a Noetherian $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 A$ be an ideal, and suppose that $\pi_0 A$ is I-adically complete: that is, that the canonical map $\pi_0 A \to \varprojlim_n (\pi_0 A)/I^n$ is an isomorphism of commutative rings. Proposition 4.3.6 implies that $\pi_0 A$ is I-complete. Using Propositions A.7.2.5.17 and 4.3.8, we deduce that every finitely generated discrete module over $\pi_0 A$ is I-complete. Combining this observation with Theorem 4.2.13 and Proposition A.7.2.5.17, we conclude that every almost perfect A-module is I-complete. In particular, A is I-complete: that is, the canonical map $A \to A_I^\wedge$ is an equivalence.

We conclude this section by reviewing a few standard facts about the completion of Noetherian rings.

Proposition 4.3.12. *Let R be a commutative ring and let $I \subseteq R$ be a finitely generated ideal. Suppose that R is I-adically complete (that is, the map $R \to \varprojlim R/I^n$ is an isomorphism). If R/I is Noetherian, then R is Noetherian.*

Proof. For each $n \geq 0$, set $A_n = I^n/I^{n+1}$, and let A denote the graded ring $\bigoplus A_n$. Choose a finite set of generators $x_1, \ldots, x_k$ for the ideal I, and let $\overline{x}_1, \ldots, \overline{x}_k$ denote their images in $A_1 = I/I^2$. The elements $\overline{x}_i$ generate A as an algebra over $A_0 = R/I$. It follows from the Hilbert basis theorem that A is Noetherian.

Let $J \subseteq R$ be an arbitrary ideal; we wish to show that J is finitely generated. For each $n \geq 0$, set $J_n = (J \cap I^n)/(J \cap I^{n+1})$, which we regard as a submodule of A_n. The direct sum $\bigoplus_{n \geq 0} J_n$ is an ideal in the commutative ring A. Since A is Noetherian, this ideal is finitely generated. Choose a finite set of homogeneous generators $\overline{y}_1, \ldots, \overline{y}_m \in \bigoplus_{n \geq 0} J_n$, where $\overline{y}_i \in J_{d_i}$. For $1 \leq i \leq m$, let y_i denote a lift of $\overline{y}_i$ to $J \cap I^n$. We claim that the elements $y_1, \ldots, y_m \in J$ generate the ideal J.

Let $d = \max\{d_i\}$. We will prove the following:

($*$) For each $z \in J \cap I^n$, we can find coefficients $c_i \in R$ such that $c_i \in I^{n-d}$ if $n > d$, and $z - \sum_{1 \leq i \leq m} c_i y_i$ belongs to I^{n+1}.

To prove ($*$), we let $\overline{z}$ denote the image of z in J_n. Since the elements $\overline{y}_i$ generate $\bigoplus J_n$ as an A-module, we can write $\overline{z} = \sum \overline{c_i y_i}$ for some homogeneous elements $\overline{c}_i \in A$ of degree $n - d_i$. For $1 \leq i \leq m$, choose $c_i \in I^{n-d_i}$ to be any lift of $\overline{c_i}$; then the elements c_i have the desired property.

Now let $z \in J$ be an arbitrary element. We will define a sequence of elements $z_0, z_1, \ldots, \in J$ such that $z - z_q \in I^q$. Set $z_0 = 0$. Assuming that z_q has been defined, we apply $(*)$ to write

$$z - z_q \equiv \sum_{1 \leq i \leq m} c_{i,q} y_i \pmod{I^{q+1}}$$

where $c_{i,q} \in I^{q-d}$ for $q \geq d$. Now set $z_{q+1} = z - z_q - \sum_{1 \leq i \leq m} c_{i,q} y_i$. For each $1 \leq i \leq m$, the sum $\sum_{q \geq 0} c_{i,q}$ converges I-adically to a unique element $c_i \in R$. We now observe that $z = \sum c_i y_i$ belongs to the ideal generated by the elements y_i, as desired. □

Corollary 4.3.13. *Let R be a Noetherian ring, let $I \subseteq R$ be an ideal, and let $Cpl(R; I)$ denote the I-adic completion of R. Then $Cpl(R; I)$ is Noetherian.*

Proof. For each integer n, let J_n denote the ideal of $Cpl(R; I)$ given by $\varprojlim I^n / I^{n+m}$, so that the canonical map $\phi : R \to Cpl(R; I)$ induces isomorphisms $R/I^n \to Cpl(R; I)/J_n$ for each $n \geq 0$. It follows that the canonical map $Cpl(R; I) \to \varprojlim Cpl(R; I)/J_n$ is an isomorphism. We will show that $J_n = I^n Cpl(R; I)$ for each $n \geq 0$. Assuming this, we deduce that J_1 is finitely generated, that $Cpl(R; I)$ is J_1-adically complete, and that $Cpl(R; I)/J_1 \simeq R/I$ is Noetherian. It then follows from Proposition 4.3.12 that $Cpl(R; I)$ is Noetherian.

Choose a finite set of generators $x_1, \ldots, x_k$ for the ideal I^n and an arbitrary $z \in J_n$, given by a compatible sequence of elements $\{\overline{z}_m \in I^n / I^{n+m}\}_{m \geq 0}$. Lift each $\overline{z_m}$ to an element $z_m \in I^n$. Then $z_{m+1} - z_m \in I^{n+m}$, so we can write

$$z_{m+1} = z_m + \sum_{1 \leq i \leq k} c_{m,i} x_i$$

for some $c_{m,i} \in I^m$. For $1 \leq i \leq k$, the residue classes of the partial sums $\{\sum_{j \leq m} c_{j,i}\}_{m \geq 0}$ determine an element $c_i \in Cpl(R; I)$. Then $z = \phi(z_0) + \sum c_i \phi(x_i)$, so that z belongs to the ideal $I^n Cpl(R; I)$. □

Corollary 4.3.14. *Let A be a Noetherian $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 A$ be an ideal. Then the completion $A^\wedge_I$ is a Noetherian $\mathbb{E}_\infty$-ring.*

Proof. Corollary 4.3.7 implies that $\pi_0 A^\wedge_I$ is the I-adic completion of the Noetherian commutative ring $\pi_0 A$, and therefore a Noetherian commutative ring (Corollary 4.3.13). To complete the proof, it will suffice to show that each $\pi_k A^\wedge_I$ is a finitely generated module over $\pi_0 A^\wedge_I$. Since $A \to A^\wedge_I$ is flat (Corollary 4.3.9), we have a canonical isomorphism

$$\pi_k A^\wedge_I \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 A^\wedge_I, \pi_k A).$$

It will therefore suffice to show that $\pi_k A$ is a finitely generated module over $\pi_0 A$, which follows from our assumption that A is Noetherian. □

5 Completions of Spectral Deligne-Mumford Stacks

In §4, we studied the operation of completing a module M over an $\mathbb{E}_\infty$-ring R along a finitely generated ideal $I \subseteq \pi_0 R$. In this section, we will study the global counterpart of this construction. Suppose we are given a spectral Deligne-Mumford stack $\mathfrak{X}$ and a (cocompact) closed subset K of the underlying topological space $|\mathfrak{X}|$. In §5.1, we will study an associated geometric object $\mathfrak{X}^\wedge_K$, which we refer to as the *formal completion* of $\mathfrak{X}$ along K (Definition 5.1.1). In the special case where $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} R$ and K is defined by a finitely generated ideal $I \subseteq \pi_0 R$, then we denote the completion $\mathfrak{X}^\wedge_K$ by $\mathrm{Spf}\, R$, and refer to it as the formal spectrum of R (with respect to I). We will see that there is a close relationship between the ∞-category Mod_R^{I-comp} of I-complete R-modules and the ∞-category $\mathrm{QCoh}(\mathrm{Spf}\, R)$ of quasi-coherent sheaves on $\mathrm{Spf}\, R$ (Lemma 5.1.10). We will then use this result to prove a global version of Proposition 4.2.5 (Theorem 5.1.9).

The operation of formal completion $\mathfrak{X} \mapsto \mathfrak{X}^\wedge_K$ is best-behaved in the case when $\mathfrak{X}$ is locally Noetherian. In §5.2, we will show that in the locally Noetherian case, the ∞-category $\mathrm{QCoh}(\mathfrak{X}^\wedge_K)^{\mathrm{aperf}}$ of almost perfect quasi-coherent sheaves on $\mathfrak{X}^\wedge_K$ admits a t-structure (Proposition 5.2.4). Moreover, the heart of $\mathrm{QCoh}(\mathfrak{X}^\wedge_K)^{\mathrm{aperf}}$ can be identified with the abelian category of coherent sheaves on the formal completion o the underlying ordinary Deligne-Mumford stack of $\mathfrak{X}$ (see Proposition 5.2.12 and Remark 5.2.13). In §5.3, we will use this result to prove an analogue of the Grothendieck existence theorem in the setting of spectral algebraic geometry: if $\mathfrak{X}$ is a spectral algebraic space which is proper and almost of finite presentation over a Noetherian $\mathbb{E}_\infty$-ring R which is complete with respect to an ideal $I \subseteq \pi_0 R$, then the restriction functor

$$\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}} \to \mathrm{QCoh}(\mathfrak{X} \times_{\mathrm{Spec}^{\text{ét}} R} \mathrm{Spf}\, R)^{\mathrm{aperf}}$$

is an equivalence of ∞-categories (Theorem 5.3.2). In §5.4, we will study some of the consequences of this result. In particular, we will prove that $\mathfrak{X}$ can be recovered (functorially) from its formal completion $\mathfrak{X} \times_{\mathrm{Spec}^{\text{ét}} R} \mathrm{Spf}\, R$ (see Corollary 5.4.3).

5.1 Formal Completions

Let A be a connective $\mathbb{E}_\infty$-ring. In §4.2 and §4.3, we studied the operation $A \mapsto A^\wedge_I$ of completing A with respect to a finitely generated ideal $I \subseteq \pi_0 A$. In this section, we will introduce the closely related operation of *formal completion* of a spectral Deligne-Mumford stack $\mathfrak{X}$ along a closed subset $K \subseteq |\mathfrak{X}|$.

Definition 5.1.1. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack. We will abuse notation by identifying $\mathfrak{X}$ with the functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ represented by $\mathfrak{X}$, so that $\mathfrak{X}(R) = \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}^{\text{ét}} R, \mathfrak{X})$. Let $|\mathfrak{X}|$ denote the underlying topological space of $\mathfrak{X}$, and let $K \subseteq |\mathfrak{X}|$ be a closed subset, so that the complement $|\mathfrak{X}| - K$ determines an open immersion of spectral Deligne-Mumford stacks $j : \mathfrak{U} \to \mathfrak{X}$. We will say that K is *cocompact* if the open immersion j is quasi-compact. We let $\mathfrak{X}^\wedge_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which assigns to each connective $\mathbb{E}_\infty$-ring R the summand of $\mathfrak{X}(R)$ spanned by those maps $\mathrm{Spec}^{\text{ét}} R \to \mathfrak{X}$ such that the fiber product $\mathfrak{U} \times_{\mathfrak{X}} \mathrm{Spec}^{\text{ét}} R$ is empty. We will refer to $\mathfrak{X}^\wedge_K$ as the *formal completion* of $\mathfrak{X}$ along the closed subset $K \subseteq |\mathfrak{X}|$.

We will say that a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ is *supported on* K if $j^* \mathcal{F} \simeq 0$. We let $\mathrm{QCoh}_K(\mathfrak{X})$ denote the full subcategory of $\mathrm{QCoh}(\mathfrak{X})$ spanned by those quasi-coherent sheaves which are supported on K.

Our main goal in this section is to prove that in the situation of Definition 5.1.1, there is a close relationship between the ∞-categories $\mathrm{QCoh}(\mathfrak{X}^\wedge_K)$ and $\mathrm{QCoh}_K(\mathfrak{X})$ (Theorem 5.1.9). We begin by studying the operation of formal completion in the affine case.

Example 5.1.2. Let $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} R$ be an affine spectral Deligne-Mumford stack, so that the underlying topological space $|\mathfrak{X}|$ is homeomorphic to the Zariski spectrum $\mathrm{Spec}^{\mathrm{Z}}\, \pi_0 R$ of the commutative ring $\pi_0 R$. There is a bijective correspondence between closed subsets $K \subseteq |\mathfrak{X}|$ and radical ideals $I \subseteq \pi_0 R$. A closed subset $K \subseteq |\mathfrak{X}|$ is cocompact if and only if I can be written as the radical of a finitely generated ideal $J \subseteq \pi_0 R$. In this case, a map $\mathrm{Spec}^{\text{ét}} A \to \mathfrak{X}$ factors through the formal completion $\mathfrak{X}^\wedge_K$ if and only if the corresponding map of $\mathbb{E}_\infty$-rings $\phi : R \to A$ induces a map of commutative rings $\pi_0 R \to \pi_0 A$ which annihilates some power of the ideal J.

Let R be an $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be a finitely generated ideal. Then I determines a closed subset $K \subseteq \mathrm{Spec}^{\mathrm{Z}} R = |\mathrm{Spec}^{\text{ét}} R|$. Note that a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ is supported on K if and only if the corresponding R-module belongs to Mod_R^{I-nil} (see Definition 4.1.3). There is a complementary description of the full subcategory $\mathrm{Mod}_R^{I-loc} \subseteq \mathrm{Mod}_R$:

Proposition 5.1.3. *Let R be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $U \subseteq \mathrm{Spec}^{\mathrm{Z}}\, \pi_0 R$ be the quasi-compact open set $\{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}}\, \pi_0 R : I \nsubseteq \mathfrak{p}\}$. Let $\mathfrak{U}$ denote the corresponding open substack of $\mathrm{Spec}^{\text{ét}} R$, and let $f : \mathfrak{U} \to \mathrm{Spec}^{\text{ét}} R$ be the inclusion. Then the pushforward functor $f_* : \mathrm{QCoh}(\mathfrak{U}) \to \mathrm{QCoh}(\mathrm{Spec}^{\text{ét}} R) \simeq \mathrm{Mod}_R$ is a fully faithful embedding, whose essential image is the full subcategory Mod_R^{I-loc} of I-local objects of Mod_R.*

Proof. The assertion that f is fully faithful follows from Corollary VIII.2.4.6. Let $M \in \mathrm{Mod}_R$ be I-nilpotent and let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{U})$. Then

$$\mathrm{Map}_{\mathrm{Mod}_R}(M, f_* \mathcal{F}) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{U})}(f^* M, \mathcal{F})$$

vanishes, since $f^* M \simeq 0$ (here f^* denotes the left adjoint to f_*, given by pullback along the open immersion f). It follows that $f_* \mathcal{F}$ is I-local. Conversely, suppose that $M \in \mathrm{Mod}_R$ is I-local. We wish to prove that the unit map $u : M \to f_* f^* M$ is an equivalence. Let N be the fiber of u. Because f_* is fully faithful, the pullback $f^* N$ vanishes. In particular, for every element $x \in I$, we have $N[\frac{1}{x}] \simeq 0$ (since $N[\frac{1}{x}]$ can be identified with the global sections of $f^* N$ over an open substack of $\mathfrak{U}$). This proves that N is I-nilpotent. Since u is a map between I-local objects of Mod_R, N is also I-local. It follows that $N \simeq 0$, so that u is an equivalence as desired. □

Notation 5.1.4. Let A be a connective $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal, defining a cocompact closed subset of $K \subseteq |\operatorname{Spec}^{\text{ét}} A| \simeq \operatorname{Spec}^{\mathrm{Z}} \pi_0 A$. We let $\operatorname{Spf} A$ denote the formal completion $(\operatorname{Spec}^{\text{ét}} A)^\vee_K$. We will refer to $\operatorname{Spf} A$ as the *formal spectrum* of A (with respect to the ideal I).

It will be useful to have a more explicit description of the formal completion of an affine spectral Deligne-Mumford stack.

Lemma 5.1.5. *Let R be a connective $\mathbb{E}_\infty$-ring, $I \subseteq \pi_0 R$ a finitely generated ideal, and $\operatorname{Spf} R$ the formal spectrum of R with respect to I. Then there exists an tower of $\mathbb{E}_\infty$-algebras over R*

$$\cdots \to A_2 \to A_1 \to A_0$$

having the following properties:

(a) Each A_i is a connective $\mathbb{E}_\infty$-ring, and each of the maps $A_{i+1} \to A_i$ determines a surjection $\pi_0 A_{i+1} \to \pi_0 A_i$.

(b) There is an equivalence of functors $\operatorname{Spf} R \simeq \varinjlim \operatorname{Spec}^{\text{ét}} A_n$. That is, for every connective $\mathbb{E}_\infty$-ring B, the canonical map

$$\varinjlim_n \mathrm{Map}_{\mathrm{CAlg}}(A_n, B) \to \mathrm{Map}_{\mathrm{CAlg}}(R, B)$$

is a fully faithful embedding, whose essential image is the collection of maps $\phi : R \to B$ which annihilate some power of I.

(c) Each of the $\mathbb{E}_\infty$-rings A_n is almost perfect when regarded as an R-module.

Proof. Choose any element $x \in \pi_0 R$. We will construct a tower of $\mathbb{E}_\infty$-algebras over R

$$\cdots \to A(x)_2 \to A(x)_1 \to A(x)_0$$

having the following properties:

(a_x) Each $A(x)_i$ is a connective $\mathbb{E}_\infty$-ring, and each of the maps $A(x)_{i+1} \to A(x)_i$ determines a surjection $\pi_0 A(x)_{i+1} \to \pi_0 A(x)_i$.

(b_x) For every connective $\mathbb{E}_\infty$-ring B, the canonical map

$$\varinjlim_n \mathrm{Map}_{\mathrm{CAlg}}(A(x)_n, B) \to \mathrm{Map}_{\mathrm{CAlg}}(R, B)$$

is a fully faithful embedding, whose essential image is the collection of maps $\phi : R \to B$ which annihilate some power of x.

(c_x) Each of $\mathbb{E}_\infty$-rings $A(x)_n$ is almost perfect, when regarded as an R-module.

Assuming that this can be done, choose a finite set of generators $x_1, \ldots, x_k$ for the ideal I. Setting $A_n = A(x_1)_n \otimes_R A(x_2)_n \otimes_R \cdots \otimes_R A(x_k)_n$, we obtain a tower of $\mathbb{E}_\infty$-algebras over R satisfying conditions (a), (b), and (c).

It remains to construct the tower $\{A(x)_n\}$. For each integer $n \geq 0$, let $R\{t_n\}$ denote a free $\mathbb{E}_\infty$-algebra over R on one generator t_n. We have R-algebra morphisms $\alpha_n : R\{t_n\} \to R$ and $\beta_n : R\{t_n\} \to R$, determined uniquely up to homotopy by the requirements that $t_n \mapsto x^n \in \pi_0 R$ and $t_n \mapsto 0 \in \pi_0 R$. Moreover, we have maps $\gamma_n : R\{t_n\} \to R\{t_{n-1}\}$ determined up to homotopy by the requirement that $t_n \mapsto x t_{n-1} \in \pi_0 R\{t_{n-1}\}$. For each $n \geq 0$, the diagram

$$\begin{array}{ccccc}
R & \xleftarrow{\alpha_n} & R\{t_n\} & \xrightarrow{\beta_n} & R \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \gamma_n} & & \downarrow{\scriptstyle \mathrm{id}} \\
R & \xleftarrow{\alpha_{n-1}} & R\{t_{n-1}\} & \xrightarrow{\beta_{n-1}} & R
\end{array}$$

commutes up to homotopy and can therefore be lifted to a commutative diagram in CAlg_R. Concatenating these, we obtain a commutative diagram

$$\begin{array}{ccccc}
R & \xleftarrow{\alpha_0} & R\{t_0\} & \xrightarrow{\beta_0} & R \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \gamma_1} & & \uparrow{\scriptstyle \mathrm{id}} \\
R & \xleftarrow{\alpha_1} & R\{t_1\} & \xrightarrow{\beta_0} & R \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \gamma_2} & & \uparrow{\scriptstyle \mathrm{id}} \\
R & \xleftarrow{\alpha_2} & R\{t_2\} & \xrightarrow{\beta_2} & R \\
\uparrow & & \uparrow & & \uparrow \\
\cdots & \longleftarrow & \cdots & \longrightarrow & \cdots
\end{array}$$

For each n, let $A(x)_n$ denote the colimit of the nth row of this diagram, so that we have a tower

$$\cdots \to A(x)_2 \to A(x)_1 \to A(x)_0$$

where $A(x)_n \simeq R \otimes_{R\{t_n\}} R$ is the R-algebra obtained by freely "dividing out" by $x^n \in \pi_0 R$. In particular, we have $\pi_0 A(x)_n \simeq (\pi_0 R)/(x^n)$, thereby verifying condition (a_x). To verify (c_x), it will suffice to show that R is almost perfect when regarded as an $R\{t_n\}$-module via β. For this, it suffices to show that the sphere spectrum is almost perfect when regarded as an $S\{t_n\}$-module via the map of $\mathbb{E}_\infty$-rings $S\{t_n\} \to S$ given by $t_n \mapsto 0 \in \pi_0 S$. Since $S\{t_n\}$ is Noetherian (Proposition A.7.2.5.31), this is equivalent to the assertion that each homotopy group $\pi_k S$ is finitely generated as a module over the commutative ring $\pi_0(S\{t_n\}) \simeq \mathbf{Z}[t_n]$ (Proposition A.7.2.5.17). This is clear, since the stable homotopy groups of spheres are finitely generated abelian groups.

To verify (b_x), we note that if $\phi : R \to B$ is a map of connective $\mathbb{E}_\infty$-rings, then the homotopy fiber of the map $\varinjlim_n \mathrm{Map}_{\mathrm{CAlg}}(A(x)_n, B) \to \mathrm{Map}_{\mathrm{CAlg}}(R, B)$ over the point ϕ is given by a sequential colimit $\varinjlim_n P_n$, where each P_n can be identified with a space of paths in $\Omega^\infty B$ joining the base point to a suitably chosen representative for the image of x^n in $\pi_0 B$. Let $y \in \pi_0 B$ be the image of x under ϕ. If y is not nilpotent, then each P_n is empty. Assume otherwise; we wish to show that $P_\infty = \varinjlim P_n$ is contractible. Note that if P_n contains some point p_n, the we have canonical isomorphisms $\pi_k(P_n, p) \simeq \pi_{k+1} B$. For $m \geq n$, let p_m denote the image of p_n in P_m, and let p_∞ denote the image of p_n in P_∞. Note that the induced map

$$\pi_{k+1} B \simeq \pi_k(P_n, p_n) \to \pi_k(P_m, p_m) \to \pi_{k+1} B$$

is given by multiplication by y^{m-n}. Since y is nilpotent, this map is trivial for $m \gg n$. It follows that $\pi_k(P_\infty, p_\infty) \simeq \varinjlim \pi_k(P_m, p_m)$ is trivial. Since p_n was chosen arbitrarily, we conclude that P_∞ is contractible as desired. $\square$

Remark 5.1.6. The notation Spf R is traditionally reserved for the formal spectrum of a ring R which is *complete* with respect to an ideal I. Notation 5.1.4 does not require this. However, there is no real gain in generality. Suppose that R is a connective $\mathbb{E}_\infty$-ring and that $I \subseteq \pi_0 R$ is a finitely generated ideal, and choose a tower of R-algebras

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma 5.1.5. Let $R^\wedge_I$ denote the I-completion of R, so that I generates an ideal $J \subseteq \pi_0 R^\wedge_I$. The formal spectrum Spf $R^\wedge_I$ of $R^\wedge_I$ with respect to J can be identified with the direct limit $\varinjlim \operatorname{Spec}^{\text{ét}}(R^\wedge_I \otimes_R A_n)$. The fiber of the completion map $u : R \to R^\wedge_I$ is I-local and each A_n is I-nilpotent, so that the tensor product $\operatorname{fib}(u) \otimes_R A_n$ vanishes and therefore u induces an equivalence $A_n \to R^\wedge_I \otimes_R A_n$. It follows that we have a canonical equivalence

$$\operatorname{Spf} R^\wedge_I \simeq \varinjlim \operatorname{Spec}^{\text{ét}}(R^\wedge_I \otimes_R A_n) \simeq \varinjlim \operatorname{Spec}^{\text{ét}} A_n \simeq \operatorname{Spf} R.$$

Remark 5.1.7. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack, let $K \subseteq |\mathfrak{X}|$ be a closed subset, and let $\mathcal{F} \in \operatorname{QCoh}(\mathfrak{X})$. Then $\mathcal{F}$ is supported on K if and only if each of the homotopy sheaves $\pi_i \mathcal{F}$ is supported on K. It follows that the full subcategories

$$\operatorname{QCoh}_K(\mathfrak{X})_{\geq 0} = \operatorname{QCoh}_K(\mathfrak{X}) \cap \operatorname{QCoh}(\mathfrak{X})_{\geq 0} \qquad \operatorname{QCoh}_K(\mathfrak{X})_{\leq 0} = \operatorname{QCoh}_K(\mathfrak{X}) \cap \operatorname{QCoh}(\mathfrak{X})_{\leq 0}$$

determine a t-structure on $\operatorname{QCoh}_K(\mathfrak{X})$.

Notation 5.1.8. Let $X : \operatorname{CAlg}^{\text{cn}} \to \mathcal{S}$ be a functor, and let $\mathcal{F} \in \operatorname{QCoh}(X)$ be a quasi-coherent sheaf on X. Recall that $\mathcal{F}$ is said to be *connective* (*almost connective*) if, for every $\mathbb{E}_\infty$-ring R and every point $\eta \in X(R)$, the R-module $\mathcal{F}(\eta)$ is connective (almost connective). We let $\operatorname{QCoh}(X)^{\text{cn}}$ and $\operatorname{QCoh}(X)^{\text{acn}}$ denote the full subcategories $\operatorname{QCoh}(X)$ spanned by the connective and almost connective objects, respectively.

If $\mathfrak{X}$ is a spectral Deligne-Mumford stack and $K \subseteq |\mathfrak{X}|$ is a closed subset, then we let $\operatorname{QCoh}_K(\mathfrak{X})^{\text{acn}}$ denote the intersection $\operatorname{QCoh}_K(\mathfrak{X}) \cap \operatorname{QCoh}(\mathfrak{X})^{\text{acn}}$. If $\mathfrak{X}$ is quasi-compact (or, more generally, if K is quasi-compact) then this subcategory coincides with the union $\bigcup_n \operatorname{QCoh}_K(\mathfrak{X})_{\geq -n}$.

We are now ready to state the main result of this section.

Theorem 5.1.9. *Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack and let $K \subseteq |\mathfrak{X}|$ be a cocompact closed subset. Then the composite functor*

$$\operatorname{QCoh}_K(\mathfrak{X})^{\text{acn}} \subseteq \operatorname{QCoh}(\mathfrak{X})^{\text{acn}} \to \operatorname{QCoh}(\mathfrak{X}^\vee_K)^{\text{acn}}$$

is an equivalence of ∞-categories.

We will deduce Theorem 5.1.9 by combining Proposition 4.2.5 with the following result:

Lemma 5.1.10. *Let R be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal. Then the composite functor*

$$(\operatorname{Mod}_R^{I-comp})_{\geq 0} \subseteq \operatorname{Mod}_R^{\text{cn}} \simeq \operatorname{QCoh}(\operatorname{Spec}^{ét} R)^{\text{cn}} \to \operatorname{QCoh}(\operatorname{Spf} R)^{\text{cn}}$$

is an equivalence of ∞-categories.

Proof. Choose a tower

$$\cdots \to A_2 \to A_1 \to A_0$$

of connective R-algebras satisfying the requirements of Lemma 5.1.5, so that the functor Spf R can be described as the filtered colimit $\varinjlim_n \operatorname{Spec}^{\text{ét}} A_n$. It follows that $\operatorname{QCoh}(\operatorname{Spf} R) \simeq \varprojlim_n \operatorname{Mod}_{A_n}$. Let $f^* : \operatorname{Mod}_R \simeq \operatorname{QCoh}(\operatorname{Spec}^{\text{ét}} R) \to \operatorname{QCoh}(\operatorname{Spf} R)$ denote the pullback functor. Then f^* admits a right adjoint U, which carries a compatible system $\{M_n \in \operatorname{Mod}_{A_n}\}_{n \geq 0}$ to the limit $U(\{M_n\}) = \varprojlim_n M_n \in \operatorname{Mod}_R$. By assumption each of the maps $A_{n+1} \to A_n$ is surjective on π_0. It follows that if each M_k is connective, then each of the maps $M_{n+1} \to A_n \otimes_{A_{n+1}} M_{n+1} \to M_n$ is surjective on π_0. In particular, $U(\{M_n\})$ is a connective R-module and each of the maps $U(\{M_n\}) \to M_n$ is surjective on π_0.

Note that if N is an I-local R-module, then $A_n \otimes_R N$ is both I-local and I-nilpotent (since A_n is I-nilpotent), and therefore vanishes. It follows every A_n-module is I-complete when viewed as an R-module. In particular, for every object $\{M_n\} \in \mathrm{QCoh}(\mathrm{Spf}\, R)$, $U(\{M_n\})$ is a limit of I-complete R-modules and therefore I-complete. It follows that f^* and U determine a pair of adjoint functors

$$(\mathrm{Mod}_R^{I-comp})_{\geq 0} \rightleftarrows \mathrm{QCoh}(\mathrm{Spf}\, R)^{\mathrm{cn}}.$$

We wish to show that these functors are mutually inverse equivalences. The main step is to prove the following:

- $(*)$ If M is a connective R-module, then the unit map $M \to U(f^* M)$ exhibits $U(f^* M)$ as an I-completion of M.

Assuming $(*)$, we deduce that f^* induces a fully faithful embedding from $(\mathrm{Mod}_R^{I-comp})_{\geq 0}$ to $\mathrm{QCoh}(\mathfrak{X}_K^{\vee})^{\mathrm{cn}}$. To complete the proof, it will therefore suffice to show that U is conservative when restricted to $\mathrm{QCoh}(\mathfrak{X}_K^{\vee})^{\mathrm{cn}}$. Since U is an exact functor between stable ∞-categories, it will suffice to show that if $\{M_n\}$ is an object of $\mathrm{QCoh}(\mathrm{Spf}\, R)^{\mathrm{cn}}$ satisfying $U(\{M_n\}) \simeq 0$, then each $M_n \in \mathrm{Mod}_{A_n}$ vanishes. We prove by induction k that $\pi_i M_n \simeq 0$ for $i \leq k$. When $k = 0$, this follows from our observation that each of the maps $\pi_0 U(\{M_n\}) \to \pi_0 M_n$ is surjective. If $k > 0$, the inductive hypothesis implies that $\{M_n\}$ is the k-fold suspension of an object $\{N_n\} \in \mathrm{QCoh}(\mathrm{Spf}\, R)^{\mathrm{cn}}$. Then $U(\{N_n\}) \simeq 0$ and we can apply the inductive hypothesis to deduce that $\pi_k M_n \simeq \pi_0 N_n \simeq 0$.

It remains to prove $(*)$. Let M be a connective R-module. Since $U(f^* M)$ is I-complete, the unit map $M \to U(f^* M)$ induces a map $\beta_M : M_I^{\wedge} \to U(f^* M)$. We wish to show that β_M is an equivalence. Choose an element $x \in I$, and let $C(x^n)$ denote the cofiber of the map of R-modules $R \to R$ given by multiplication by x^n. Since $\mathrm{fib}(\beta_M)$ is I-complete, we have

$$\mathrm{fib}(\beta_M) \simeq \varprojlim \mathrm{fib}(\beta_M) \otimes_R C(x^n);$$

it will therefore suffice to show that each tensor product $\mathrm{fib}(\beta_M) \otimes_R C(x^n)$ vanishes. Since $C(x^n)$ can be obtained as a successive extension of n copies of $C(x)$, we may suppose that $n = 1$. Note that $\mathrm{fib}(\beta_M) \otimes_R C(x) \simeq \mathrm{fib}(\beta_{M \otimes_R C(x)})$. Consequently, to show that β_M is an equivalence, it suffices to show that $\beta_{M \otimes_R C(x)}$ is an equivalence.

Choose generators $x_1, \ldots, x_n \in I$ for the ideal I. Using the above argument repeatedly, we are reduced to proving that β_N is an equivalence when $N = M \otimes_R C(x_1) \otimes_R C(x_2) \otimes \cdots \otimes_R C(x_n)$. For $1 \leq i \leq n$, we observe that N can be obtained as a successive extension of 2^{n-1} copies of $M \otimes_R C(x_i)$. Since the homotopy groups of $M \otimes_R C(x_i)$ are annihilated by multiplication by x_i^2, we conclude that each of the homotopy groups of N is annihilated by multiplication by $x_i^{2^n}$. We are therefore reduced to proving the following special case of $(*)$:

- $(*')$ Let M be a connective R-module, and suppose that there exists an integer k such that each homotopy group $\pi_i M$ is annihilated by the ideal $I^k \subseteq \pi_0 R$. Then $\beta_M : M_I^{\wedge} \to U(f^* M)$ is an equivalence.

To prove $(*')$, it suffices to show that for every integer $j \geq 0$, the map $\pi_j M_I^{\wedge} \to \pi_j U(f^* M)$ is an isomorphism of abelian groups. Both $M_I^{\wedge}$ and $U(f^* M)$ are right t-exact functors of M. We may therefore replace M by $\tau_{\leq j} M$ and thereby reduce to prove $(*')$ under the additional assumption that M is p-truncated for some integer p. We now proceed by induction on p. If $p < 0$, then $M \simeq 0$ and there is nothing to prove. Otherwise, we have a map of fiber sequences

$$\begin{array}{ccccc}
(\tau_{\leq p-1} M)_I^{\wedge} & \longrightarrow & M_I^{\vee} & \longrightarrow & (\pi_p M)_I^{\wedge}[p] \\
\downarrow & & \downarrow & & \downarrow \\
U(f^* \tau_{\leq p-1} M) & \longrightarrow & U(f^* M) & \longrightarrow & U(f^* \pi_p M)[p]
\end{array}$$

where the left map is an equivalence by the inductive hypothesis. We may therefore replace M by $\pi_p M$ and thereby reduce to the case where M is discrete. In this case, M has the structure of a module over the discrete R-algebra $R' = \pi_0 R/I^k$.

Note that $\mathrm{Spf}\, R \times_{\mathfrak{X}} \mathrm{Spec}^{\text{ét}} R' \simeq \mathrm{Spec}^{\text{ét}} R'$, so that the tower of R'-algebras $\{R' \otimes_R A_n\}_{n \geq 0}$ is equivalent (as a pro-object of $\mathrm{CAlg}^{\mathrm{cn}}_{R'}$) to the constant diagram taking the value R'. Since M is I-complete, we can identify β_M with the unit map

$$M \to U(f^* M) \simeq \varprojlim \{M \otimes_R A_n\} \simeq \varprojlim \{M \otimes_{R'} (R' \otimes_R A_n)\} \simeq M \otimes_{R'} R',$$

which is evidently an equivalence. □

Remark 5.1.11. Let R be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and choose a tower

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma 5.1.5. The proof of Lemma 5.1.10 shows that for every connective A-module M, the I-completion $M^\wedge_I$ can be identified with the inverse limit of the tower

$$\cdots \to A_2 \otimes_R M \to A_1 \otimes_R M \to A_0 \otimes_R M.$$

In particular, the I-completion $R^\wedge_I$ of R is given by $\varprojlim A_i$.

Remark 5.1.12. Let R be a connective $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be a finitely generated ideal, so the I-completion functor induces an equivalence $\mathrm{Mod}_R^{I-nil} \to \mathrm{Mod}_R^{I-comp}$ (see Proposition 4.2.5). This functor restricts to an equivalence

$$\theta : \mathrm{Mod}_R^{I-nil} \cap \mathrm{Mod}_R^{\mathrm{acn}} \to \mathrm{Mod}_R^{I-comp} \cap \mathrm{Mod}_R^{\mathrm{acn}}.$$

Indeed, the functor of I-completion is right t-exact, so that θ is well-defined. It will therefore suffice to show that if M is an I-nilpotent R-module such that $M^\wedge_I$ is almost connective, then M is almost connective. We can recover M as $V \otimes_R M^\vee_I$, where V is R-module of Proposition 4.1.12. It now suffices to observe that V is almost connective (this follows from the proof of Proposition 4.1.12).

Proof of Theorem 5.1.9. The assertion is local on $\mathfrak{X}$. We may therefore reduce to the case where $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} R$ is affine. Since K is cocompact, it corresponds to the radical of a finitely generated ideal $I \subseteq \pi_0 R$. Let $f^* : \mathrm{Mod}_R \simeq \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{X}^\vee_K)$ be the restriction functor. We wish to show that the composite functor

$$\theta : \mathrm{Mod}_R^{I-nil} \cap \mathrm{Mod}_R^{\mathrm{acn}} \subseteq \mathrm{Mod}_R^{\mathrm{acn}} \xrightarrow{f^*} \mathrm{QCoh}(\mathrm{Spf}\, R)^{\mathrm{acn}}$$

is an equivalence of ∞-categories. Note that an R-module M satisfies $f^* M \simeq 0$ if and only if $A \otimes_R M \simeq 0$ whenever $\phi : R \to A$ is a map of connective $\mathbb{E}_\infty$-rings which annihilates a power of I. In particular, this condition is satisfied whenever M is I-local. It follows that for any $M \in \mathrm{Mod}_R$, the canonical map $f^* M \to f^* M^\wedge_I$ is an equivalence. We may therefore factor θ as a composition

$$\mathrm{Mod}_R^{I-nil} \cap \mathrm{Mod}_R^{\mathrm{acn}} \xrightarrow{\theta'} \mathrm{Mod}_R^{I-comp} \cap \mathrm{Mod}_R^{\mathrm{acn}} \xrightarrow{\theta''} \mathrm{QCoh}(\mathfrak{X}^\vee_{\mathrm{acn}})^{\mathrm{acn}}.$$

Here θ' is the equivalence of ∞-categories of Remark 5.1.12 (given by I-completion) and θ'' is an equivalence of ∞-categories by Lemma 5.1.10. □

We close this section be recording a few other applications of Lemma 5.1.10.

Proposition 5.1.13. *Let R be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 R$, and let $f : \mathrm{Spf}\, R \to \mathrm{Spec}^{\text{ét}} R$ be the canonical map. Let $M \in \mathrm{Mod}_R \simeq \mathrm{QCoh}(\mathrm{Spec}^{\text{ét}} R)$ be an almost connective R-module. Let n be an integer. The following conditions are equivalent:*

(1) *The R-module M is perfect to order n.*

(2) *The pullback f^*M is perfect to order n (as a quasi-coherent sheaf on* $\operatorname{Spf} R$*).*

Corollary 5.1.14. *Let R be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 R$, and let $f : \operatorname{Spf} R \to \operatorname{Spec}^{\acute{e}t} R$ be the canonical map. Let $M \in \operatorname{Mod}_R \simeq \operatorname{QCoh}(\operatorname{Spec}^{\acute{e}t} R)$ be an almost connective R-module which is I-complete. The following conditions are equivalent:*

(1) *The R-module M is almost perfect.*

(2) *The pullback f^*M is almost perfect (as a quasi-coherent sheaf on* $\operatorname{Spf} R$*).*

Proof of Proposition 5.1.13. The implication (1) ⇒ (2) is obvious. We will prove that (2) ⇒ (1). Replacing M by a shift if necessary, we may suppose that M is connective. We now proceed by induction on n. We begin by treating the case $n = 0$. Assume that f^*M is perfect to order 0. We wish to show that M is perfect to order zero: that is, that $\pi_0 M$ is finitely generated as a module over the commutative ring $\pi_0 R$. Choose a tower of R-algebras

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma 5.1.5. Then $A_0 \otimes_R M$ is perfect to order 0, so that $\pi_0(A_0 \otimes_R M) \simeq \operatorname{Tor}_0^{\pi_0 R}(\pi_0 A_0, \pi_0 M)$ is finitely generated as a module over $\pi_0 A_0$. Since M is I-complete (Proposition 4.3.8), Lemma 5.1.10 implies that the map $M \to \varprojlim A_n \otimes_R M$ is an equivalence. Since each $A_n \otimes_R M$ is connective and each of the maps

$$\pi_0(A_n \otimes_R M) \to \pi_0(A_{n-1} \otimes_R M)$$

is surjective, we deduce that $\pi_0 M \to \pi_0(A_0 \otimes_R M)$ is surjective. In particular, we can choose a map of R-modules $\alpha : R^k \to M$ such that the composite map

$$\pi_0 R^k \to \pi_0 M \to \pi_0(A_0 \otimes_R M)$$

is surjective. We claim that α induces a surjection $\pi_0 R^k \to \pi_0 M$. To prove this, let K denote the fiber of α; we wish to show that $\pi_{-1} K \simeq 0$. In fact, we claim that K is connective. Since K is almost perfect as an R-module, it is I-complete (Proposition 4.3.8); it will therefore suffice to show that $f^*K \in \operatorname{QCoh}(\operatorname{Spf} R)$ is connective (Lemma 5.1.10). Equivalently, we must show that each tensor product $A_n \otimes_R K$ is connective. It is clear that $A_n \otimes_R K$ is (-1)-connective. Let $P = \pi_{-1}(A_n \otimes_R K)$, and let J denote the kernel of the map $\pi_0 A_n \to \pi_0 A_0$. Then $P/JP \simeq \pi_{-1}(A_0 \otimes_R K) \simeq 0$ by construction. Since J is a nilpotent ideal in $\pi_0 A$, it follows from Nakayama's lemma that $P \simeq 0$, as desired. □

Proposition 5.1.15. *Let R be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 R$, let $f : \operatorname{Spf} R \to \operatorname{Spec}^{\acute{e}t} R$ be the inclusion, and let $M \in \operatorname{Mod}_R \simeq \operatorname{QCoh}(\operatorname{Spec}^{\acute{e}t} R)$ be almost perfect. The following conditions are equivalent:*

(1) *As an R-module, M is locally free of finite rank.*

(2) *The pullback $f^*M \in \operatorname{QCoh}(\operatorname{Spf} R)$ is locally free of finite rank.*

Proof. The implication (1) ⇒ (2) is obvious. Suppose that (2) is satisfied. Proposition 4.3.8 shows that M is I-complete. Using Lemma 5.1.10, we deduce that M is connective. Since M is almost perfect, we conclude that $\pi_0 M$ is finitely presented as a module over $\pi_0 R$. We may therefore choose a map $u : R^n \to M$ which induces a surjection $\pi_0 R^n \to \pi_0 M$. To prove (1), it will suffice to show that u admits a section. For this, it suffices to show that the map

$$\phi : \operatorname{Map}_{\operatorname{Mod}_R}(M, R^n) \to \operatorname{Map}_{\operatorname{Mod}_R}(M, M)$$

is surjective on π_0. Letting K denote the cofiber of u, we are reduced to proving that $\operatorname{Map}_{\operatorname{Mod}_R}(M, K)$ is connected. Choose a tower of $\mathbb{E}_\infty$-algebras

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma 5.1.5. The proof of Lemma 5.1.10 shows that we can recover $K \simeq K_I^\vee$ as the limit of the tower $\{A_i \otimes_R K\}$. Then $\mathrm{Map}_{\mathrm{Mod}_R}(M, K)$ is the limit of the tower $\mathrm{Map}_{\mathrm{Mod}_R}(M, A_i \times_R K)$. It will therefore suffice to prove the following:

(a) Each of the mapping spaces $\mathrm{Map}_{\mathrm{Mod}_R}(M, A_i \otimes_R K)$ is connected.

(b) Each of the maps $\psi_i : \mathrm{Map}_{\mathrm{Mod}_R}(M, A_i \otimes_R K) \to \mathrm{Map}_{\mathrm{Mod}_R}(M, A_{i-1} \otimes_R K)$ induces a surjection of fundamental groups.

Note that K is 1-connective, so that $A_i \otimes_R K$ is a 1-connective module over A_i. We have a homotopy equivalence $\mathrm{Map}_{\mathrm{Mod}_R}(M, A_i \otimes_R K) \simeq \mathrm{Map}_{\mathrm{Mod}_{A_i}}(A_i \otimes_R M, A_i \otimes_R K)$. Consequently, assertion (a) follows immediately from assumption (2). To prove (b), we note that the homotopy fiber of ψ_i (over the base point) can be identified with $\mathrm{Map}_{\mathrm{Mod}_{A_i}}(A_i \otimes_R M, J \otimes_R K)$, where $J = \mathrm{fib}(A_i \to A_{i-1})$. Since J is connective, $J \otimes_R K$ is 1-connective, and the desired result follows from the projectivity of $A_i \otimes_R M$. □

Corollary 5.1.16. *Let R be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 R$, let $f : \mathrm{Spf}\, R \to \mathrm{Spec}^{\acute{e}t}\, R$ be the inclusion, and let $M \in \mathrm{Mod}_R \simeq \mathrm{QCoh}(\mathrm{Spec}^{\acute{e}t}\, R)$ be almost perfect. Let n be an integer. The following conditions are equivalent:*

(1) *As an R-module, M has* Tor-*amplitude* $\leq n$.

(2) *The pullback $f^* M \in \mathrm{QCoh}(\mathrm{Spf}\, R)$ has* Tor-*amplitude* $\leq n$.

Proof. Choose k such that $M \in (\mathrm{Mod}_R)_{\geq -k}$. Replacing M by $M[k]$ and n by $n+k$, we may reduce to the case where M is connective. The implication (1) ⇒ (2) is obvious. We will prove the converse using induction on n. When $n = 0$, the desired result follows from Propositions 5.1.15 and A.7.2.5.20. If $n > 0$, we can choose a fiber sequence

$$N \to R^m \to M,$$

where N is connective. Then $f^* N$ has Tor-amplitude $\leq n-1$, so the inductive hypothesis implies that N has Tor-amplitude $\leq n$. Using Proposition A.7.2.5.23, we deduce that M has Tor-amplitude $\leq n$. □

Corollary 5.1.17. *Let R be a connective $\mathbb{E}_\infty$-ring which is complete with respect to a finitely generated ideal $I \subseteq \pi_0 R$, let $f : \mathrm{Spf}\, R \to \mathrm{Spec}^{\acute{e}t}\, R$ be the inclusion, and let $M \in \mathrm{Mod}_R \simeq \mathrm{QCoh}(\mathrm{Spec}^{\acute{e}t}\, R)$ be almost perfect. Let n be an integer. The following conditions are equivalent:*

(1) *As an R-module, M is perfect*

(2) *The pullback $f^* M \in \mathrm{QCoh}(\mathrm{Spf}\, R)$ is perfect.*

Proof. Combine Corollary 5.1.16 with the characterization of perfect modules given in Proposition A.7.2.5.23. □

5.2 Truncations in $\mathrm{QCoh}(\mathfrak{X}^\wedge_K)$

Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack and let $K \subseteq |\mathfrak{X}|$ be a cocompact closed subset. In §5.1, we defined the formal completion $\mathfrak{X}^\wedge_K$ of $\mathfrak{X}$ along K and studied the ∞-category $\mathrm{QCoh}(\mathfrak{X}^\wedge_K)$ of quasi-coherent sheaves on $\mathfrak{X}^\wedge_K$. Our goal in this section is to study the exactness properties of the restriction functor

$$\mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{X}^\wedge_K).$$

In order to obtain reasonable results, it is necessary to make some assumption about $\mathfrak{X}$ and the class of quasi-coherent sheaves under consideration. We will restrict our attention to the case where $\mathfrak{X}$ is locally Noetherian, and to the study of almost perfect objects of $\mathrm{QCoh}(\mathfrak{X}^\wedge_K)$.

Definition 5.2.1. Let R be a Noetherian $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be an ideal, and let M be an R-module. We will say that M is *formally n-truncated along I* if the I-completion $M^\wedge_I$ is a n-truncated and almost perfect when regarded as a module over the I-completion $R^\wedge_I$.

Let $\mathfrak{X}$ be a locally Noetherian spectral Deligne-Mumford stack, $K \subseteq |\mathfrak{X}|$ a closed subset, and $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ a quasi-coherent sheaf. We will say that $\mathcal{F}$ is *formally n-truncated along K* if the following condition is satisfied:

(∗) Let $f : \mathrm{Spec}^{\text{ét}} R \to \mathfrak{X}$ be an étale map and $I \subseteq \pi_0 R$ an ideal defining the inverse image of K in $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)$, and identify $f^* \mathcal{F} \in \mathrm{QCoh}(\mathrm{Spec}^{\text{ét}} R)$ with an R-module M. Then M is formally n-truncated along I.

Notation 5.2.2. Let $\mathfrak{X}$ be a locally Noetherian spectral Deligne-Mumford stack, and let $K \subseteq |\mathfrak{X}|$ be a closed subset, let $f : \mathfrak{X}^\wedge_K \to \mathfrak{X}$ be the inclusion, and let n be an integer. We let $\mathrm{QCoh}(\mathfrak{X}^\vee_K)^{\mathrm{aperf}}_{\leq n}$ denote the full subcategory of $\mathrm{QCoh}(\mathfrak{X}^\vee_K)^{\mathrm{aperf}}$ spanned by those quasi-coherent sheaves $\mathcal{F}$ which are of the form $f^* \mathcal{F}'$, where $\mathcal{F}' \in \mathrm{QCoh}_K(\mathfrak{X})^{\mathrm{acn}}$ is formally n-truncated along K.

Remark 5.2.3. The object $\mathcal{F}' \in \mathrm{QCoh}_K(\mathfrak{X})^{\mathrm{acn}}$ appearing in the statement of Notation 5.2.2 is determined by $\mathcal{F}$ up to canonical equivalence, by virtue of Theorem 5.1.9.

The first main result of this section can be stated as follows.

Proposition 5.2.4. *Let $\mathfrak{X}$ be a locally Noetherian spectral Deligne-Mumford stack and let $K \subseteq |\mathfrak{X}|$ be a closed subset. Then the pair of full subcategories $(\mathrm{QCoh}(\mathfrak{X}^\wedge_K)^{\mathrm{aperf}} \cap \mathrm{QCoh}(\mathfrak{X}^\wedge_K)^{\mathrm{cn}}, \mathrm{QCoh}(\mathfrak{X}^\wedge_K)^{\mathrm{aperf}}_{\leq 0})$ determine a t-structure on the stable ∞-category $\mathrm{QCoh}(\mathfrak{X}^\wedge_K)^{\mathrm{aperf}}$.*

We begin by showing that Definition 5.2.1 behaves well with respect to the étale localization.

Lemma 5.2.5. *Let $f : R \to A$ be an étale map of Noetherian $\mathbb{E}_\infty$-rings, let $I \subseteq \pi_0 R$ be an ideal, and let $J \subseteq \pi_0 A$ be the image of I. Then:*

(1) *The induced map of completions $R^\wedge_I \to A^\wedge_J$ is flat.*

(2) *If $M \in \mathrm{Mod}_R$ is formally n-truncated along I, then $A \otimes_R M$ is formally n-truncated along J.*

Proof. We begin by proving (1). Let $A' = A \otimes_R R^\wedge_I$, so that A' is étale over $R^\wedge_I$. Since $R^\wedge_I$ is Noetherian (Corollary 4.3.14), Theorem A.7.2.5.31 implies that A' is Noetherian. Since $R \to R^\wedge_I$ is an I-equivalence, the induced map $A \to A'$ is an J-equivalence. It follows that the induced map $A' \to A^\wedge_J$ is a J-equivalence, and therefore exhibits $A^\wedge_J$ as the J'-completion of A', where J' denotes the ideal in $\pi_0 A'$ generated by J. Using Corollary 4.3.9, we deduce that $A^\wedge_J$ is flat over A'. Since A' is étale over $R^\wedge_I$, we conclude that $A^\wedge_J$ is flat over $R^\vee_I$.

Now suppose that $M \in \mathrm{Mod}_R$ is formally n-truncated along I. Then $M^\wedge_I$ is almost perfect and n-truncated. It follows from (1) $A^\wedge_J \otimes_{R^\wedge_I} M^\wedge_I$ is an almost perfect, n-truncated $A^\wedge_J$-module. Using Proposition 4.3.8, we deduce that $A^\wedge_J \otimes_{R^\wedge_I} M^\wedge_I$ is J-complete (when regarded as an A-module). Since the map $u : A \otimes_R M \to A^\vee_J \otimes_{R^\wedge_I} M^\wedge_I$ is a J-equivalence, it exhibits $A^\wedge_J \otimes_{R^\wedge_I} M^\wedge_I$ as a J-completion of $A \otimes_R M$. It follows that $(A \otimes_R M)^\wedge_J$ is n-truncated and almost perfect, as desired. ☐

Lemma 5.2.6. *Let R be a Noetherian $\mathbb{E}_\infty$-ring, let $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} R$ be its spectrum, and let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ be a quasi-coherent sheaf corresponding to an R-module M. Let $I \subseteq \pi_0 R$ be an ideal and K the corresponding closed subset of $|\mathfrak{X}|$. Then $\mathcal{F}$ is formally n-truncated along K if and only if M is formally n-truncated along I.*

Proof. The "only if" direction follows immediately from the definitions, and the converse follows from Lemma 5.2.5. ☐

Lemma 5.2.7. *Let $f : R \to A$ be a faithfully flat étale of Noetherian $\mathbb{E}_\infty$-rings, let $I \subseteq \pi_0 R$ be an ideal, and let $J \subseteq \pi_0 A$ be the image of I. Then:*

(1) *The induced map of completions* $R^{\wedge}_{I} \to A^{\wedge}_{J}$ *is faithfully flat.*

(2) *If* $M \in \mathrm{Mod}_R$ *is almost connective and* $A \otimes_R M$ *is formally* n*-truncated along* J*, then* M *is formally* n*-truncated along* I*.*

Proof. We first prove (1). Lemma 5.2.5 implies that $A^{\wedge}_{J}$ is flat over $R^{\wedge}_{J}$. It will therefore suffice to show that every maximal ideal of $\pi_0 R^{\wedge}_{J}$ can be lifted to a prime ideal in $\pi_0 A^{\wedge}_{J}$. Without loss of generality, we may replace R by $\pi_0 R$ and thereby reduce to the case where R is discrete. Let $\mathfrak{m}$ be a maximal ideal in $R^{\wedge}_{I}$. Let I' denote the ideal in $R^{\wedge}_{I}$ generated by I, and let $x \in I'$. If $x \notin \mathfrak{m}$, then x is invertible in $R^{\wedge}_{I}/\mathfrak{m}$, so we can choose an element $y \in R^{\wedge}_{I}$ such that $1 - xy \in \mathfrak{m}$ is not invertible. This is impossible, since $R^{\wedge}_{I}$ is I'-adically complete (the element $1 - xy$ has a multiplicative inverse given by the sum of the I'-adically convergent series $1 + xy + x^2y^2 + \cdots$). It follows that $\mathfrak{m}$ contains the ideal I. Consequently, to show that $\mathfrak{m}$ can be lifted to a prime ideal of in $\pi_0 A^{\wedge}_{I}$, it suffices to show that the map

$$\theta : \mathrm{Spec}^{\mathrm{Z}} A^{\wedge}_{I}/I' A^{\vee}_{J} \to \mathrm{Spec}^{\mathrm{Z}} R^{\vee}_{I}/I'$$

is surjective. We can identify θ with the map $\mathrm{Spec}^{\mathrm{Z}} A/J \to \mathrm{Spec}^{\mathrm{Z}} R/I$, which is a pullback of the surjective map $\mathrm{Spec}^{\mathrm{Z}} A \to \mathrm{Spec}^{\mathrm{Z}} R$.

We now prove (2). We first claim that $M^{\wedge}_{I}$ is almost perfect as an $R^{\wedge}_{I}$-module. Since M is almost connective, the proof of Lemma 5.1.10 shows that we can identify $M^{\wedge}_{I}$ with the global sections over $\mathrm{Spf}\, R$ of the quasi-coherent sheaf $\mathcal{F}$ associated to M. Using Corollary 5.1.14, we are reduced to showing that $\mathcal{F}$ is almost perfect. Let $R \to R'$ be a map of connective $\mathbb{E}_\infty$-rings which carries I to a nilpotent ideal in $\pi_0 R'$; we wish to show that $R' \otimes_R M$ is almost perfect as an R'-module. Since A is faithfully flat over R, we can use Proposition VIII.2.6.15 to reduce to showing that $(A \otimes_R R') \otimes_R M$ is almost perfect over $(A \otimes_R R')$, which follows from our assumption that $A \otimes_R M$ is J-truncated along n.

To complete the proof of (2), we must show that $M^{\wedge}_{I}$ is n-truncated. The proof of Lemma 5.2.5 furnishes an equivalence

$$(A \otimes_R M)^{\wedge}_{J} \simeq A^{\wedge}_{J} \otimes_{R^{\wedge}_{I}} M^{\wedge}_{I}.$$

Since $A^{\wedge}_{J}$ is faithfully flat over $R^{\wedge}_{I}$, we are reduced to proving that $(A \otimes_R M)^{\wedge}_{J}$ is n-truncated (Proposition VIII.2.6.15). This follows from our assumption that $A \otimes_R M$ is n-truncated along J. □

Lemmas 5.2.5 and 5.2.7 immediately imply the following global assertion for a locally Noetherian spectral Deligne-Mumford stack:

Lemma 5.2.8. *Let* $f : \mathfrak{X} \to \mathfrak{Y}$ *be a map of locally Noetherian spectral Deligne-Mumford stacks, let* $K \subseteq |\mathfrak{Y}|$ *be a closed subset, and let* $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{Y})$*. Then:*

(1) *If* $\mathcal{F}$ *is formally* n*-truncated along* K*, then* $f^* \mathcal{F}$ *is formally* n*-truncated along* $f^{-1}K$*.*

(2) *If* f *is an étale surjection and* $f^* \mathcal{F}$ *is formally* n*-truncated along* $f^{-1}K$*, then* $\mathcal{F}$ *is formally* n*-truncated along* K*.*

Proof of Proposition 5.2.4. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. For every object $U \in \mathcal{X}$, let $\mathfrak{X}_U$ denote the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|U)$, and let K_U denote the inverse image of K in the topological space $|\mathfrak{X}_U|$. Let us say that the object $U \in \mathcal{X}$ is *good* if the pair of full subcategories

$$(\mathrm{QCoh}((\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{aperf}} \cap \mathrm{QCoh}((\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{cn}}, \mathrm{QCoh}((\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{aperf}}_{\leq 0})$$

determines a t-structure on $\mathrm{QCoh}((\mathfrak{X}_U)^{\vee}_{K_U})^{\mathrm{aperf}}$. To check that U is good, we must verify two conditions:

(*a*) If $\mathcal{F}, \mathcal{F}' \in \mathrm{QCoh}((\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{aperf}}$ are such that $\mathcal{F}$ is connective and $\mathcal{F}' \in \mathrm{QCoh}((\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{aperf}}_{\leq -1}$, then $\mathrm{Map}_{\mathrm{QCoh}((\mathfrak{X}_U)^{\wedge}_{K_U})}(\mathcal{F}, \mathcal{F}')$ is contractible.

(b) For every object $\mathcal{F} \in \mathrm{QCoh}((\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{aperf}}$, there exists a fiber sequence

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

where $\mathcal{F}'$ is connective and almost perfect and $\mathcal{F}'' \in \mathrm{QCoh}((\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{aperf}}_{\leq -1}$.

We will prove that every object $U \in \mathcal{X}$ is good. Let us first suppose that U is affine, so that we can write $\mathfrak{X}_U \simeq \mathrm{Spec}^{\text{ét}} R$ for some Noetherian $\mathbb{E}_\infty$-ring R. Let $I \subseteq \pi_0 R$ be an ideal defining the closed subset $K_U \subseteq |\mathfrak{X}_U| \simeq \mathrm{Spec}^{\mathrm{Z}}\, \pi_0 R$, and let $A = R^{\wedge}_I$ be the I-completion of R. Using Corollary 5.1.14, we can identify $\mathrm{QCoh}((\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{aperf}}$ with the ∞-category $\mathrm{Mod}_A^{\mathrm{aperf}}$ of almost perfect A-modules. Under this equivalence, the full subcategory $(\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{aperf}} \cap \mathrm{QCoh}((\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{cn}}$ corresponds to $(\mathrm{Mod}_A^{\mathrm{aperf}})_{\geq 0}$ (Lemma 5.1.10), while $\mathrm{QCoh}(\mathfrak{X}_U)^{\mathrm{aperf}}_{\leq 0}$ corresponds to $(\mathrm{Mod}_A^{\mathrm{aperf}})_{\leq 0}$ (Lemma 5.2.6). It is now clear that U satisfies (a), and assertion (b) follows from Proposition A.7.2.5.18.

To complete the proof that every object of $\mathcal{X}$ is good, it will suffice to show that the full subcategory of $\mathcal{X}$ spanned by the good objects is closed under small colimits (Lemma V.2.3.11). Let us therefore suppose that we are given a small diagram $u : \mathcal{J} \to \mathcal{X}$ having colimit U. Assume that $u(J)$ is good for each object $J \in \mathcal{J}$; we wish to show that U is good. For each $J \in \mathcal{J}$, let $\mathcal{C}(J)$ denote the ∞-category $\mathrm{QCoh}((\mathfrak{X}_{u(J)})^{\wedge}_{K_{u(J)}})^{\mathrm{aperf}}$. Since $u(J) \in \mathcal{X}$ is good, we have a t-structure $(\mathcal{C}(J)_{\geq 0}, \mathcal{C}(J)_{\leq 0})$ on $\mathcal{C}(J)$, where $\mathcal{C}(J)_{\geq 0} = \mathrm{QCoh}((\mathfrak{X}_{u(J)})^{\wedge}_{K_{u(J)}})^{\mathrm{aperf}} \cap \mathrm{QCoh}((\mathfrak{X}_{u(J)})^{\wedge}_{K_{u(J)}})^{\mathrm{cn}}$ and $\mathcal{C}(J)_{\leq 0} = \mathrm{QCoh}((\mathfrak{X}_{u(J)})^{\wedge}_{K_{u(J)}})^{\mathrm{aperf}}_{\leq 0}$. The explicit characterizations of the subcategories $\mathcal{C}(J)_{\geq 0}$ and $\mathcal{C}(J)_{\leq 0}$ shows that for every morphism $\alpha : J \to J'$ in $\mathcal{J}$, the induced map $\mathcal{C}(J') \to \mathcal{C}(J)$ is t-exact. It follows that the limits $\varprojlim_{J \in \mathcal{J}} \mathcal{C}(J)_{\geq 0}$ and $\varprojlim_{J \in \mathcal{J}} \mathcal{C}(J)_{\leq 0}$ determine a t-structure on $\varprojlim_{J \in \mathcal{J}} \mathcal{C}(J) \simeq \mathrm{QCoh}((\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{aperf}}$. Since the map $\coprod_{J \in \mathcal{J}} u(J) \to U$ is an effective epimorphism, Proposition VIII.2.6.15 allows us to identify $\varprojlim_{J \in \mathcal{J}} \mathcal{C}(J)_{\geq 0}$ with the intersection $(\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{aperf}} \cap \mathrm{QCoh}((\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{cn}}$, and Lemma 5.2.8 allows us to identify $\varprojlim_{J \in \mathcal{J}} \mathcal{C}(J)_{\leq 0}$ with $\mathrm{QCoh}((\mathfrak{X}_U)^{\wedge}_{K_U})^{\mathrm{aperf}}_{\leq 0}$. □

Remark 5.2.9. Let $\mathfrak{X}$ be a locally Noetherian spectral Deligne-Mumford stack, and let $K \subseteq |\mathfrak{X}|$ be a closed subset. Then the t-structure on $\mathrm{QCoh}(\mathfrak{X}^{\wedge}_K)^{\mathrm{aperf}}$ described in Proposition 5.2.4 is left complete. To prove this, we may work locally on $\mathfrak{X}$: we may therefore suppose that $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} R$ for some Noetherian $\mathbb{E}_\infty$-ring R. Let $I \subseteq \pi_0 R$ be an ideal defining the closed subset $K \subseteq |\mathfrak{X}| \simeq \mathrm{Spec}^{\mathrm{Z}}\, \pi_0 R$, and let $A = R^{\wedge}_I$ denote the I-completion of R. Then Corollary 5.1.14 gives a t-exact identification of $\mathrm{QCoh}(\mathfrak{X}^{\wedge}_K)^{\mathrm{aperf}}$ with the ∞-category of almost perfect A-modules, which is evidently left complete (see Proposition A.7.2.5.17).

Note that $\mathrm{QCoh}(\mathfrak{X}^{\wedge}_K)^{\mathrm{aperf}}$ is never right complete (unless the set K is empty). However, it is right bounded (see §A.1.2.1) when K is quasi-compact.

Our next goal is to describe the heart of the t-structure appearing in Proposition 5.2.4. More generally, we will describe the intersection $\mathrm{QCoh}(\mathfrak{X}^{\wedge}_K)^{\mathrm{cn}} \cap \mathrm{QCoh}(\mathfrak{X}^{\wedge}_K)^{\mathrm{aperf}}_{\leq n}$, for every integer $n \geq 0$.

Notation 5.2.10. Let $\mathrm{Mod} = \mathrm{Mod}(\mathrm{Sp})$ denote the ∞-category of pairs (A, M), where A is an $\mathbb{E}_\infty$-ring and M is an A-module spectrum. Fix an integer $n \geq 0$, and let $\mathcal{C}$ denote the full subcategory of Mod spanned by those pairs (A, M), where A is connective and M is finitely n-presented over A. The forgetful functor $\mathcal{C} \to \mathrm{CAlg}^{\mathrm{cn}}$ is a coCartesian fibration, classified by a functor $\chi : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. The functor χ carries every connective $\mathbb{E}_\infty$-ring A to the full subcategory $\mathrm{Mod}_A^{n-fp} \subseteq \mathrm{Mod}_A$ spanned by the finitely n-presented A-modules. If $f : A \to B$ is a map of connective $\mathbb{E}_\infty$-rings, then the induced functor $\mathrm{Mod}_A^{n-fp} \to \mathrm{Mod}_B^{n-fp}$ is given by $M \mapsto \tau_{\leq n}(B \otimes_A M)$. We let $\mathrm{QCoh}^{n-fp} : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ denote a right Kan extension of χ along the Yoneda embedding $\mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op}$. More informally, if $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is a functor, then an object $\mathcal{F} \in \mathrm{QCoh}^{n-fp}(X)$ can be viewed as a functor which assigns to each point $\eta \in X(A)$ an A-module $\mathcal{F}(\eta) \in \mathrm{Mod}_A^{n-fp}$, which is functorial in the sense that if $f : A \to B$ is a map of connective $\mathbb{E}_\infty$-rings and η' denotes the image of η in $X(B)$, then we have a canonical equivalence $\mathcal{F}(\eta') \simeq \tau_{\leq n}(B \otimes_A \mathcal{F}(\eta))$. We refer the reader to §VIII.2.7 for a more detailed discussion of this construction.

Remark 5.2.11. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack representing a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Then the ∞-category $\mathrm{QCoh}^{n-fp}(\mathfrak{X})$ of Construction 2.4.3 can be identified with the ∞-category $\mathrm{QCoh}^{n-fp}(X)$ of Notation 5.2.10.

For any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, we have an evident functor $\mathrm{QCoh}(X)^{\mathrm{cn},\mathrm{aperf}} \to \mathrm{QCoh}(X)^{n-fp}$, which is given pointwise by the construction $(M \in \mathrm{Mod}_A) \mapsto \tau_{\leq n} M$.

We can now state the other main result of this section:

Proposition 5.2.12. *Let $\mathfrak{X}$ be a locally Noetherian spectral Deligne-Mumford stack and let $K \subseteq |\mathfrak{X}|$ be a closed subset. For every integer $n \geq 0$, the composite functor*

$$\mathrm{QCoh}(\mathfrak{X}^{\wedge}_K)^{\mathrm{cn}} \cap \mathrm{QCoh}(\mathfrak{X}^{\wedge}_K)^{\mathrm{aperf}}_{\leq n} \hookrightarrow \mathrm{QCoh}(\mathfrak{X}^{\vee}_K)^{\mathrm{cn}} \cap \mathrm{QCoh}(\mathfrak{X}^{\wedge}_K)^{\mathrm{aperf}} \to \mathrm{QCoh}^{n-fp}(\mathfrak{X}^{\wedge}_K)$$

is an equivalence of ∞-categories.

Remark 5.2.13. Let $K \subseteq |\mathfrak{X}|$ be as in Proposition 5.2.12. Taking $n = 0$, we deduce that giving an object $\mathcal{F}$ in the heart of the stable ∞-category $\mathrm{QCoh}(\mathfrak{X}^{\wedge}_K)^{\mathrm{aperf}}$ can be described by specifying, for every commutative ring R equipped with a map $\eta : \mathrm{Spec}^{\text{ét}} R \to \mathfrak{X}$ for which the induced map $\mathrm{Spec}^{\mathrm{Z}} R \to |\mathfrak{X}|$ factors through K, a finitely presented discrete R-module (which is given by $\pi_0 \eta^* \mathcal{F}$). If $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} A$ is affine and K is defined by an ideal $I \subseteq \pi_0 A$, then the heart of $\mathrm{QCoh}(\mathfrak{X}^{\wedge}_K)^{\mathrm{aperf}}$ can be identified with the abelian category of finitely generated discrete modules over the Noetherian ring $\pi_0 A^{\wedge}_I$. More generally, if $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a spectral algebraic space, then we can identify the heart of $\mathrm{QCoh}(\mathfrak{X}^{\wedge}_K)^{\mathrm{aperf}}$ with the abelian category of coherent sheaves on the formal completion of the ordinary algebraic space $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}})$ along K.

The proof of Proposition 5.2.12 will require some preliminaries.

Lemma 5.2.14. *Suppose we are given a tower of connective $\mathbb{E}_\infty$-rings*

$$\cdots \to A_2 \to A_1 \to A_0$$

having limit A, where each of the maps $\pi_0 A_{i+1} \to \pi_0 A_i$ is surjection whose kernel is a nilpotent ideal of $\pi_0 A_{i+1}$. For every integer $i \geq 0$, suppose we are given a connective A_i-module M_i, and if $i > 0$ a map of A_{i-1}-modules

$$\phi_i : A_{i-1} \otimes_{A_i} M_i \to M_{i-1}.$$

Let $n \geq 0$ be an integer. Suppose that each of the spectra $\mathrm{fib}(\phi_i)$ is n-connective, and that M_0 is perfect to order $(n-1)$ if $n > 0$. Then:

1. *If $n > 0$, then $M = \varprojlim M_i$ is perfect to order $(n-1)$, when regarded as an A-module.*
2. *For every integer i, let $\psi_i : A_i \otimes_A M \to M_i$ be the canonical map. Then $\mathrm{fib}(\psi_i)$ is n-connective.*

Proof. Since each M_i is connective and each of the maps $\pi_0 M_{i+1} \to \pi_0 M_i$ is surjective, we deduce that M is connective and that each of the maps $\pi_0 M \to \pi_0 M_i$ is surjective. This proves (2) in the case $n = 0$ (and condition (1) is automatic). We handle the general case using induction on n. Assume that $n > 0$. Then $\pi_0 M_0$ is finitely generated as a module over $\pi_0 A_0$. We may therefore choose finitely many elements $x_1, \ldots, x_k \in \pi_0 M$ whose images generator $\pi_0 M_0$. The elements x_i determine a map of A-modules $A^k \to M$, which in turn determines a compatible family of A_i-module maps $\theta_i : A_i^k \to M_i$. We claim that each of the maps θ_i is surjective on connected components. This holds by hypothesis when $i = 0$. If $i > 0$, then the image of θ_i generates $\pi_0 M_i / J \pi_0 M_i \simeq \pi_0 M_{i-1}$, where J denotes the kernel of $\pi_0 A_i \to \pi_0 A_{i-1}$, and therefore generates $\pi_0 M_i$ by Nakayama's lemma (since J is a nilpotent ideal).

For $i \geq 0$, form a fiber sequence

$$N_i \to A_i^k \to M_i,$$

so that each N_i is connective. If $n \geq 2$, then N_0 is perfect to order $n-2$ as an A_0-module. Moreover, we have maps $\phi'_i : A_{i-1} \otimes_{A_i} N_i \to N_{i-1}$ such that $\mathrm{fib}(\phi'_i) \simeq \mathrm{fib}(\phi_i)[-1]$ is $(n-1)$-connective for each i. Let

$N = \varprojlim N_i$. Applying the inductive hypothesis, we deduce that each of the maps $\psi'_i : A_i \otimes_A N \to N_i$ is $(n-1)$-connective. This proves (2), since $\mathrm{fib}(\psi_i) \simeq \mathrm{fib}(\psi'_i)[1]$. Note that N is connective, and is perfect to order $n-2$ if $n \geq 2$. Using the fiber sequence

$$N \to A^k \to M,$$

we deduce that M is perfect to order $n-1$. □

Lemma 5.2.15. *Let R be a Noetherian $\mathbb{E}_\infty$-ring, and let M be a connective R-module. If M is perfect to order n, then $\tau_{\leq n} M$ is almost perfect.*

Proof. According to Remark A.7.2.5.19, it will suffice to prove that $\pi_i M$ is a finitely generated module over $\pi_0 R$ for $0 \leq i \leq n$. We proceed by induction on n. When $n = 0$, the result is obvious. Assume therefore that $n > 0$. Then there exists a fiber sequence

$$N \to R^k \to M$$

where N is connective and perfect to order $(n-1)$. For $i \leq n$, we have an exact sequence

$$(\pi_i R)^k \to \pi_i M \to \pi_{i-1} N$$

of modules over $\pi_0 R$. Since $\pi_{i-1} N$ is finitely generated by the inductive hypothesis and $(\pi_i R)^k$ is finitely generated (by virtue of our assumption that R is Noetherian), we conclude that $\pi_i M$ is finitely generated, as desired. □

Lemma 5.2.16. *Let R be a Noetherian commutative ring, let $I \subseteq R$ be an ideal, and let M and N be discrete R-modules. Assume that N is I-nilpotent and that M is finitely generated. Then every class $\eta \in \mathrm{Ext}^p_R(M, N)$ vanishes when restricted to $\mathrm{Ext}^p_R(I^m M, N)$ for $m \gg 0$.*

Proof. We proceed by induction on p. If $p = 0$, the result is obvious. Otherwise, choose an injective map $u : N \to Q$, where Q is an injective R-module. Let $Q_0 \subseteq Q$ be the submodule consisting of elements which are annihilated by I^k for $k \gg 0$. We claim that Q_0 is injective. To prove this, it suffices to show that for every inclusion of finitely generated R-modules $P_0 \subseteq P$, every map $\alpha_0 : P_0 \to Q_0$ can be extended to a map $\alpha : P \to Q_0$. Since P_0 is finitely generated and Q_0 is I-nilpotent, there exists an integer $k \geq 0$ such that α_0 annihilates $I^k P_0$. Since R is Noetherian and P is finitely generated, the Artin-Rees lemma implies that there is an integer k' such that $I^{k'} P \cap P_0 \subseteq I^k P_0$. Then α_0 determines a map $\beta_0 : P_0/(I^{k'} P \cap P_0) \to Q_0$. Since Q is injective, we can extend β_0 to a map $\beta : P/I^{k'} P \to Q$. The map β evidently factors through Q_0, and the composite map

$$P \to P/I^{k'} P \xrightarrow{\beta} Q_0$$

is an extension of α_0.

Replacing Q by Q_0, we can assume that Q is I-nilpotent. We then have an exact sequence of I-nilpotent R-modules

$$0 \to N \to Q \to N' \to 0.$$

Since $p > 0$, the we have $\mathrm{Ext}^p_R(M, Q) \simeq 0$, so the boundary map $\partial : \mathrm{Ext}^{p-1}_R(M, N') \to \mathrm{Ext}^p_R(M, N)$ is surjective. Write $\eta = \partial(\overline{\eta})$ for some class $\overline{\eta} \in \mathrm{Ext}^{p-1}_R(M, N')$. Applying the inductive hypothesis, we deduce that $\overline{\eta}$ has trivial image in $\mathrm{Ext}^{p-1}_R(I^m M, N')$ for $m \gg 0$. It follows that the image of η in $\mathrm{Ext}^p_R(M, N)$ vanishes as well. □

Lemma 5.2.17. *Let R be a Noetherian commutative ring and let M be a finitely generated discrete R-module. Let $I \subseteq R$ be an ideal, and choose a tower*

$$\cdots \to A_3 \to A_2 \to A_1 \to A_0$$

of $\mathbb{E}_\infty$-algebras over R satisfying the requirements of Lemma 5.1.5. For every integer $n \geq 0$, the canonical map

$$\theta : \{\tau_{\leq n} A_i \otimes_R M\}_{i \geq 0} \to \{\pi_0(A_i \otimes_R M)\}_{i \geq 0} \simeq \{M/I^j M\}_{j \geq 0}$$

is an equivalence of Pro-*objects of the ∞-category* Mod_R.

Proof. Let $\mathcal{C}$ be the full subcategory of Mod_R spanned by those objects which are connective, almost perfect, n-truncated, and I-nilpotent. Then the domain and codomain of θ can be identified with Pro-objects of $\mathcal{C}$. It will therefore suffice to show that θ induces a homotopy equivalence

$$\alpha_N : \varinjlim_{j \geq 0} \mathrm{Map}_{\mathrm{Mod}_R}(M/I^j M, N) \to \varinjlim_{i \geq 0} \mathrm{Map}_{\mathrm{Mod}_R}(\tau_{\leq n}(A_i \otimes_R M), N)$$

for every object $N \in \mathcal{C}$. Since N is n-truncated, we can identify the codomain of α with

$$\varinjlim_{i \geq 0} \mathrm{Map}_{\mathrm{Mod}_R}(A_i \otimes_R M, N).$$

The collection of those objects $N \in \mathcal{C}$ for which α_N is a homotopy equivalence is closed under extensions; we may therefore suppose that $N = N_0[k]$, where N_0 is a finitely generated discrete R-module. Since N is I-nilpotent, N_0 is a module over the quotient ring R/I^k for $k \gg 0$. It follows that the codomain of α_N can be rewritten as $\varinjlim \mathrm{Map}_{\mathrm{Mod}_{R/I^k}}((R/I^k \otimes_R A_i) \otimes_R M, N)$. Since the projection map $\mathrm{Spf}\, R \times_{\mathrm{Spec}^{\text{ét}} R} \mathrm{Spec}^{\text{ét}}\, R/I^k \to \mathrm{Spec}^{\text{ét}}\, R/I^k$ is an equivalence, the tower $\{R/I^k \otimes_R A_i\}$ is equivalent to R/I^k in the ∞-category $\mathrm{Pro}(\mathrm{CAlg})$. It follows that we can identify the codomain of α_N with $\mathrm{Map}_{\mathrm{Mod}_{R/I^k}}(R/I^k \otimes_R M, N) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(M, N)$. To prove that α_N is a homotopy equivalence, it will suffice to show that the direct limit $\varinjlim_{j \geq 0} \mathrm{Map}_{\mathrm{Mod}_R}(I^j M, N)$ vanishes. For this, it suffices to show for every integer p, the abelian group $\varinjlim_{j \geq 0} \mathrm{Ext}^p_R(I^j M, N_0)$ vanishes. This follows immediately from Lemma 5.2.16. □

Notation 5.2.18. Let $\mathcal{A}b$ denote the category of abelian groups, and $\mathrm{Pro}(\mathcal{A}b)$ the category of Pro-objects of $\mathcal{A}b$. Let R be a commutative ring and $I \subseteq R$ an ideal. To any discrete R-module M, we can associate an object of $\mathrm{Pro}(\mathcal{A}b)$, represented by the inverse system $\{M/I^n M\}_{n \geq 0}$. Given an exact sequence of discrete R-modules

$$0 \to M' \xrightarrow{\phi} M \to M'' \to 0,$$

we obtain an exact sequence of Pro-objects

$$0 \to \{M'/\phi^{-1}(I^n M)\}_{n \geq 0} \to \{M/I^n M\}_{n \geq 0} \to \{M''/I^n M''\}_{n \geq 0} \to 0.$$

If R is Noetherian and M is a finitely generated R-module, then the Artin-Rees lemma allows us to identify the term on the left side with the Pro-abelian group $\{M'/I^n M'\}_{n \geq 0}$. It follows that we have an exact sequence

$$0 \to \{M'/I^n M'\}_{n \geq 0} \to \{M/I^n M\}_{n \geq 0} \to \{M''/I^n M''\}_{n \geq 0} \to 0$$

in the abelian category $\mathrm{Pro}(\mathcal{A}b)$. We can summarize the above discussion as follows: if R is a Noetherian commutative ring and $I \subseteq R$ is an ideal, then the construction $M \mapsto \{M/I^n M\}_{n \geq 0}$ determines an exact functor from the category of finitely generated R-modules to the category $\mathrm{Pro}(\mathcal{A}b)$.

Lemma 5.2.19. *Let R be a Noetherian $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and choose a tower of R-algebras*

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma 5.1.5. Let M be an almost perfect R-module. For every integer n, the canonical map

$$\theta^M_n : \{\pi_n(A_i \otimes_R M)\}_{i \geq 0} \to \{\mathrm{Tor}_0^{\pi_0 R}(\pi_0 A_i, \pi_n M)\}_{i \geq 0} \simeq \{(\pi_n M)/I^j(\pi_n M)\}_{j \geq 0}$$

is an isomorphism in the category $\mathrm{Pro}(\mathcal{A}b)$ *of* Pro-*abelian groups.*

Proof. Let us say that an R-module M is *n-good* if the map θ_n^M is an isomorphism, and that M is *good* if it is n-good for every integer n. Note that M is n-good if and only if the truncation $\tau_{\leq n} M$ is n-good. Consequently, to prove that every almost perfect R-module M is good, it will suffice to treat the case where M is truncated.

Suppose we are given a fiber sequence of R-modules

$$M' \to M \to M''.$$

We then obtain a commutative diagram

$$\begin{array}{ccc}
\{\pi_{n+1}(A_i \otimes_R M'')\}_{i \geq 0} & \xrightarrow{\theta_{n+1}^{M''}} & \{(\pi_{n+1} M'')/I^j(\pi_{n+1} M'')\}_{j \geq 0} \\
\downarrow & & \downarrow \\
\{\pi_n(A_i \otimes_R M')\}_{i \geq 0} & \xrightarrow{\theta_n^{M'}} & \{(\pi_n M')/I^j(\pi_n M')\}_{j \geq 0} \\
\downarrow & & \downarrow \\
\{\pi_n(A_i \otimes_R M)\}_{i \geq 0} & \xrightarrow{\theta_n^{M}} & \{(\pi_n M)/I^j(\pi_n M)\}_{j \geq 0} \\
\downarrow & & \downarrow \\
\{\pi_n(A_i \otimes_R M'')\}_{i \geq 0} & \xrightarrow{\theta_n^{M''}} & \{(\pi_n M'')/I^j(\pi_n M'')\}_{j \geq 0} \\
\downarrow & & \downarrow \\
\{\pi_{n-1}(A_i \otimes_R M')\}_{i \geq 0} & \xrightarrow{\theta_{n-1}^{M'}} & \{(\pi_{n-1} M')/I^j(\pi_{n-1} M')\}_{j \geq 0}
\end{array}$$

in the category $\mathrm{Pro}(\mathcal{A}b)$. The left column is obviously exact. If M, M', and M'' are almost perfect, then the discussion of Notation 5.2.18 shows that the right column is also exact. Applying the five lemma, we deduce that if M' and M'' are good, then M is also good. Consequently, the collection of almost perfect good R-modules is closed under extensions. To prove that every truncated almost perfect R-module M is good, it will suffice to treat the case where M is discrete. In this case, we can regard M as a module over the discrete commutative ring $\pi_0 R$. Replacing R by $\pi_0 R$ (and the tower $\{A_i\}_{i \geq 0}$ with $\{\pi_0 R \otimes_R A_i\}_{i \geq 0}$), we can assume that R is also discrete. In this case, the desired result follows immediately from Lemma 5.2.17. □

Lemma 5.2.20. *Let R be a Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to a ideal $I \subseteq \pi_0 R$. Then for every integer n, the canonical map*

$$f : \mathrm{Mod}_R^{n-fp} \to \mathrm{QCoh}^{n-fp}(\mathrm{Spf}\, R)$$

is an equivalence of ∞-categories.

Proof. Choose a tower of R-algebras

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma 5.1.5, so that $R \simeq \varprojlim A_i$ and $\mathrm{Spf}\, R \simeq \varinjlim \mathrm{Spec}^{\text{ét}} A_i$. Then the ∞-category $\mathrm{QCoh}^{n-fp}(\mathrm{Spf}\, R)$ can be identified with the limit of the tower $\{\mathrm{Mod}_{A_i}^{n-fp}\}_{i \geq 0}$. The functor f is given by the restriction of a functor $F : (\mathrm{Mod}_R)_{\leq n} \to \varprojlim (\mathrm{Mod}_{A_i})_{\leq n}$. The functor F admits a right adjoint G, which carries a compatible family of n-truncated A_i-modules $\{M_i\}$ to the limit $\varprojlim M_i$. If each M_i is connective, then the maps

$$\pi_0 M_i \to \mathrm{Tor}_0^{\pi_0 A_i}(\pi_0 A_{i-1}, \pi_0 M_i) \simeq \pi_0 M_{i-1}$$

are surjective, so that $G\{M_i\} = \varprojlim M_i$ is also connective. If, in addition, each M_i is almost perfect, then Lemma 5.2.14 implies that $G\{M_i\}$ is perfect to order n. Since $G\{M_i\}$ is n-truncated, we conclude that $G\{M_i\}$

is almost perfect (Lemma 5.2.15). It follows that the functor G restricts to a functor $g : \varprojlim \mathrm{Mod}_{A_i}^{n-fp} \to \mathrm{Mod}_R^{n-fp}$, so we adjoint functors

$$\mathrm{Mod}_R^{n-fp} \underset{g}{\overset{f}{\rightleftarrows}} \varprojlim \mathrm{Mod}_{A_i}^{n-fp}.$$

It follows immediately from Lemma 5.2.14 that the counit map $f \circ g \to \mathrm{id}$ is an equivalence. We wish to prove that the unit map $\mathrm{id} \to g \circ f$ is also an equivalence. In other words, we wish to show that if $M \in \mathrm{Mod}_R^{n-fp}$, then the map $u_M : M \to \varprojlim \tau_{\leq n}(A_i \otimes_R M)$ is an equivalence. Let K denote the fiber of u, and note that K is n-truncated. The proof of Lemma 5.1.10 shows that $M \simeq \varprojlim(A_i \otimes_R M)$, so that $K \simeq \varprojlim \tau_{\geq n+1}(A_i \otimes_R M)$. It follows that K is n-connective, and that $\pi_n K \simeq \varprojlim^1 \pi_{n+1}(A_i \otimes_R M)$. It will therefore suffice to show that the abelian group $\varprojlim^1 \pi_{n+1}(A_i \otimes_R M)$. This follows from the observation that the inverse system $\{\pi_{n+1}(A_i \otimes_R M)\}_{i \geq 0}$ is trivial as an object of $\mathrm{Pro}(\mathcal{A}b)$, because $\pi_{n+1} M \simeq 0$ (Lemma 5.2.19). □

Proof of Proposition 5.2.12. The assertion is local on $\mathfrak{X}$. We may therefore assume without loss of generality that $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} R$ for some Noetherian $\mathbb{E}_\infty$-ring R. Let $I \subseteq \pi_0 R$ be an ideal defining the closed subset $K \subseteq |\mathfrak{X}|$. The desired result now follows immediately by applying Lemma 5.2.20 to the completion $R_I^\wedge$. □

5.3 The Grothendieck Existence Theorem

Let R be a Noetherian ring which is complete with respect to an ideal I. Let X be an R-scheme, and let $\mathfrak{X}$ denote the formal completion of X along the closed subscheme $\mathrm{Spec}\, R/I \times_{\mathrm{Spec}\, R} X$. There is an evident restriction functor from the category of coherent sheaves on X to the category of coherent sheaves on $\mathfrak{X}$. If X is proper, then we have the following fundamental result (see Theorem 5.1.4 and Corollary 5.1.6 of [8]):

Theorem 5.3.1 (Grothendieck Existence Theorem)**.** *In the above situation, if X is proper, then the restriction functor induces an equivalence from the category of coherent sheaves on X to the category of coherent sheaves on $\mathfrak{X}$.*

Our goal in this section is to prove an analogue of Theorem 5.3.1 in the setting of spectral algebraic geometry. Our result can be stated as follows:

Theorem 5.3.2. *Let R be a Noetherian $\mathbb{E}_\infty$-ring which is I-complete for some ideal $I \subseteq \pi_0 R$. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a spectral algebraic space which is proper and locally almost of finite presentation over R, let $\mathfrak{X}^\wedge = \mathfrak{X} \times_{\mathrm{Spec}^{ét} R} \mathrm{Spf}\, R$, and let $f : \mathfrak{X}^\wedge \to \mathfrak{X}$ be the inclusion map. Then f induces a t-exact equivalence of ∞-categories*

$$f^* : \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}} \to \mathrm{QCoh}(\mathfrak{X}^\wedge)^{\mathrm{aperf}}.$$

We begin by proving that thepullback functor f^* in Theorem 5.3.2 is fully faithful. This does not require any Noetherian hypotheses on R.

Proposition 5.3.3. *Let R be a connective $\mathbb{E}_\infty$-ring which is I-complete for some finitely generated ideal $I \subseteq \pi_0 R$. Let $\mathfrak{X}$ be a spectral algebraic space which is proper and locally almost of finite presentation over $\mathrm{Spec}^{ét} R$, and let $\mathfrak{X}^\wedge = \mathrm{Spf}\, R \times_{\mathrm{Spec}^{ét} R} \mathfrak{X}$ denote the formal completion of $\mathfrak{X}$ along the closed substack determined by I, and let $f : \mathfrak{X}^\wedge \to \mathfrak{X}$ denote the inclusion map. Let $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}(\mathfrak{X})$, and assume that $\mathcal{G}$ is almost perfect. Then the canonical map map*

$$\mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{F}, \mathcal{G}) \to \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X}^\wedge)}(f^* \mathcal{F}, f^* \mathcal{G})$$

is a homotopy equivalence.

Corollary 5.3.4. *In the situation of Proposition 5.3.3, the pullback functor $f^* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{X}^\wedge)$ is fully faithful when restricted to the full subcategory $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}} \subseteq \mathrm{QCoh}(\mathfrak{X})$ spanned by the almost perfect objects.*

We first treat the following special case of Proposition 5.3.3

Lemma 5.3.5. *Let R be a connective $\mathbb{E}_\infty$-ring which is I-complete for some finitely generated ideal $I \subseteq \pi_0 R$. Let $\mathfrak{X}$ be a spectral algebraic space which is proper and locally almost of finite presentation over* $\operatorname{Spec}^{\acute{e}t} R$*, and let $f : \mathfrak{X} \times_{\operatorname{Spec}^{\acute{e}t} R} \operatorname{Spf} R \to \mathfrak{X}$ be the inclusion map. If $\mathcal{G} \in \operatorname{QCoh}(\mathfrak{X})$ is almost perfect, then the restriction map*

$$\Gamma(\mathfrak{X}; \mathcal{G}) \to \Gamma(\mathfrak{X} \times_{\operatorname{Spec}^{\acute{e}t} R} \operatorname{Spf} R; f^* \mathcal{G})$$

is an equivalence of spectra.

Proof. Choose a tower of R-algebras

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma 5.1.5. Then the functor $\mathfrak{X} \times_{\operatorname{Spec}^{\acute{e}t} R} \operatorname{Spf} R$ can be identified with the colimit of the sequence of functors represented by the spectral algebraic spaces $\mathfrak{X}_n = \mathfrak{X} \times_{\operatorname{Spec}^{\acute{e}t} R} \operatorname{Spec}^{\acute{e}t} A_n$. Using Proposition 1.5.14, we obtain equivalences

$$\Gamma(\mathfrak{X} \times_{\operatorname{Spec}^{\acute{e}t} R} \operatorname{Spf} R; f^* \mathcal{G}) \simeq \varprojlim_n \Gamma(\mathfrak{X}_n; \mathcal{G} \,|\, \mathfrak{X}_n) \simeq \varprojlim_n (\Gamma(\mathfrak{X}, \mathcal{G}) \otimes_R A_n).$$

Let $M = \Gamma(\mathfrak{X}; \mathcal{G}) \in \operatorname{Mod}_R$, so that $\theta_{\mathcal{F}}$ can be identified with the canonical map

$$\theta : M \to \varprojlim_n (M \otimes_R A_n).$$

Since $\mathcal{G}$ is almost perfect, Theorem 3.2.2 implies that M is almost perfect as an R-module. In particular, M is connective, so that the proof of Lemma 5.1.10 shows that θ exhibits $\varprojlim_n (M \otimes_R A_n)$ as the I-completion $M^\vee_I$ of M. Since R is I-complete and M is almost perfect, Proposition 4.3.8 guarantees that θ is an equivalence. □

Proof of Proposition 5.3.3. Let us first consider $\mathcal{G}$ as fixed, and regard the morphism

$$\theta_{\mathcal{F}} : \operatorname{Map}_{\operatorname{QCoh}(\mathfrak{X})}(\mathcal{F}, \mathcal{G}) \to \operatorname{Map}_{\operatorname{QCoh}(\mathfrak{X}^\wedge)}(f^* \mathcal{F}, f^* \mathcal{G})$$

as a functor of $\mathcal{F}$. This functor carries colimits in $\operatorname{QCoh}(\mathfrak{X})$ to limits in $\operatorname{Fun}(\Delta^1, \mathcal{S})$. Consequently, the collection of those objects $\mathcal{F} \in \operatorname{QCoh}(\mathfrak{X})$ for which $\theta_{\mathcal{F}}$ is a homotopy equivalence is closed under colimits. Using Theorem 1.5.10, we are reduced to proving that $\theta_{\mathcal{F}}$ is an equivalence in the special case where $\mathcal{F}$ is perfect. In this case, $\mathcal{F}$ is a dualizable object of $\operatorname{QCoh}(\mathfrak{X})$; let us denote its dual by $\mathcal{F}^\vee$. Replacing $\mathcal{G}$ by $\mathcal{F}^\vee \otimes \mathcal{G}$, we can reduce to the case where $\mathcal{F}$ is the structure sheaf of $\mathfrak{X}$. In this case, we can identify $\theta_{\mathcal{F}}$ with the restriction map

$$\Gamma(\mathfrak{X}; \mathcal{G}) \to \Gamma(\mathfrak{X}^\wedge; f^* \mathcal{G}).$$

The desired result now follows from Lemma 5.3.5. □

We also have the following relative version of Proposition 5.3.3:

Proposition 5.3.6. *Let R be a connective $\mathbb{E}_\infty$-ring which is I-complete for some finitely generated ideal $I \subseteq \pi_0 R$. Let $\mathfrak{X}$ be a spectral algebraic space which is proper and locally almost of finite presentation over* $\operatorname{Spec}^{\acute{e}t} R$*, let $\mathfrak{X}^\wedge = \mathfrak{X} \times_{\operatorname{Spec}^{\acute{e}t} R} \operatorname{Spf} R$ denote the formal completion of $\mathfrak{X}$ along the closed substack determined by I, and let $f : \mathfrak{X}^\wedge \to \mathfrak{X}$ denote the inclusion map. Let $\mathcal{C}$ be a locally proper quasi-coherent stack on $\mathfrak{X}$ (see Definition 3.3.6), and let $\mathcal{F}, \mathcal{G} \in \operatorname{QCoh}(\mathfrak{X}; \mathcal{C})$ be objects such that $\mathcal{G}$ is locally compact. Then the evident map*

$$\operatorname{Map}_{\operatorname{QCoh}(\mathfrak{X};\mathcal{C})}(\mathcal{F}, \mathcal{G}) \to \operatorname{Map}_{\operatorname{QCoh}(\mathfrak{X}^\wedge;\mathcal{C})}(f^* \mathcal{F}, f^* \mathcal{G})$$

is a homotopy equivalence. In particular, the functor f^* is fully faithful when restricted to locally compact objects of $\operatorname{QCoh}(\mathfrak{X}; \mathcal{C})$.

Proof. Let us regard $\mathcal{G}$ as fixed, and consider the morphism

$$\theta_{\mathcal{F}} : \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X};\mathcal{C})}(\mathcal{F}, \mathcal{G}) \to \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X}^{\wedge};\mathcal{C})}(f^* \mathcal{F}, f^* \mathcal{G})$$

as a functor of $\mathcal{F}$. This functor carries colimits in $\mathrm{QCoh}(\mathfrak{X};\mathcal{C})$ to limits in $\mathrm{Fun}(\Delta^1, \mathcal{S})$. Consequently, the collection of those objects $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X};\mathcal{C})$ for which $\theta_{\mathcal{F}}$ is a homotopy equivalence is closed under colimits. Using Theorem 1.5.10, we are reduced to proving that $\theta_{\mathcal{F}}$ is an equivalence in the special case where $\mathcal{F}$ is locally compact. Then $\mathcal{F}$ corepresents a map of quasi-coherent stacks $e_{\mathcal{F}} : \mathcal{C} \to \mathcal{Q}$ (where $\mathcal{Q}$ denotes the quasi-coherent stack given by $R \mapsto \mathrm{Mod}_R$); see Remark 3.3.21. Then $\theta_{\mathcal{F}}$ can be identified with the restriction map

$$\Gamma(\mathfrak{X}; e_{\mathcal{F}}(\mathcal{G})) \to \Gamma(\mathfrak{X} \times_{\mathrm{Spec}^{\text{ét}} R} \mathrm{Spf}\, R; f^* e_{\mathcal{F}}(\mathcal{G})).$$

Since $\mathcal{C}$ is locally proper and both $\mathcal{F}$ and $\mathcal{G}$ are locally compact, $e_{\mathcal{F}}(\mathcal{G})$ is a perfect object of $\mathrm{QCoh}(\mathfrak{X})$. In particular, $e_{\mathcal{F}}(\mathcal{G})$ is almost perfect, so that Lemma 5.3.5 implies that $\theta_{\mathcal{F}}$ is a homotopy equivalence. □

We now turn to the proof of Theorem 5.3.2 itself.

Proof of Theorem 5.3.2. It is clear that the pullback functor f^* is right t-exact. To verify that f^* is left t-exact, suppose that $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\leq 0}$; we wish to show that $\mathcal{F}$ is formally 0-truncated along the closed subset $K \subseteq |\,\mathfrak{X}\,|$ given by the inverse image of $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)/I \subseteq \mathrm{Spec}^{\mathrm{Z}}\, \pi_0 R$. Choose an étale map $u : \mathrm{Spec}^{\text{ét}} A \to \mathfrak{X}$, so that $u^* \mathcal{F}$ corresponds to a 0-truncated, almost perfect A-module M. Let J denote the image of I in $\pi_0 A$; we wish to show that the formal completion $M^{\vee}_J$ is 0-truncated and almost perfect over $A^{\vee}_J$. Since M is almost perfect over A, Proposition 4.3.8 furnishes an equivalence $M^{\vee}_J \simeq A^{\vee}_J \otimes_A M$. The desired result now follows from the fact that $A^{\vee}_J$ is flat over A (Corollary 4.3.9).

Since $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$ and $\mathrm{QCoh}(\mathfrak{X}^{\wedge})^{\mathrm{aperf}}$ are both left complete and right bounded (Remark 5.2.9), it will suffice to show that for every pair of integers m and n, the pullback functor f^* induces an equivalence of ∞-categories

$$\theta : \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\leq n} \cap \mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}_{\geq m} \to \mathrm{QCoh}(\mathfrak{X}^{\wedge})^{\mathrm{aperf}}_{\leq n} \cap \mathrm{Coh}(\mathfrak{X}^{\wedge})^{\mathrm{aperf}}_{\geq m}.$$

Proposition 5.3.3 implies that θ is fully faithful. To verify the essential surjectivity, we proceed by induction on the difference $n - m$. If $n - m < 0$, then the intersection $\mathrm{QCoh}(\mathfrak{X}^{\wedge})^{\mathrm{aperf}}_{\leq n} \cap \mathrm{Coh}(\mathfrak{X}^{\vee})^{\mathrm{aperf}}_{\geq m}$ consists of zero objects and there is nothing to prove. Let us therefore assume that $n - m \geq 0$ and that $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}^{\wedge})^{\mathrm{aperf}}_{\leq n} \cap \mathrm{Coh}(\mathfrak{X}^{\wedge})^{\mathrm{aperf}}_{\geq m}$. We have a fiber sequence

$$\tau_{\leq n-1}\, \mathcal{F} \to \mathcal{F} \to (\pi_n\, \mathcal{F})[n].$$

The inductive hypothesis implies that $\tau_{\leq n-1}\, \mathcal{F}$ belongs to the essential image of f^*. It will therefore suffice to show that $\pi_n\, \mathcal{F}$ belongs to the essential image of f^*. Note that $\pi_n\, \mathcal{F}$ can be identified with a coherent sheaf (in the sense of classical algebraic geometry on the formal algebraic space given by completing $(\mathcal{X}, \pi_0\, \mathcal{O}_{\mathcal{X}})$ along K (Remark 5.2.13). The classical Grothendieck existence theorem (for algebraic spaces; see [31]) implies that $\mathcal{F}$ is the restriction of a coherent sheaf on the algebraic space $(\mathcal{X}, \pi_0\, \mathcal{O}_{\mathcal{X}})$, which we can identify with an object belonging to the heart of $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}}$. □

5.4 Algebraizability of Formal Stacks

Let R be a Noetherian commutative ring which is complete with respect to an ideal $I \subseteq R$. Suppose we are given schemes X and Y which are of finite type over R, and let $\mathfrak{X}$ and $\mathfrak{Y}$ denote their formal completions along the closed subsets defined by I. Every map of R-schemes $f : X \to Y$ determines a map of formal schemes $\mathfrak{X} \to \mathfrak{Y}$. If X is proper over R and Y is separated, then the converse holds: every map $f_0 : \mathfrak{X} \to \mathfrak{Y}$ arises by formally completing a map $f : X \to Y$. This can be deduced by applying the Grothendieck existence theorem to the structure sheaf of $\mathfrak{X}$, regarded as a closed formal subscheme of the fiber product

$$\mathfrak{X} \times_{\mathrm{Spf}\, R} \mathfrak{Y}.$$

Our goal in this section is to prove an analogous result in the setting of spectral algebraic geometry. We can state our main result as follows:

Theorem 5.4.1. *Let R be a Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to an ideal $I \subseteq \pi_0 R$, let $\mathfrak{X}$ be a spectral algebraic space which is proper and locally almost of finite presentation over R, and let $\mathfrak{X}^\wedge = \mathfrak{X} \times_{\operatorname{Spec}^{\text{ét}} R} \operatorname{Spf} R$. Let $\mathfrak{Y}$ be a locally Noetherian geometric spectral Deligne-Mumford stack, and identify $\mathfrak{Y}$ with the functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ represented by $\mathfrak{Y}$.*

Then the restriction map

$$\operatorname{Map}_{\mathrm{Stk}}(\mathfrak{X}, \mathfrak{Y}) \to \operatorname{Map}_{\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathfrak{X}^\wedge, \mathfrak{Y})$$

is a homotopy equivalence.

Corollary 5.4.2. *Let R be a Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to an ideal $I \subseteq \pi_0 R$. Let $\mathrm{Stk}_{/ \operatorname{Spf} R}$ denote the full subcategory of $\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/ \operatorname{Spf} R}$ spanned those natural transformations of functors $X \to \operatorname{Spf} R$ which are representable by spectral Deligne-Mumford stacks, so that the construction $\mathfrak{X} \mapsto \mathfrak{X} \times_{\operatorname{Spec}^{\text{ét}} R} \operatorname{Spf} R$ defines a functor $\mathrm{Stk}_{/ \operatorname{Spec}^{\text{ét}} R} \to \mathrm{Stk}_{/ \operatorname{Spf} R}$, which we will denote by $\mathfrak{X} \mapsto \mathfrak{X}^\wedge$. Let $\mathfrak{X}, \mathfrak{Y} \in \mathrm{Stk}_{/ \operatorname{Spec}^{\text{ét}} R}$. Assume that $\mathfrak{X}$ is a spectral algebraic space which is proper and locally almost of finite presentation over $\operatorname{Spec}^{\text{ét}} R$, and that $\mathfrak{Y}$ is geometric. Then the restriction map*

$$\operatorname{Map}_{\mathrm{Stk}_{/ \operatorname{Spec}^{\text{ét}} R}}(\mathfrak{X}, \mathfrak{Y}) \to \operatorname{Map}_{\mathrm{Stk}_{/ \operatorname{Spf} R}}(\mathfrak{X}^\wedge, \mathfrak{Y}^\wedge)$$

is a homotopy equivalence.

Corollary 5.4.3. *Let R be a Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to an ideal $I \subseteq \pi_0 R$, and let $\phi : \mathrm{Stk}_{/ \operatorname{Spec}^{\text{ét}} R} \to \mathrm{Stk}_{/ \operatorname{Spf} R}$ be the pullback functor of Corollary 5.4.2. Then ϕ is fully faithful when restricted to the full subcategory of $\mathrm{Stk}_{/ \operatorname{Spec}^{\text{ét}} R}$ spanned by the spectral algebraic spaces which are proper and locally almost of finite presentation over R.*

Remark 5.4.4. Let R be a Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to an ideal $I \subseteq \pi_0 R$. Let $f : \mathfrak{X}^\wedge \to \operatorname{Spf} R$ be a natural transformation of functors which is representable by spectral algebraic spaces which are proper and locally almost of finite presentation. We will say that $\mathfrak{X}^\vee$ is *algebraizable* if it lies in the essential image of the functor ϕ of Corollary 5.4.3: that is, if $\mathfrak{X}^\wedge = \mathfrak{X} \times_{\operatorname{Spec}^{\text{ét}} R} \operatorname{Spf} R$ for some spectral algebraic space $\mathfrak{X}$ which is proper and locally almost of finite presentation over R. Corollary 5.4.3 implies that if $\mathfrak{X}$ exists, then it is unique (up to a contractible space of choices).

The proof of Theorem 5.4.1 will require some preliminaries.

Notation 5.4.5. Let $\mathcal{C}$ and $\mathcal{D}$ be presentable symmetric monoidal ∞-categories, and assume that the tensor product functors

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$$

preserve colimits separately in each variable. We let $\operatorname{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ denote the ∞-category of symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$, and $\operatorname{Fun}^{\mathrm{L}^{\otimes}}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\operatorname{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ spanned by those symmetric monoidal functors which preserve small colimits.

Lemma 5.4.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable symmetric monoidal ∞-categories. Assume that $\mathcal{C}$ and $\mathcal{D}$ are stable and equipped with t-structures for which the tensor product functors*

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$$

are right t-exact and preserve small colimits in each variable. Let $\mathcal{E} \subseteq \operatorname{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ be the full subcategory spanned by those symmetric monoidal functors $F : \mathcal{C} \to \mathcal{D}$ which are right t-exact and preserve small colimits. If the t-structure on $\mathcal{C}$ is right complete, then the restriction functor

$$\theta : \mathcal{E} \to \operatorname{Fun}^{\mathrm{L}^{\otimes}}(\mathcal{C}_{\geq 0}, \mathcal{D}_{\geq 0})$$

is an equivalence of ∞-categories.

Proof. It will suffice to show that, for every ∞-category K, the induced map

$$\mathrm{Map}_{\widehat{\mathcal{C}\mathrm{at}}_\infty}(K, \mathcal{E}) \to \mathrm{Map}_{\widehat{\mathcal{C}\mathrm{at}}_\infty}(K, \mathrm{Fun}^{\mathrm{L}^{\otimes}}(\mathcal{C}_{\geq 0}, \mathcal{D}_{\geq 0}))$$

is a homotopy equivalence. The collection of ∞-categories K which satisfy this condition is stable under colimits. We may therefore assume without loss of generality that K is small. Replacing $\mathcal{D}$ by $\mathrm{Fun}(K, \mathcal{D})$, we are reduced to proving that θ induces an equivalence $\mathcal{E}^{\simeq} \to \mathrm{Fun}^{\mathrm{L}^{\otimes}}(\mathcal{C}_{\geq 0}, \mathcal{D}_{\geq 0})$ on the level of the underlying Kan complexes.

Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category of presentable ∞-categories, endowed with the symmetric monoidal structure described in §A.6.3.1. Since $\mathcal{C}$ is stable, we have a symmetric monoidal functor $\mathrm{Sp} \to \mathcal{C}$, which induces a symmetric monoidal functor $\phi : \mathcal{C}_{\geq 0} \otimes \mathrm{Sp} \to \mathcal{C}$. The assumption that $\mathcal{C}$ is right complete implies that ϕ is an equivalence (that is, we can identify $\mathcal{C}$ with $\mathcal{C}_{\geq 0} \otimes \mathrm{Sp} \simeq \mathrm{Sp}(\mathcal{C}_{\geq 0}) \simeq \varprojlim \mathcal{C}_{\geq -n}$). Since $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})}(\mathrm{Sp}, \mathcal{D})$ is contractible, we deduce that the restriction map

$$\mathrm{Fun}^{\mathrm{L}^{\otimes}}(\mathcal{C}, \mathcal{D})^{\simeq} = \mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})}(\mathcal{C}_{\geq 0}, \mathcal{D}) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})}(\mathcal{C}_{\geq 0}, \mathcal{D}) = \mathrm{Fun}^{\otimes, L}(\mathcal{C}_{\geq 0}, \mathcal{D})^{\simeq}$$

is a homotopy equivalence. It now suffices to observe that under this homotopy equivalence, $\mathcal{E}^{\simeq}$ is the preimage of the summand $\mathrm{Fun}^{\mathrm{L}^{\otimes}}(\mathcal{C}_{\geq 0}, \mathcal{D}_{\geq 0})^{\simeq} \subseteq \mathrm{Fun}^{\mathrm{L}^{\otimes}}(\mathcal{C}_{\geq 0}, \mathcal{D})^{\simeq}$. □

Lemma 5.4.7. *Let $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a spectral Deligne-Mumford stack. Let $n \geq 0$ be an integer, and assume that $\mathfrak{Y}$ is $(n+1)$-quasi-compact. Then:*

(1) *If $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{Y})$ is finitely n-presented, then $\mathcal{F}$ is a compact object of $\mathrm{QCoh}(\mathfrak{X})_{\leq n}$.*

(2) *The inclusion $\mathrm{QCoh}^{n-fp}(\mathfrak{Y}) \hookrightarrow \mathrm{QCoh}(\mathfrak{Y})$ extends to a fully faithful embedding*

$$\theta : \mathrm{Ind}(\mathrm{QCoh}^{n-fp}(\mathfrak{Y})) \to \mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}.$$

(3) *Assume that $\mathfrak{Y}$ is locally Noetherian and $(n+2)$-quasi-compact. Then θ is an equivalence of ∞-categories.*

Proof. We first prove (1). We will prove the following:

$(*)$ Let $U \in \mathcal{Y}$ and let $\mathfrak{Y}_U = (\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}} | U)$. Suppose that we are given a filtered diagram $\{\mathcal{G}_\alpha\}$ in $\mathrm{QCoh}(\mathfrak{Y})_{\leq n}$ having colimit $\mathcal{G}$. If U is m-coherent for some integer $m \geq 0$, then the canonical map

$$\varinjlim_\alpha \mathrm{Ext}^p_{\mathrm{QCoh}(\mathfrak{Y}_U)}(\mathcal{F} | U, \mathcal{G}_\alpha | U) \to \mathrm{Ext}^p_{\mathrm{QCoh}(\mathfrak{Y}_U)}(\mathcal{F} | U, \mathcal{G} | U)$$

is an isomorphism for $p < m - n$ and an injection when $p = m - n$.

Assertion (1) follows from $(*)$ by taking U to be the final object of $\mathcal{Y}$ and $m = n+1$. We will prove $(*)$ by induction on m. We observe that the conclusion of $(*)$ holds when $m = -1$ for *every* object $U \in \mathcal{Y}$, since $\mathrm{Ext}^p_{\mathrm{QCoh}(\mathfrak{Y}_U)}(\mathcal{F} | U, \mathcal{F}' | U) \simeq 0$ for $p < -n$ provided that $\mathcal{F}$ is connective and $\mathcal{F}'$ is n-truncated. To handle the inductive step, we invoke the assumption that U is m-coherent to choose an effective epimorphism $u : V_0 \to U$, where V_0 is affine. Let $V_\bullet$ be the Čech nerve of u. If $m > 0$, then each V_i is $(m-1)$-coherent. For every object $\mathcal{F}' \in \mathrm{QCoh}(\mathfrak{Y})$, we have a spectral sequence $\{E_r^{p,q}\}_{r \geq 1}$ with $E_1^{p,q} = \mathrm{Ext}^p_{\mathrm{QCoh}(\mathfrak{Y}_{V_q})}(\mathcal{F} | V_q, \mathcal{F}' | V_q)$, which converges to $\mathrm{Ext}^{p+q}_{\mathrm{QCoh}(\mathfrak{Y}_U)}(\mathcal{F} | U, \mathcal{F}' | U)$ provided that $\mathcal{F}'$ is truncated. Consequently, to prove assertion $(*)$, it will suffice to show that the maps

$$\varinjlim_\alpha \mathrm{Ext}^p_{\mathrm{QCoh}(\mathfrak{Y}_{V_q})}(\mathcal{F} | V_q, \mathcal{G}_\alpha | V_q) \to \mathrm{Ext}^p_{\mathrm{QCoh}(\mathfrak{Y}_{V_q})}(\mathcal{F} | V_q, \mathcal{G} | V_q)$$

isomorphisms for $p + q < m - n$ and injections when $p + q = m - n$. If $q > 0$, this follows from the inductive hypothesis. It therefore suffices to treat the case $q = 0$. That is, we may replace U by V_0 and thereby reduce to the case where U is affine. In this case, the desired result follows from our assumption that $\mathcal{F}$ is finitely n-presented.

Assertion (2) follows from (1) and Proposition T.5.3.5.11. Let us prove (3). The proof proceeds by induction on n. We begin with the case $n = 0$, using an argument of Deligne. Let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{Y})$ be discrete. Since $\mathfrak{Y}$ is 2-quasi-compact, we can choose an étale surjection $f : \mathrm{Spec}^{\text{ét}} R \to \mathfrak{Y}$, where f is quasi-compact and quasi-separated. We can identify the pullback $f^* \mathcal{F} \in \mathrm{QCoh}(\mathrm{Spec}^{\text{ét}} R) \simeq \mathrm{Mod}_R$ with a discrete R-module M. Write $M = \varinjlim M_\alpha$, where each M_α is a finitely presented R-module. Theorem VIII.2.5.18 implies that $\mathcal{F}' = \pi_0 f_* M$ is quasi-coherent, and the proof of Theorem VIII.2.5.18 shows that $\mathcal{F}' \simeq \varinjlim_\alpha \mathcal{F}'_\alpha$, where $\mathcal{F}'_\alpha = \pi_0 f_* M_\alpha$. For each index α, let $\mathcal{F}_\alpha$ denote the fiber product $\mathcal{F}'_\alpha \times_{\mathcal{F}'} \mathcal{F}$ in the abelian category $\mathrm{QCoh}(\mathfrak{Y})^\heartsuit$. Then $\mathcal{F} \simeq \varinjlim \mathcal{F}_\alpha$. Moreover, for each index α the map $f^* \mathcal{F}_\alpha \to f^* \mathcal{F}$ (which is a monomorphism in the abelian category $\mathrm{QCoh}(\mathrm{Spec}^{\text{ét}} R)^\heartsuit$) factors through M_α. Since $\mathfrak{Y}$ is locally Noetherian, R is Noetherian, so that $f^* \mathcal{F}_\alpha$ corresponds (under the equivalence $\mathrm{QCoh}(\mathrm{Spec}^{\text{ét}} R) \simeq \mathrm{Mod}_R$) to a finitely presented (discrete) R-module. Since f is an étale surjection, we deduce that $\mathcal{F}_\alpha \in \mathrm{QCoh}^{0-fp}(\mathfrak{Y})$, so that $\mathcal{F}$ belongs to the essential image of θ.

We now treat the case $n > 0$. Let $\mathcal{C} \subseteq \mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}$ denote the essential image of θ, so that $\mathcal{C}$ contains all finitely n-presented quasi-coherent sheaves and is stable under filtered colimits. Let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}$; we wish to show that $\mathcal{F} \in \mathcal{C}$. Choose a fiber sequence

$$\mathcal{F}'[1] \to \mathcal{F} \to \mathcal{F}''$$

where $\mathcal{F}''$ is discrete and $\mathcal{F}' \in \mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n-1}$. The argument above shows that we can write $\mathcal{F}''$ as a filtered colimit $\varinjlim \mathcal{F}''_\alpha$, where each $\mathcal{F}''_\alpha$ is finitely 0-presented. Then $\mathcal{F} \simeq \varinjlim (\mathcal{F} \times_{\mathcal{F}''} \mathcal{F}''_\alpha)$. Since $\mathcal{C}$ is closed under filtered colimits, it will suffice to show that each fiber product $\mathcal{F} \times_{\mathcal{F}''} \mathcal{F}''_\alpha$ belongs to $\mathcal{C}$. Replacing $\mathcal{F}$ by $\mathcal{F} \times_{\mathcal{F}''} \mathcal{F}''_\alpha$, we can reduce to the case where $\mathcal{F}''$ is finitely 0-presented. Applying the inductive hypothesis, we can write $\mathcal{F}'$ as the colimit of a diagram $\{\mathcal{F}'_\beta\}_{\beta \in B}$ indexed by some filtered partially ordered set B, where each $\mathcal{F}'_\beta$ is finitely $(n-1)$-presented. The above fiber sequence is classified by a map $v : \mathcal{F}'' \to \varinjlim_{\beta \in B} \mathcal{F}'_\beta[2]$. Since $\mathfrak{Y}$ is locally Noetherian, the sheaf $\mathcal{F}''$ is finitely $(n+1)$-presented. Because $\mathfrak{Y}$ is $(n+2)$-quasi-compact, assertion (1) implies that v factors through a map $v_0 : \mathcal{F}'' \to \mathcal{F}'_{\beta_0}[2]$ for some $\beta_0 \in B$. For $\beta \geq \beta_0$ in B, let v_β be the induced map $\mathcal{F}'' \to \mathcal{F}'_\beta[2]$. Then $\mathcal{F} \simeq \varinjlim_{\beta \geq \beta_0} \mathrm{fib}(v_\beta)$. Since each fiber $\mathrm{fib}(v_\beta)$ is finitely n-presented, we conclude that $\mathcal{F} \in \mathcal{C}$ as desired. □

Lemma 5.4.8. *Let $\mathfrak{X}$ be a locally Noetherian geometric spectral Deligne-Mumford stack, and let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}$. The following conditions are equivalent:*

1. *The sheaf $\mathcal{F}$ is flat.*
2. *For every object $\mathcal{F}' \in \mathrm{QCoh}(\mathfrak{X})^\heartsuit$, the tensor product $\mathcal{F} \otimes \mathcal{F}'$ belongs to $\mathrm{QCoh}(\mathfrak{X})^\heartsuit$.*
3. *For every object $\mathcal{F}' \in \mathrm{Coh}(\mathfrak{X})^\heartsuit$, the tensor product $\mathcal{F} \otimes \mathcal{F}'$ belongs to $\mathrm{QCoh}(\mathfrak{X})^\heartsuit$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. The implication (3) $\Rightarrow$ (2) follows from Lemma 5.4.7 (which guarantees that every object of $\mathrm{QCoh}(\mathfrak{X})^\heartsuit$ can be obtained as a filtered colimit of objects of $\mathrm{Coh}(\mathfrak{X})^\heartsuit$). We will complete the proof by showing that (2) $\Rightarrow$ (1). Assume that $\mathcal{F}$ satisfies condition (2), let $u : \mathrm{Spec}^{\text{ét}} R \to \mathfrak{X}$ be an étale map, and let $M \in \mathrm{Mod}_R$ be the R-module corresponding to $u^* \mathcal{F}$. We wish to show that M is flat. Equivalently, we wish to show that $M \otimes_R N$ is discrete, whenever N is a discrete R-module. It is clear that $M \otimes_R N$ is connective (since M, N, and R are connective); it will therefore suffice to show that $M \otimes_R N$ is 0-truncated. As a spectrum, we can identify $M \otimes_R N$ with the global sections of the coherent sheaf $\mathcal{F} \otimes u_* N$ on $\mathfrak{X}$. It will therefore suffice to show that $\mathcal{F} \otimes u_* N$ belongs to $\mathrm{QCoh}(\mathfrak{X})^\heartsuit$. Since $\mathfrak{X}$ is geometric, the morphism u is affine. It follows that the pushforward functor u_* is t-exact. In particular, $u_* N$ belongs to the heart of $\mathrm{QCoh}(\mathfrak{X})$, so that the desired result follows from (2). □

Proof of Theorem 5.4.1. Choose a tower of $\mathbb{E}_\infty$-algebras

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma 5.1.5 and define $\mathfrak{X}_i = \mathfrak{X} \times_{\operatorname{Spec}^{\text{ét}} R} \operatorname{Spec}^{\text{ét}} A_i$ for $i \geq 0$. Then we can write $\mathfrak{X}^\wedge = \varinjlim_{i \geq 0} \mathfrak{X}_i$. We wish to show that the canonical map

$$\operatorname{Map}_{\operatorname{Stk}}(\mathfrak{X}, \mathfrak{Y}) \to \varprojlim_{i \geq 0} \operatorname{Map}_{\operatorname{Stk}}(\mathfrak{X}_i, \mathfrak{Y})$$

is an equivalence. For every spectral Deligne-Mumford stack $\mathfrak{Z}$, let $\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y}), \operatorname{QCoh}(\mathfrak{Z}))'$ denote the full subcategory of $\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y}), \operatorname{QCoh}(\mathfrak{Z}))$ spanned by those symmetric monoidal functors $F : \operatorname{QCoh}(\mathfrak{Y}) \to \operatorname{QCoh}(\mathfrak{Z})$ which are right t-exact, preserve small colimits, carry flat objects to flat objects, and carry almost perfect objects to almost perfect objects. For every map $f : \mathfrak{Z} \to \mathfrak{Y}$, we can regard the pullback functor $f^* : \operatorname{QCoh}(\mathfrak{Y}) \to \operatorname{QCoh}(\mathfrak{Z})$ as an object of $\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y}), \operatorname{QCoh}(\mathfrak{Z}))'$. Since $\mathfrak{Y}$ is geometric, Theorem VIII.3.4.2 implies that the construction $f \mapsto f^*$ induces an equivalence of ∞-categories $\operatorname{Map}_{\operatorname{Stk}}(\mathfrak{Z}, \mathfrak{Y}) \to \operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y}), \operatorname{QCoh}(\mathfrak{Z}))'$. It will therefore suffice to show that the functor

$$\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y}), \operatorname{QCoh}(\mathfrak{X}))' \to \varprojlim \operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y}), \operatorname{QCoh}(\mathfrak{X}_i))$$

is an equivalence of ∞-categories.

For every spectral Deligne-Mumford stack $\mathfrak{Z}$, let $\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y}), \operatorname{QCoh}(\mathfrak{Z}))''$ denote the full subcategory of $\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y}), \operatorname{QCoh}(\mathfrak{Z}))$ spanned by those symmetric monoidal functors which are right t-exact, preserve small colimits, and preserve almost perfect objects. We will prove the following assertions:

- (a) The functor
$$\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y}), \operatorname{QCoh}(\mathfrak{X}))'' \to \varprojlim \operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y}), \operatorname{QCoh}(\mathfrak{X}_i))''$$
is an equivalence of ∞-categories.
- (b) Let $F : \operatorname{QCoh}(\mathfrak{Y}) \to \operatorname{QCoh}(\mathfrak{X})$ be a symmetric monoidal functor which is right t-exact, preserves small colimits, and carries almost perfect objects to almost perfect objects. Suppose that, for every flat sheaf $\mathcal{F} \in \operatorname{QCoh}(\mathfrak{Y})$, the image of $F(\mathcal{F})$ in $\operatorname{QCoh}(\mathfrak{X}^\vee)$ is flat. Then for every flat sheaf $\mathcal{F} \in \operatorname{QCoh}(\mathfrak{Y})$, $F(\mathcal{F})$ is flat.

We begin with (a). For every spectral Deligne-Mumford stack $\mathfrak{Z}$, let $\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y})^{\text{cn}}, \operatorname{QCoh}(\mathfrak{Z})^{\text{cn}})''$ denote the full subcategory of $\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y})^{\text{cn}}, \operatorname{QCoh}(\mathfrak{Z})^{\text{cn}})$ spanned by those symmetric monoidal functors $F : \operatorname{QCoh}(\mathfrak{Y})^{\text{cn}} \to \operatorname{QCoh}(\mathfrak{Z})^{\text{cn}}$ which preserve small colimits and almost perfect objects. We have an evident commutative diagra

$$\begin{array}{ccc}
\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y}), \operatorname{QCoh}(\mathfrak{X}))'' & \longrightarrow & \varprojlim \operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y}), \operatorname{QCoh}(\mathfrak{X}_i))'' \\
\downarrow & & \downarrow \\
\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y})^{\text{cn}}, \operatorname{QCoh}(\mathfrak{X})^{\text{cn}})'' & \longrightarrow & \varprojlim \operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y})^{\text{cn}}, \operatorname{QCoh}(\mathfrak{X}_i)^{\text{cn}})''.
\end{array}$$

Lemma 5.4.6 implies that the vertical morphisms in this diagram are equivalences. We are therefore reduced to proving that the lower horizontal map is an equivalence of ∞-categories.

For every spectral Deligne-Mumford stack $\mathfrak{Z}$ and every integer n, the ∞-category $\operatorname{QCoh}(\mathfrak{Z})^{\text{cn}}_{\leq n}$ is a localization of $\operatorname{QCoh}(\mathfrak{Z})^{\text{cn}}$ which inherits a symmetric monoidal structure. Let $\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y})^{\text{cn}}, \operatorname{QCoh}(\mathfrak{Z})^{\text{cn}}_{\leq n})''$ denote the full subcategory of $\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y})^{\text{cn}}, \operatorname{QCoh}(\mathfrak{Z})^{\text{cn}}_{\leq n})$ spanned by those functors which preserve small colimits, and carry almost perfect objects of $\operatorname{QCoh}(\mathfrak{Y})^{\text{cn}}$ to finitely n-presented objects of $\operatorname{QCoh}(\mathfrak{Z})^{\text{cn}}_{\leq n}$. Since the t-structure on $\operatorname{QCoh}(\mathfrak{Z})$ is left complete (Proposition VIII.2.3.18), we have $\operatorname{QCoh}(\mathfrak{Z})^{\text{cn}} \simeq \varprojlim \operatorname{QCoh}(\mathfrak{Z})^{\text{cn}}_{\leq n}$. It will therefore suffice to show that each of the functors

$$\operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y})^{\text{cn}}, \operatorname{QCoh}(\mathfrak{X})^{\text{cn}}_{\leq n})'' \to \varprojlim \operatorname{Fun}^{\otimes}(\operatorname{QCoh}(\mathfrak{Y})^{\text{cn}}, \operatorname{QCoh}(\mathfrak{X}_i)^{\text{cn}}_{\leq n})''$$

is an equivalence of ∞-categories.

Note that each of the presentable ∞-categories $\mathrm{QCoh}(\mathfrak{Z})^{\mathrm{cn}}_{\leq n}$ is equivalent to an $(n+1)$-category, and therefore has the structure of a module over the presentable ∞-category $\tau_{\leq n}\,\mathcal{S}$ of n-truncated spaces (Proposition A.6.3.2.13). Consequently, every colimit-preserving symmetric monoidal functor $\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}} \to \mathrm{QCoh}(\mathfrak{Z})^{\mathrm{cn}}_{\leq n}$ factors (uniquely) through $\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}} \otimes \tau_{\leq n}\,\mathcal{S} \simeq \mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}$. Let $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}, \mathrm{QCoh}(\mathfrak{Z})^{\mathrm{cn}}_{\leq n})''$ denote the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}, \mathrm{QCoh}(\mathfrak{Z})^{\mathrm{cn}}_{\leq n})$ spanned by those functors which preserve small colimits and carry almost perfect objects of $\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}$ to finitely n-presented objects of $\mathrm{QCoh}(\mathfrak{Z})^{\mathrm{cn}}_{\leq n}$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}, \mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}_{\leq n})'' & \longrightarrow & \varprojlim \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}, \mathrm{QCoh}(\mathfrak{X}_i)^{\mathrm{cn}}_{\leq n})'' \\ \downarrow & & \downarrow \\ \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}, \mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}_{\leq n})'' & \longrightarrow & \varprojlim \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}, \mathrm{QCoh}(\mathfrak{X}_i)^{\mathrm{cn}}_{\leq n})'' \end{array}$$

where the vertical maps are equivalences of ∞-categories. It will therefore suffice to show that the upper horizontal map is an equivalence.

Lemma 5.4.7 gives an equivalence of symmetric monoidal ∞-categories

$$\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n} \simeq \mathrm{Ind}(\mathrm{QCoh}^{n-fp}(\mathfrak{Y})).$$

It follows that for every spectral Deligne-Mumford stack $\mathfrak{Z}$, the canonical map

$$\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}, \mathrm{QCoh}(\mathfrak{Z})^{\mathrm{cn}}_{\leq n})'' \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}^{n-fp}(\mathfrak{Y}), \mathrm{QCoh}^{n-fp}(\mathfrak{Z}))$$

is a fully faithful embedding, whose essential image is the full subcategory

$$\mathrm{Fun}^{\otimes}(\mathrm{QCoh}^{n-fp}(\mathfrak{Y}), \mathrm{QCoh}^{n-fp}(\mathfrak{Z}))'' \subseteq \mathrm{Fun}^{\otimes}(\mathrm{QCoh}^{n-fp}(\mathfrak{Y}), \mathrm{QCoh}^{n-fp}(\mathfrak{Z}))$$

spanned by those symmetric monoidal functors $F : \mathrm{QCoh}^{n-fp}(\mathfrak{Y}) \to \mathrm{QCoh}^{n-fp}(\mathfrak{Z})$ which preserve finite colimits. We are therefore reduced to proving that the functor

$$\mathrm{Fun}^{\otimes}(\mathrm{QCoh}^{n-fp}(\mathfrak{Y}), \mathrm{QCoh}^{n-fp}(\mathfrak{X}))'' \to \varprojlim \mathrm{Fun}^{\otimes}(\mathrm{QCoh}^{n-fp}(\mathfrak{Y}), \mathrm{QCoh}^{n-fp}(\mathfrak{X}_i)).$$

is an equivalence of ∞-categories. For this, it suffices to show that the functor $\theta : \mathrm{QCoh}^{n-fp}(\mathfrak{X}) \to \varprojlim_i \mathrm{QCoh}^{n-fp}(\mathfrak{X}_i)$ is an equivalence of ∞-categories. Proposition 5.2.12 allows us to identify the ∞-category $\varprojlim_i \mathrm{QCoh}^{n-fp}(\mathfrak{X}_i)$ with $\mathrm{QCoh}(\mathfrak{X}^{\wedge})^{\mathrm{aperf}}_{\leq n} \cap \mathrm{QCoh}(\mathfrak{X}^{\wedge})^{\mathrm{cn}}$. It follows that θ is given by the restriction of the t-exact equivalence $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{aperf}} \to \mathrm{QCoh}(\mathfrak{X}^{\wedge})^{\mathrm{aperf}}$ of Theorem 5.3.2. This completes the proof of (a).

We now prove (b). Assume that $F : \mathrm{QCoh}(\mathfrak{Y}) \to \mathrm{QCoh}(\mathfrak{X})$ is a symmetric monoidal functor which is right t-exact, preserves small colimits, and preserves almost perfect objects. Assume further that each of the composite functors $\mathrm{QCoh}(\mathfrak{Y}) \to \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{X}_i)$ preserves flat objects. Using Theorem VIII.3.4.2, we deduce that each of these composite functors is given by pullback along a map of spectral Deligne-Mumford stacks $f_i : \mathfrak{X}_i \to \mathfrak{Y}$. Together, these functors determine a map $f : \mathfrak{X}^{\wedge} \to \mathfrak{Y}$. Let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{Y})$ be flat; we wish to show that $F(\mathcal{F}) \in \mathrm{QCoh}(\mathfrak{X})$ is flat. According to Lemma 5.4.8, it will suffice to show that if $\mathcal{F}' \in \mathrm{Coh}(\mathfrak{X})^{\heartsuit}$, then $F(\mathcal{F}) \otimes \mathcal{F}' = \varinjlim_{\alpha} F(\mathcal{F}_{\alpha}) \otimes \mathcal{F}'$ belongs to $\mathrm{QCoh}(\mathfrak{X})^{\heartsuit}$. Fix an integer $m > 0$; we wish to show that $\pi_m(F(\mathcal{F}) \otimes \mathcal{F}')$ is trivial. Since F is right t-exact, the map $F(\mathcal{F}) \to F(\tau_{\leq m}\,\mathcal{F})$ is $(m+1)$-connective. Since $\mathcal{F}'$ is connective, the map $F(\mathcal{F}) \otimes \mathcal{F}' \to F(\tau_{\leq m}\,\mathcal{F}) \otimes \mathcal{F}'$ is $(m+1)$-connective. In particular, we obtain an isomorphism

$$\pi_m(F(\mathcal{F}) \otimes \mathcal{F}') \to \pi_m(F(\tau_{\leq m}\,\mathcal{F}) \otimes \mathcal{F}').$$

We will prove that $\pi_m(F(\tau_{\leq m}\,\mathcal{F}) \otimes \mathcal{F}') \simeq 0$.

Using Lemma 5.4.7, we can write the truncation $\tau_{\leq m}\,\mathcal{F}$ as the colimit of a diagram $\{\mathcal{F}_{\alpha}\}_{\alpha \in A}$ with values in $\mathrm{QCoh}^{m-fp}(\mathfrak{Y})$, indexed by a filtered partially ordered set A. Since the t-structure on $\mathrm{QCoh}(\mathfrak{X})$ is compatible with filtered colimits, we have $\pi_m(F(\tau_{\leq m}\mathcal{F}) \otimes \mathcal{F}') = \varinjlim \pi_m(F(\mathcal{F}_\alpha) \otimes \mathcal{F}')$. It will therefore suffice to show that for every element $\alpha \in A$, the map

$$\theta : \pi_m(F(\mathcal{F}_\alpha) \otimes \mathcal{F}') \to \pi_m(F(\tau_{\leq m}\mathcal{F}) \otimes \mathcal{F}')$$

vanishes.

Choose an étale surjection $\mathrm{Spec}^{\text{ét}} B \to \mathfrak{X}$, and let $J \subseteq \pi_0 B$ be the ideal generated by the image of I. Then f determines a map of spectral Deligne-Mumford stacks $\mathrm{Spec}^{\text{ét}}(\pi_0 B)/J \to \mathfrak{Y}$. Choose an étale surjection $u : \mathfrak{U} \to \mathfrak{Y}$, where $\mathfrak{U}$ is affine. Since $\mathfrak{Y}$ is geometric, the fiber product $\mathrm{Spec}^{\text{ét}}(\pi_0 B)/J \times_{\mathfrak{Y}} \mathfrak{U}$ is affine, hence of the form $\mathrm{Spec}^{\text{ét}} B'_0$ for some étale $(\pi_0 B)/J$-algebra B'_0. Using the structure theory of étale morphisms of $\mathbb{E}_\infty$-rings (Proposition VII.8.10), we can write $B'_0 = B' \otimes_B (\pi_0 B)/J$ for some étale B-algebra B'. We then have an étale map $v : \mathrm{Spec}^{\text{ét}} B' \to \mathfrak{X}$. Let J' be the ideal in $\pi_0 B'$ generated by I and $\mathrm{Spf}\, B'$ the associated formal scheme. By construction, the composite map $\mathrm{Spf}\, B' \to \mathfrak{X}^\wedge \xrightarrow{f} \mathfrak{Y}$ factors through u.

Write $\mathfrak{U} = \mathrm{Spec}^{\text{ét}} C$ and identify $u^* \mathcal{F}$ with a flat C-module M. Using Theorem A.7.2.2.15, we can write M as the colimit of a diagram $\{M_{\alpha'}\}_{\alpha' \in A'}$ indexed by a filtered partially ordered set A', where each $M_{\alpha'}$ is a free C-module of finite rank. Then $u^* \tau_{\leq n} \mathcal{F} \simeq \varinjlim \tau_{\leq n} M_{\alpha'}$. Since $u^* \mathcal{F}_\alpha$ is a compact object of $\mathrm{QCoh}^{m-fp}(\mathfrak{U})$ (Lemma 5.4.7), the map $u^* \mathcal{F}_\alpha \to u^* \tau_{\leq m} \mathcal{F}$ factors through $\tau_{\leq m} M_{\alpha'}$ for some index $\alpha' \in A'$. The same reasoning shows that there exists an index β such that the map $\tau_{\leq m} M_{\alpha'} \to u^* \tau_{\leq m} \mathcal{F}$ factors through $u^* \mathcal{F}_\beta$. Enlarging β if necessary, we can assume that $\beta \geq \alpha$ and that the composite map $u^* \mathcal{F}_\alpha \to \tau_{\leq m} M_{\alpha'} \to u^* \mathcal{F}_\beta$ is homotopic to the transition map appearing in our filtered system $\{\mathcal{F}_\alpha\}_{\alpha \in A}$.

Let $\hat{v} : \mathrm{Spf}\, B' \to \mathfrak{X}$ denote the restriction of v to the formal spectrum of B'. The above argument shows that the map $\hat{v}^* F(\mathcal{F}_\alpha) \to \hat{v}^* F(\mathcal{F}_\beta)$ factors through $\tau_{\leq m} \mathcal{O}^k_{\mathrm{Spf}\, B'}$ for some integer k. It follows that the map

$$\theta' : \hat{v}^* \pi_m(F(\mathcal{F}_\alpha) \otimes \mathcal{F}') \to \hat{v}^* \pi_m(F(\mathcal{F}_\beta) \otimes \mathcal{F}')$$

factors through $\pi_m(\tau_{\leq m} \mathcal{O}^k_{\mathrm{Spf}\, B'} \otimes v^* \mathcal{F}')$ for some integer k. We have isomorphisms

$$0 \simeq \hat{v}^*(\pi_m \mathcal{F}')^k \simeq \pi_m(\mathcal{O}^k_{\mathrm{Spf}\, B'} \otimes \hat{v}^* \mathcal{F}') \to \pi_m(\tau_{\leq m} \mathcal{O}^k_{\mathrm{Spf}\, B'} \otimes \hat{v}^* \mathcal{F}').$$

We conclude that θ' is the zero map.

Let $\mathcal{G}$ denote the image of the map

$$\pi_m(F(\mathcal{F}_\alpha) \otimes \mathcal{F}) \to \pi_m(F(\mathcal{F}_\beta) \otimes \mathcal{F}')$$

(formed in the abelian category $\mathrm{QCoh}(\mathfrak{X})^\heartsuit$). To complete the proof, it will suffice to show that $\mathcal{G} \simeq 0$. Let us identify $v^* \mathcal{G}$ with a discrete B'-module N. Then N is finitely generated as a module over the Noetherian commutative ring $\pi_0 B'$, and the restriction of N to $\mathrm{Spf}\, B'$ vanishes. It follows that $N = J'N$. Let $S \subseteq \mathrm{Spec}^{\mathrm{Z}}(\pi_0 B')$ denote the support of N. Then S is a closed set which does not intersect the image of $\mathrm{Spec}^{\mathrm{Z}} B'_0$. Let $U \subseteq |\mathfrak{X}|$ denote the image of the open set $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 B') - S$. Since v is étale, the set U is open and $\mathcal{G}$ vanishes on U. We will prove that $U = |\mathfrak{X}|$.

Let K denote the closed subset of $|\mathfrak{X}|$ given by the image of $|\mathfrak{X}_0|$, so that $\mathfrak{X}^\wedge = \mathfrak{X}^\wedge_K$. By construction, we have a surjection $\mathrm{Spec}^{\mathrm{Z}} B'_0 \to K$. Since S does not intersect the image of $\mathrm{Spec}^{\mathrm{Z}} B'_0$, we have $K \subseteq U$. Let $Z \subseteq \mathrm{Spec}^{\mathrm{Z}} \pi_0 R$ denote the image of the closed set $|\mathfrak{X}| - U$. Since $\mathfrak{X} \to \mathrm{Spec}^{\text{ét}} R$ is a proper map, Z is a closed subset corresponding to some ideal $I' \subseteq \pi_0 R$. The assumption that $K \subseteq U$ implies that Z does not meet the image of $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)/I$. It follows that I generates the unit ideal in $(\pi_0 R)/I'$. Since R is I-adically complete, $(\pi_0 R)/I'$ is also I-adically complete and therefore $(\pi_0 R)/I' \simeq 0$. It follows that $Z = \emptyset$, so that $U = |\mathfrak{X}|$ as desired. □

6 Relationship with Formal Moduli Problems

Let A be an $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 A$ be a finitely generated ideal. In §5.1, we saw that the formal completion $A^\wedge_I$ can be identified with the $\mathbb{E}_\infty$-ring of functions on the formal spectrum $\mathrm{Spf}\, A$, obtained by completing

$\mathrm{Spec}^{\acute{e}t} A$ along the closed subset $K \subseteq \mathrm{Spec}^{\mathrm{Z}} A$ determined by the ideal I. Let us now suppose that A is Noetherian and that I is a maximal ideal in $\pi_0 A$. In this case, we can think of the formal spectrum $\mathrm{Spf}\, A$ as the union of all infinitesimal neighborhoods of the closed point of $\mathrm{Spec}^{\acute{e}t} A$. The language of formal moduli problems developed in [46] suggests another way of describing the same mathematical object. Namely, let $k = \pi_0 A/I$ denote the residue field of A, and let $\mathrm{CAlg}^{\mathrm{sm}}_{/k}$ denote the ∞-category of local Artinian $\mathbb{E}_\infty$-rings with residue field k (see Notation 6.1.3 for a precise definition). Then A determines a functor

$$\mathrm{Spec_f}\, A : \mathrm{CAlg}^{\mathrm{sm}}_{/k} \to \mathcal{S},$$

given by the formula $(\mathrm{Spec_f}\, A)(R) = \mathrm{Map}_{\mathrm{CAlg}_{/k}}(A, R)$. The main result of this section is that the construction $A \mapsto \mathrm{Spec_f}\, A$ is fully faithful when restricted to complete local Noetherian $\mathbb{E}_\infty$-rings with residue field k. Moreover, the essential image consists precisely of those functors $X : \mathrm{CAlg}^{\mathrm{sm}}_{/k} \to \mathcal{S}$ which are formal moduli problems whose tangent complexes satisfy a certain finite-dimensionality condition (see Theorem 6.2.2). After reviewing the relevant definitions in §6.1, we will formulate our result precisely in §6.2. The proof relies on a spectral version of Schlessinger's criterion for formal representability (Theorem 6.2.5), which we prove in §6.3.

6.1 Deformation Theory of Formal Thickenings

Let X_0 be a smooth projcetive variety defined over the field $\mathbf{Z}/p\,\mathbf{Z}$. In some cases, one can obtain information about X_0 by *lifting* the variety X_0 to positive characteristic. That is, suppose that X_0 is given as the special fiber of a morphism of schemes $\pi : X \to \mathrm{Spec}\,\mathbf{Z}_p$. Under some reasonable assumptions (for example, if π is proper and smooth), there is a close relationship between X_0 and the generic fiber $X_{\mathbf{Q}_p}$ of the morphism π. One can sometimes exploit this relationship to reduce questions about X_0 to questions about $X_{\mathbf{Q}_p}$, which may be more amenable to attack (since $X_{\mathbf{Q}_p}$ is defined over a field of characteristic zero). In applications of this technique, one frequently encounters the following question:

(Q) Given a smooth projective variety X_0 over the field $\mathbf{Z}/p\,\mathbf{Z}$, when does there exist a proper flat $\mathbf{Z}_p$-scheme X having special fiber X_0?

This question can naturally be broken into two parts:

(Q') Given a smooth projective variety X_0 over the field $\mathbf{Z}/p\,\mathbf{Z}$, when does there exist a proper flat formal $\mathbf{Z}_p$-scheme $\mathfrak{X}$ having special fiber X_0?

(Q'') Given a formal scheme $\mathfrak{X}$ as above, under what circumstances does it arise as the formal completion of a proper flat $\mathbf{Z}_p$-scheme?

Question (Q'') can often be attacked by means of the Grothendieck existence theorem, which we have discussed in §5.3. Question (Q') asks about the existence of a compatible family of proper flat morphisms

$$X_n \to \mathrm{Spec}\,\mathbf{Z}/p^{n+1}\,\mathbf{Z}.$$

This is a question of deformation theory, which can be phrased naturally using the language developed in [46].

Recall that to every field k, we can associate an ∞-category Moduli_k of *formal moduli problems* over k. By definition, an object $Z \in \mathrm{Moduli}_k$ is a functor from the ∞-category $\mathrm{CAlg}^{\mathrm{sm}}_k$ of small $\mathbb{E}_\infty$-algebras over k to the ∞-category of spaces, satisfying some natural gluing conditions (see Definition X.1.1.14). However, this definition is not really suitable for thinking about questions like (Q'): the rings $\mathbf{Z}/p^{n+1}\,\mathbf{Z}$ are are local Artin rings with residue field $\mathbf{Z}/p\,\mathbf{Z}$, but they are not algebras over $\mathbf{Z}/p\,\mathbf{Z}$, and therefore cannot be regarded as objects of $\mathrm{CAlg}^{\mathrm{sm}}_{\mathbf{Z}/p\,\mathbf{Z}}$. Our goal in this section is to address this deficiency by introducing a variant of the ∞-category Moduli_k, which we will denote by $\mathrm{Moduli}_{/k}$. The objects of $\mathrm{Moduli}_{/k}$ are functors $Z : \mathrm{CAlg}^{\mathrm{sm}}_{/k} \to \mathcal{S}$ satisfying a mild gluing condition (see Proposition 6.1.5), where $\mathrm{CAlg}^{\mathrm{sm}}_{/k}$ is an appropriately defined full subcategory of $\mathrm{CAlg}_{/k}$ (whose objects can be regarded as "infinitesimal thickenings" of k). For example, if X_0 is a smooth projective variety over $k = \mathbf{Z}/p\mathbf{Z}$, we can define an object $Z \in \mathrm{Moduli}_{/k}$ which assigns to each object $R \in \mathrm{CAlg}^{\mathrm{sm}}_{/k}$ a classifying space for Deligne-Mumford stacks X_R equipped with an equivalence

$$\mathrm{Spec}^{\text{ét}}\, k \times_{\mathrm{Spec}^{\text{ét}}\, R} X_R \simeq X_0.$$

In this case, we can regard (Q') as a question about the functor Z: namely, the question of whether or not the space $\varprojlim_n Z(\mathbf{Z}/p^{n+1}\mathbf{Z})$ is nonempty. In §6.2, we will see that questions of this sort can often be reduced to problems in commutative algebra.

Notation 6.1.1. Throughout this section, we will assume that the reader is familiar with the theory of formal moduli problems developed in §X.1 of [46]. Let k be a field. We let $\mathrm{CAlg}_{/k}$ denote the ∞-category of $\mathbb{E}_\infty$-rings A equipped with a map $A \to k$. We have a canonical equialence of ∞-categories $\mathrm{Stab}(\mathrm{CAlg}_{/k}) \simeq \mathrm{Mod}_k$. In particular, the object $k \in \mathrm{Mod}_k$ determines a spectrum object of $E \in \mathrm{Stab}(\mathrm{CAlg}_{/k})$, whose nth space $\Omega^{\infty - n} E$ is given by the square-zero extension $k \oplus k[n]$. We will regard the pair $(\mathrm{CAlg}_{/k}, \{E\})$ as a deformation context, in the sense of Definition X.1.1.3.

We begin by dispensing with some formalities.

Proposition 6.1.2. *Let k be a field and let $A \in \mathrm{CAlg}_{/k}$. The following conditions are equivalent:*

(1) *The object A is small (in the sense of Definition X.1.1.8). That is, the map $A \to k$ factors as a composition*

$$A = A_0 \to A_1 \to \cdots \to A_n = k$$

where each of the maps $A_i \to A_{i+1}$ exhibits A_i as a square-zero extension of A_{i+1} by $k[m_i]$, for some $m_i \geq 0$.

(2) *The following axioms are satisfied:*

  - (*i*) *The underlying map $\pi_0 A \to k$ is surjective.*
  - (*ii*) *The commutative ring $\pi_0 A$ is local. We will denote its maximal ideal by $\mathfrak{m}_A$, so that we have a canonical isomorphism $A/\mathfrak{m}_A \simeq k$.*
  - (*iii*) *The $\mathbb{E}_\infty$-ring A is connective. That is, we have $\pi_n A \simeq 0$ for $n < 0$.*
  - (*iv*) *For each $n \geq 0$, the homotopy group $\pi_n A$ has finite length as a module over $\pi_0 A$.*
  - (*v*) *The homotopy groups $\pi_n A$ vanish for $n \gg 0$.*

Proof. Suppose first that A is small, so that there there exists a finite sequence of maps

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq k$$

where each A_i is a square-zero extension of A_{i+1} by $k[m_i]$, for some $m_i \geq 0$. We prove that each A_i satisfies conditions (i) through (v) by descending induction on i. The case $i = n$ is obvious, so let us assume that $i < n$ and that A_{i+1} is known to satisfy conditions (i) through (v). We have a fiber sequence of k-module spectra

$$k[n_i] \to A_i \to A_{i+1}$$

which immediately implies that A_i satisfies (i), (iii), (iv), and (v). To prove (ii), we note that the map $\phi : \pi_0 A_i \to \pi_0 A_{i+1}$ is surjective and $\ker(\phi)^2 = 0$, from which it follows immediately that $\pi_0 A_i$ is local.

Now suppose that the map $A \to k$ satisfies axioms (i) through (v). We will prove that A is small by induction on the length of $\pi_* A$ (regarded as a module over $\pi_0 A$). If follows from (v) that there exists a largest integer n such that $\pi_n A \neq 0$. We first treat the case $n = 0$. We will abuse notation by identifying A with the underlying commutative ring $\pi_0 A$. Condition (ii) asserts that A is a local ring; let $\mathfrak{m}$ denote its maximal ideal. Since $\pi_0 A$ has finite length as a module over itself, we have $\mathfrak{m}^{i+1} \simeq 0$ for $i \gg 0$. Choose i as small as possible. If $i = 0$, then $\mathfrak{m} \simeq 0$ and $A \simeq k$, in which case there is nothing to prove. Otherwise, we can choose a nonzero element $x \in \mathfrak{m}^i \subseteq \mathfrak{m}$. Let A' denote the quotient ring $A/(x)$. Since $x^2 = 0$, Theorem A.7.4.1.26 implies that A is a square-zero extension of $A/(x)$ by k. The inductive hypothesis implies that A' is small, so that A is small.

Now suppose that $n > 0$ and let $M = \pi_n A$. Since M has finite length over $\pi_0 A$, we can find an element $x \in M$ which is annihilated by $\mathfrak{m}$. We therefore have an exact sequence

$$0 \to k \xrightarrow{x} M \to M' \to 0$$

of modules over $\pi_0 A$. We will abuse notation by viewing this sequence as a fiber sequence of A''-modules, where $A'' = \tau_{\leq n-1} A$. It follows from Theorem A.7.4.1.26 that there is a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & k \\ \downarrow & & \downarrow \\ A'' & \longrightarrow & k \oplus M[n+1]. \end{array}$$

Set $A' = A'' \times_{k \oplus M'[n+1]} k$. Then $A \simeq A' \times_{k \oplus k[n+1]} k$ so that A is a square-zero extension of A' by $k[n]$. Using the inductive hypothesis we deduce that A' is small, so that A is also small. □

Notation 6.1.3. Let k be a field. We let $\mathrm{CAlg}^{\mathrm{sm}}_{/k}$ denote the full subcategory of $\mathrm{CAlg}_{/k}$ spanned by the small objects: that is, those objects $A \in \mathrm{CAlg}_{/k}$ which satisfy conditions (i) through (v) of Proposition 6.1.2.

Proposition 6.1.2 has the following relative version:

Proposition 6.1.4. *Let k be a field and let $f : A \to B$ be a morphism in $\mathrm{CAlg}^{\mathrm{sm}}_{/k}$. The following conditions are equivalent:*

(1) *The morphism f is small. That is, f factors as a composition*

$$A = A_0 \to A_1 \to \cdots \to A_n = B,$$

where each A_i is a square-zero extension of A_{i+1} by $k[m_i]$ for some $m_i \geq 0$.

(2) *The map $\pi_0 A \to \pi_0 B$ is surjective.*

Proof. Let K be the fiber of f, regarded as an A-module. If $\pi_0 A \to \pi_0 B$ is surjective, then K is connective. Since $\pi_* B$ has finite length as a module over $\pi_0 B$, it has finite length as a module over $\pi_0 A$ (note that the residue fields of $\pi_0 A$ and $\pi_0 B$ are both isomorphic to k). It follows from the exact sequence

$$\pi_{*+1} B \to \pi_* K \to \pi_* A$$

that $\pi_* K$ has finite length as a module over $\pi_0 A$. We will prove that f is small by induction on the length of $\pi_* K$ as a module over $\pi_0 A$. If this length is zero, then $K \simeq 0$ and f is an equivalence. Assume therefore that $\pi_* K \neq 0$, and let n be the smallest integer such that $\pi_n K \neq 0$. Let $\mathfrak{m}$ denote the maximal ideal of $\pi_0 A$. Then $\mathfrak{m}$ is nilpotent, so $\mathfrak{m}(\pi_n K) \neq \pi_n K$ and we can choose a map of $\pi_0 A$-modules $\phi : \pi_n K \to k$. According to Theorem A.7.4.3.1, we have $(2n+1)$-connective map $K \otimes_A B \to L_{B/A}[-1]$. In particular, we have an isomorphism $\pi_{n+1} L_{B/A} \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_n K)$ so that ϕ determines a map $L_{B/A} \to k[n+1]$. We can interpret this map as a derivation $B \to B \oplus k[n+1]$; let $B' = B \times_{B \oplus k[n+1]} k$. Then f factors as a composition

$$A \xrightarrow{f'} B' \xrightarrow{f''} B.$$

Since f'' exhibits B' as a square-zero extension of B by $k[n]$, we are reduced to proving that f' is small. This follows from the inductive hypothesis. □

Proposition 6.1.5. *Let k be a field and let $X : \mathrm{CAlg}^{\mathrm{sm}}_{/k} \to \mathcal{S}$ be a functor. The following conditions are equivalent:*

(1) *The functor X is a formal moduli problem (in the sense of Definition X.1.1.14).*

(2) *The space $X(k)$ is contractible and, for every pullback diagram*

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ A & \longrightarrow & B \end{array}$$

in $\mathrm{CAlg}^{\mathrm{sm}}_{/k}$, if ϕ induces a surjection $\pi_0 B' \to \pi_0 B$, then the induced diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(B') \\ \downarrow & & \downarrow \\ X(A) & \longrightarrow & X(B) \end{array}$$

is a pullback square in $\mathcal{S}$.

(3) *The space $X(k)$ is contractible and, for every pullback diagram*

$$\begin{array}{ccc} A' & \longrightarrow & k \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \oplus k[n] \end{array}$$

in $\mathrm{CAlg}^{\mathrm{sm}}_{/k}$ where $n > 0$, the induced diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(k) \\ \downarrow & & \downarrow \\ X(A) & \longrightarrow & X(k \oplus k[n]) \end{array}$$

is a pullback square in $\mathcal{S}$.

(4) *The space $X(k)$ is contractible and, for every*

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ A & \xrightarrow{\psi} & B \end{array}$$

in $\mathrm{CAlg}^{\mathrm{sm}}_{/k}$, if ϕ and ψ induce surjections $\pi_0 B' \to \pi_0 B \leftarrow \pi_0 A$, then the induced diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(B') \\ \downarrow & & \downarrow \\ X(A) & \longrightarrow & X(B) \end{array}$$

is a pullback square in $\mathcal{S}$.

Proof. The equivalence of (1) and (2) follows immediately from the definitions (and the characterization of small morphisms given in Proposition 6.1.4). The implications (2) $\Rightarrow$ (4) $\Rightarrow$ (3) are obvious, and the implication (3) $\Rightarrow$ (2) follows from Proposition X.1.1.15. □

Notation 6.1.6. Let k be a field. We let $\mathrm{Moduli}_{/k}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm}}_{/k}, \mathcal{S})$ spanned by the formal moduli problems (that is, spanned by those functors $X : \mathrm{CAlg}^{\mathrm{sm}}_{/k} \to \mathcal{S}$ satisfying the equivalent conditions of Proposition 6.1.5). For each $A \in \mathrm{CAlg}_{/k}$, we let $\mathrm{Spec_f}\, A \in \mathrm{Moduli}_{/k}$ denote the functor given by the formula $(\mathrm{Spec_f}\, A)(R) = \mathrm{Map}_{\mathrm{CAlg}_{/k}}(A, R)$.

Remark 6.1.7. Let k be a field. Since $X(k)$ is contractible for each $X \in \mathrm{Moduli}_{/k}$, the object $\mathrm{Spec_f}\, k \in \mathrm{Moduli}_{/k}$ is initial. It is a final object of $\mathrm{Moduli}_{/k}$ if and only if the mapping space $\mathrm{Map}_{\mathrm{CAlg}_{/k}}(k, A)$ is contractible for every $A \in \mathrm{CAlg}^{\mathrm{sm}}_{/k}$: that is, if and only if every object $A \in \mathrm{CAlg}^{\mathrm{sm}}_{/k}$ admits a contractible space of k-algebra structures. This condition is satisfied if and only if k is an algebraic extension of the field $\mathbf{Q}$ of rational numbers.

Suppose $X \in \mathrm{Moduli}_{/k}$ is equipped with a map $\phi : X \to \mathrm{Spec_f}\, k$. We can associate to the pair (X, ϕ) a functor $X_0 : \mathrm{CAlg}^{\mathrm{sm}}_{k} \to \mathcal{S}$, which carries a small $\mathbb{E}_\infty$-algebra R over k to the fiber of the induced map $X(R) \stackrel{\phi_R}{\to} \mathrm{Map}_{\mathrm{CAlg}_{/k}}(k, R)$. The construction $(X, \phi) \mapsto X_0$ determines an equivalence of ∞-categories $(\mathrm{Moduli}_{/k})_{/\mathrm{Spec_f}\, k} \to \mathrm{Moduli}_k$, where Moduli_k denotes the ∞-category of formal moduli problems over k introduced in §X.2. This functor is an equivalence of ∞-categories if and only if k is an algebraic extension of the field $\mathbf{Q}$ of rational numbers.

Notation 6.1.8. Let k be a field and let $X \in \mathrm{Moduli}_{/k}$. We let $T_X = X(E)$ denote its tangent complex (Definition X.1.2.5). This is a spectrum whose nth space is given by $\Omega^{\infty - n} T_X = X(k \oplus k[n])$ for each $n \geq 0$.

Unwinding the definitions, we see that the tangent complex to $\mathrm{Spec_f}\, A$ is a classifying spectrum for A-module maps from the absolute cotangent complex L_A into k: that is, it is the k-linear dual of the k-module spectrum $k \otimes_A L_A$. In particular, for each $n \in \mathbf{Z}$ we have a canonical isomorphism of k-vector spaces $\pi_n T_{\mathrm{Spec_f}\, A} \simeq \pi_{-n}(k \otimes_A L_A)^\vee$.

The functor $\mathrm{Spec_f}\, k$ is an initial object of $\mathrm{Moduli}_{/k}$. In particular, to every formal moduli problem $X \in \mathrm{Moduli}_{/k}$, the canonical map $\mathrm{Spec_f}\, k \to X$ induces a map of tangent complexes $L_k^\vee = T_{\mathrm{Spec_f}\, k} \to T_X$. We will denote the cofiber of this map by T_X^{red}, and refer to it as the *reduced tangent complex* of X. If $X = \mathrm{Spec_f}\, A$ for some $A \in \mathrm{CAlg}_{/k}$, then the fiber sequence $T_{\mathrm{Spec_f}\, k} \to T_X \to T_X^{\mathrm{red}}$ is just given by the k-linear dual of the fiber sequence of k-module spectra $L_{k/A}[-1] \to k \otimes_A L_A \to L_k$. That is, we have an equivalence $T^{\mathrm{red}}_{\mathrm{Spec_f}\, A} \simeq L^\vee_{k/A}[1]$.

Remark 6.1.9. Let k be a field and let $\mathcal{C} \subseteq \mathrm{Mod}_k$ denote the full subcategory spanned by those k-module spectra which are perfect and connective. The construction $V \mapsto k \oplus V$ determines functors

$$\theta_0 : \mathcal{C} \to \mathrm{CAlg}_{/k} \qquad \theta_1 : \mathcal{C} \to \mathrm{Alg}_k^{(0),\mathrm{sm}},$$

where θ_1 is an equivalence of ∞-categories. Composition with the fucntor $\theta_0 \circ \theta_1^{-1}$ induces a forgetful functor

$$\Phi : \mathrm{Moduli}_{/k} \to \mathrm{Moduli}_k^{(0)},$$

where $\mathrm{Moduli}_k^{(0)}$ denotes the ∞-category of formal $\mathbb{E}_0$ moduli problems over k (see Definition X.3.0.3). According to Theorem X.4.0.8, there is an equivalence of ∞-categories $\Psi : \mathrm{Mod}_k \to \mathrm{Moduli}_k^{(0)}$, such that the composition of Ψ^{-1}. Then $\Psi^{-1} \circ \Phi$ determines a functor $\mathrm{Moduli}_{/k} \to \mathrm{Mod}_k$. It follows from Theorem X.3.0.4 that this functor *refines* the tangent complex functor: that is, the composite functor

$$\mathrm{Moduli}_{/k} \stackrel{\Phi}{\to} \mathrm{Moduli}_k^{(0)} \simeq \mathrm{Mod}_k = \mathrm{Mod}_k(\mathrm{Sp}) \to \mathrm{Sp}$$

carries a formal moduli problem X to its tangent complex T_X. We can informally summarize the situation by saying that for $X \in \mathrm{Moduli}_{/k}$, the tangent complex T_X is equipped with a k-module structure, depending functorially on X. In particular, we can regard each of the homotopy groups $\pi_n T_X$ as a vector space over k.

6.2 Formal Spectra as Formal Moduli Problems

Let k be a field and let A be an $\mathbb{E}_\infty$-ring equipped with a map $A \to k$. We can associate to A a formal moduli problem

$$\mathrm{Spec}_{\mathrm{f}}\, A : \mathrm{CAlg}^{\mathrm{sm}}_{/k} \to \mathcal{S}\,.$$

We now ask the following question: how close is the functor $\mathrm{Spec}_{\mathrm{f}}\, A$ to being an equivalence? For example:

(*a*) Given an object $A \in \mathrm{CAlg}_{/k}$, can we recover A from the formal moduli problem $\mathrm{Spec}_{\mathrm{f}}\, A$?

(*b*) Given a formal moduli problem $X : \mathrm{CAlg}^{\mathrm{sm}}_{/k} \to \mathcal{S}$, can we find an object $A \in \mathrm{CAlg}_{/k}$ such that $X \simeq \mathrm{Spec}_{\mathrm{f}}\, A$?

The answers to both of these questions are generally negative. For example, if we are given an object $A \in \mathrm{CAlg}_{/k}$ and we let $\mathfrak{m}$ denote the kernel of the commutative ring homomorphism $\pi_0 A \to k$, then for $R \in \mathrm{CAlg}^{\mathrm{sm}}_{/k}$, every morphism $A \to R$ must automatically annihilate some power of the ideal $\mathfrak{m}$. It follows that the formal moduli problem $\mathrm{Spec}_{\mathrm{f}}\, A$ depends only on the formal completion $\mathrm{Spf}\, A$ of $\mathrm{Spec}^{\text{ét}}\, A$ along the ideal $\mathfrak{m}$. To have any hope of recovering A from $\mathrm{Spec}_{\mathrm{f}}\, A$, we need to assume that A is $\mathfrak{m}$-complete. This motivates the following definition:

Notation 6.2.1. Let k be a field. We let $\mathrm{CAlg}^{\mathrm{cg}}_{/k}$ denote the full subcategory $\mathrm{CAlg}_{/k}$ spanned by those morphisms $A \to k$ satisfying the following conditions:

(*i*) The $\mathbb{E}_\infty$-ring A is connective and Noetherian.

(*ii*) The map $\pi_0 A \to k$ is surjective.

(*iii*) The commutative ring $\pi_0 A$ is local and complete with respect to its maximal ideal $\mathfrak{m}_A \subseteq \pi_0 A$.

We can now state the main result of this section, which gives an affirmative answer to questions (*a*) and (*b*) under some reasonable hypotheses:

Theorem 6.2.2. *Let k be a field. Then the functor* $\mathrm{Spec}_{\mathrm{f}} : \mathrm{CAlg}^{op}_{/k} \to \mathrm{Moduli}_{/k}$ *restricts to a fully faithful embedding*

$$\theta : (\mathrm{CAlg}^{\mathrm{cg}}_{/k})^{op} \to \mathrm{Moduli}_{/k}\,.$$

Moreover, a formal moduli problem X : $\mathrm{Moduli}_{/k}$ belongs to the essential image of θ if and only if it satisfies the following conditions:

(*i*) *For every integer n, the homotopy group $\pi_n T^{\mathrm{red}}_X$ is finite dimensional (as a vector space over k).*

(*ii*) *The groups $\pi_n T^{\mathrm{red}}_X$ vanish for $n > 0$.*

Remark 6.2.3. Let A be a Noetherian $\mathbb{E}_\infty$-ring, let $\mathfrak{m} \subseteq \pi_0 A$ be a maximal ideal, and let $k = (\pi_0 A)/\mathfrak{m}$ be the residue field. Then the formal moduli problem $X = \mathrm{Spec}_{\mathrm{f}}\, A \in \mathrm{Moduli}_{/k}$ satisfies conditions (*i*) and (*ii*) of Theorem 6.2.2. To see this, we note that T^{red}_X is given by the k-linear dual of the shifted relative cotangent complex $L_{k/A}[1]$ (see Notation 6.1.8). It therefore suffices to show that each $\pi_n L_{k/A}$ is a finite dimensional vector space over k, and that $\pi_n L_{k/A} \simeq 0$ vanishes for $n \leq 0$. The vanishing for $n < 0$ follows from the fact that k and A are connective. Moreover, $\pi_0 L_{k/A}$ is the module of Kähler differentials of k over $\pi_0 A$ (Proposition A.7.4.3.9), which vanishes because the map $\pi_0 A \to k$ is surjective. The finite-dimensionality is equivalent to the assertion that $L_{k/A}$ is almost perfect as a k-module, which follows from the fact that k is almost of finite presentation over A (see Theorems A.7.4.3.18 and A.7.2.5.31).

The proof of Theorem 6.2.2 will require some preliminary results. Recall that the Yoneda embedding $\mathrm{Spec}_{\mathrm{f}} : (\mathrm{CAlg}^{\mathrm{sm}}_{/k})^{op} \to \mathrm{Moduli}_{/k}$ extends to a fully faithful embedding

$$\mathrm{Pro}(\mathrm{CAlg}^{\mathrm{sm}}_{/k})^{op} \to \mathrm{Moduli}_{/k}\,.$$

We say that a formal moduli problem $X \in \mathrm{Moduli}_{/k}$ is *prorepresentable* if it belongs to the essential image of this embedding.

Proposition 6.2.4. *Let A be a Noetherian $\mathbb{E}_\infty$-ring, let $\mathfrak{m} \subseteq \pi_0 A$ be a maximal ideal, and let k denote the residue field $(\pi_0 A)/\mathfrak{m}$. Then there exists a tower of A-algebras*

$$\cdots \to A_2 \to A_1 \to A_0 \to k$$

satisfying the following conditions:

- (a) *Each $A_i \in \mathrm{CAlg}_{/k}$ is small.*
- (b) *The induced map*
$$\varinjlim \mathrm{Spec_f}\, A_i \to \mathrm{Spec_f}\, A$$
is an equivalence in $\mathrm{Moduli}_{/k}$
- (c) *The canonical map $A \to \varprojlim A_i$ exhibits $\varprojlim A_i$ as an $\mathfrak{m}$-completion of A.*

Proof. Choose a tower
$$\cdots \to B_2 \to B_1 \to B_0$$
satisfying the requirements of Lemma 5.1.5. Set $A_i = \tau_{\leq i} B_i$; we will prove that the tower $\{A_i\}_{i \geq 0}$ has the desired properties. Each of the maps $\pi_0 A \to \pi_0 A_i$ is annihilated by some power of the maximal ideal $\mathfrak{m}$. It follows that each homotopy group $\pi_n A_i$ can be regarded as a module over $(\pi_0 A)/\mathfrak{m}^k$ for $k \gg 0$. Since A_i is almost perfect as a module over A, each $\pi_n A_i$ is finitely generated as a module over $(\pi_0 A)/\mathfrak{m}^k$, and therefore of finite length. Since $\pi_0 A \to \pi_0 A_i$ is surjective, $\pi_0 A_i$ is a local Artinian ring with residue field k. Using the criterion of Proposition 6.1.2, we see that each A_i is small. This proves (a), and (c) follows from Remark 5.1.11. To prove (b), we note that if $R \in \mathrm{CAlg}_{/k}$ is small, then any map $\pi_0 A \to \pi_0 R$ automatically annihilates some power of the maximal ideal $\mathfrak{m}$. It follows that the canonical map

$$\varinjlim \mathrm{Map}_{\mathrm{CAlg}_{/k}}(B_i, R) \to \mathrm{Map}_{\mathrm{CAlg}_{/k}}(A, R)$$

is a homotopy equivalence. Choose an integer m such that R is m-truncated. Then the map

$$\mathrm{Map}_{\mathrm{CAlg}_{/k}}(A_i, R) \to \mathrm{Map}_{\mathrm{CAlg}_{/k}}(B_i, R)$$

is a homotopy equivalence for $i \geq m$. It follows that the composite map

$$\varinjlim \mathrm{Map}_{\mathrm{CAlg}_{/k}}(A_i, R) \to \varinjlim \mathrm{Map}_{\mathrm{CAlg}_{/k}}(B_i, R) \to \mathrm{Map}_{\mathrm{CAlg}_{/k}}(A, R)$$

is a homotopy equivalence. □

The essential surjectivity of the functor θ appearing in Theorem 6.2.2 is a consequence of the following more general result, which we will prove in §6.3:

Theorem 6.2.5 (Spectral Schlessinger Criterion)**.** *Let k be a field, let $X \in \mathrm{Moduli}_{/k}$ be a formal moduli problem, and assume that $\pi_n T_X^{\mathrm{red}}$ is finite dimensional as a k-vector space for each $n \leq 0$. Then there exists a Noetherian $\mathbb{E}_\infty$-ring A, where $\pi_0 A$ is a complete local Noetherian ring with residue field k, and a map $\eta : \mathrm{Spec_f}\, A \to X$ which induces isomorphisms*

$$\pi_n T^{\mathrm{red}}_{\mathrm{Spec_f}\, A} \to \pi_n T^{\mathrm{red}}_X$$

for $n \leq 0$.

Remark 6.2.6. In the situation of Theorem 6.2.5, let K denote the fiber of the map $T^{\mathrm{red}}_{\mathrm{Spec_f}\, A} \to T^{\mathrm{red}}_X$. We have exact sequences

$$\pi_{n+1} T^{\mathrm{red}}_{\mathrm{Spec_f}\, A} \to \pi_{n+1} T^{\mathrm{red}}_X \to \pi_n K \to \pi_n T^{\mathrm{red}}_{\mathrm{Spec_f}\, A} \to \pi_n T^{\mathrm{red}}_X$$

which show that $\pi_n K \simeq 0$ for $n < 0$: that is, K is connective. We have a homotopy pullback diagram of spectra

$$\begin{array}{ccc} T_{\mathrm{Spec_f}\,A} & \longrightarrow & T^{\mathrm{red}}_{\mathrm{Spec_f}\,A} \\ \downarrow & & \downarrow \\ T_X & \longrightarrow & T^{\mathrm{red}}_X. \end{array}$$

It follows that K can also be identified with the fiber of the map $T_{\mathrm{Spec_f}\,A} \to T_X$. Since K is connective, we deduce that the map $\mathrm{Spec_f}\,A \to X$ is smooth (in the sense of Definition X.1.5.6).

Proof of Theorem 6.2.2. We first show that the functor θ is fully faithful. Fix objects $A, B \in \mathrm{CAlg}^{\mathrm{cg}}_{/k}$. We wish to show that θ induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_{/k}}(B, A) \to \mathrm{Map}_{\mathrm{Moduli}_{/k}}(\mathrm{Spec_f}\,A, \mathrm{Spec_f}\,B).$$

Choose a tower of A-algebras $\{A_i\}$ satisfying the requirements of Proposition 6.2.4. In particular, we have $\mathrm{Spec_f}\,A \simeq \varinjlim \mathrm{Spec_f}\,A_i$, so that we have homotopy equivalences

$$\begin{aligned} \mathrm{Map}_{\mathrm{Moduli}_{/k}}(\mathrm{Spec_f}\,A, \mathrm{Spec_f}\,B) &\to \varprojlim \mathrm{Map}_{\mathrm{Moduli}_{/k}}(\mathrm{Spec_f}\,A_i, \mathrm{Spec_f}\,B) \\ &\simeq \varprojlim (\mathrm{Spec_f}\,B)(A_i) \\ &\simeq \varprojlim \mathrm{Map}_{\mathrm{CAlg}_{/k}}(B, A_i) \\ &\simeq \mathrm{Map}_{\mathrm{CAlg}_{/k}}(B, \varprojlim A_i). \end{aligned}$$

It will therefore suffice to show that the canonical map

$$\mathrm{Map}_{\mathrm{CAlg}_{/k}}(B, A) \to \mathrm{Map}_{\mathrm{CAlg}_{/k}}(B, \varprojlim A_i)$$

is a homotopy equivalence. This follows from Proposition 6.2.4 together with our assumption that A is complete with respect to the maximal ideal of $\pi_0 A$.

Let $\mathrm{Moduli}^0_{/k}$ denote the full subcategory of $\mathrm{Moduli}_{/k}$ spanned by those formal moduli problems X for which the homotopy groups $\pi_n T^{\mathrm{red}}_X$ are finite dimensional for each n and vanish for $n > 0$. It follows from Remark 6.2.3 that θ restricts to a functor $(\mathrm{CAlg}^{\mathrm{cg}}_{/k})^{op} \to \mathrm{Moduli}^0_{/k}$. It remains to prove that this functor is essentially surjective. Let $X \in \mathrm{Moduli}^0_{/k}$, and choose a map $u : \mathrm{Spec_f}\,A \to X$ satisfying the requirements of Theorem 6.2.5. Since $\pi_n T^{\mathrm{red}}_X \simeq 0$ for $n > 0$, the map u induces an equivalence of reduced tangent complexes $T^{\mathrm{red}}_{\mathrm{Spec_f}\,A} \to T^{\mathrm{red}}_X$. Since the diagram

$$\begin{array}{ccc} T_{\mathrm{Spec_f}\,A} & \longrightarrow & T^{\mathrm{red}}_{\mathrm{Spec_f}\,A} \\ \downarrow & & \downarrow \\ T_X & \longrightarrow & T^{\mathrm{red}}_X. \end{array}$$

is a pullback, we see that u induces an equivalence on tangent complexes $T_{\mathrm{Spec_f}\,A} \to T_X$, and is therefore an equivalence by Proposition X.1.2.10. □

We close this section by establishing a formal property of the construction $A \mapsto \mathrm{Spec_f}\,A$ which will be needed in §6.3.

Proposition 6.2.7. *Suppose we are given a pullback diagram of connective $\mathbb{E}_\infty$-rings σ :*

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow f \\ R_1 & \xrightarrow{g} & R_{01}, \end{array}$$

where f and g induce surjections $\pi_0 R_1 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$. Assume that R_0, R_1, and R_{01} are Noetherian. Then:

(1) *The $\mathbb{E}_\infty$-ring R is Noetherian.*

(2) *Let k denote the residue field of $\pi_0 R_{01}$ at some maximal ideal, so that we can regard σ as a pullback diagram in $\mathrm{CAlg}_{/k}$. Then the induced map*

$$\mathrm{Spec}_{\mathrm{f}}\, R_0 \coprod_{\mathrm{Spec}_{\mathrm{f}}\, R_{01}} \mathrm{Spec}_{\mathrm{f}}\, R_1 \to \mathrm{Spec}_{\mathrm{f}}\, R$$

is an equivalence in $\mathrm{Moduli}_{/k}$.

Proof. Let A denote the fiber product $\pi_0 R_0 \times_{\pi_0 R_{01}} \pi_0 R_1$. We first prove that A is Noetherian. Let $I \subseteq A$ be an ideal, and let I_0 and I_1 denote the images of I in $\pi_0 R_0$ and $\pi_0 R_1$. Let J denote the intersection of I_1 with the kernel of the map $\pi_0 R_1 \to \pi_0 R_{01}$. Since $\pi_0 R_0$ and $\pi_0 R_1$ is Noetherian, the ideals I_0 and J are finitely generated. We have an exact sequence of A-modules

$$0 \to J \to I \to I_0 \to 0$$

which proves that I is also finitely generated.

Let K denote the kernel of the map $\pi_0 R \to A$. Using the exact sequence

$$\pi_1 R_{01} \to \pi_0 R \to \pi_0 R_0 \times \pi_0 R_1 \to \pi_0 R_{01},$$

we see that K can be regarded as a quotient of $\pi_1 R_{01}$. In particular, K is annihilated by the kernel of the map $\pi_0 R \to \pi_0 R_{01}$, and is therefore a square-zero ideal in $\pi_0 R$. It follows that $\pi_0 R$ is K-adically complete. Since R_{01} is Noetherian, K is finitely generated. Applying Proposition 4.3.12, we deduce that $\pi_0 R$ is Noetherian.

For every integer n, we have an exact sequence

$$\pi_{n+1} R_{01} \to \pi_n R \to \pi_n R_0 \times \pi_n R_1.$$

Since R_0, R_1, and R_{01} are Noetherian, the modules $\pi_{n+1} R_{01}$, $\pi_n R_0$, and $\pi_n R_1$ are finitely generated over $\pi_0 R_{01}$, $\pi_0 R_0$, and $\pi_0 R_1$, respectively. It follows that each of these modules is finitely generated over $\pi_0 R$. Since $\pi_0 R$ is Noetherian, we deduce that $\pi_n R$ is finitely generated as a module over $\pi_0 R$. This proves (1).

The proof of (2) will proceed in several steps.

- Suppose first that R_0, R_1, and R_{01} belong to $\mathrm{CAlg}^{\mathrm{sm}}_{/k}$. Then $R \in \mathrm{CAlg}^{\mathrm{sm}}_{/k}$. We wish to show that, for every $X \in \mathrm{Moduli}_{/k}$, the diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Moduli}_{/k}}(\mathrm{Spec}_{\mathrm{f}}\, R, X) & \longrightarrow & \mathrm{Map}_{\mathrm{Moduli}_{/k}}(\mathrm{Spec}_{\mathrm{f}}\, R_0, X) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{Moduli}_{/k}}(\mathrm{Spec}_{\mathrm{f}}\, R_1, X) & \longrightarrow & \mathrm{Map}_{\mathrm{Moduli}_{/k}}(\mathrm{Spec}_{\mathrm{f}}\, R_{01}, X) \end{array}$$

  is a pullback square. This is equivalent to the requirement that the map

$$X(R) \to X(R_0) \times_{X(R_{01})} X(R_1)$$

  is a homotopy equivalence, which follows from Proposition 6.1.5.

- Suppose that $\pi_0 R$ is a local Artin ring (with residue field k). Then $\pi_0 R_0$, $\pi_0 R_1$, and $\pi_0 R_{01}$ are also local Artin rings. It follows that for each integer $n \geq 0$, the truncations $\tau_{\leq n} R_0$, $\tau_{\leq n} R_1$, and $\tau_{\leq n} R_{01}$ belong to $\mathrm{CAlg}^{\mathrm{sm}}_{/k}$. Let $R(n)$ denote the fiber product $\tau_{\leq n} R_0 \times_{\tau_{\leq n} R_{01}} \tau_{\leq n} R_1$. It follows from the previous step that the canonical map

$$\mathrm{Spec_f}(\tau_{\leq n} R_0) \coprod_{\mathrm{Spec_f}(\tau_{\leq n} R_{01})} \mathrm{Spec_f}(\tau_{\leq n} R_1) \to \mathrm{Spec_f}\, R(n)$$

is an equivalence. Passing to the filtered colimit over n, we deduce that the upper horizontal map in the diagram

$$\begin{array}{ccc} \varinjlim \mathrm{Spec_f}(\tau_{\leq n} R_0) \coprod_{\mathrm{Spec_f}(\tau_{\leq n} R_{01})} \mathrm{Spec_f}(\tau_{\leq n} R_1) & \longrightarrow & \varinjlim \mathrm{Spec_f}\, R(n) \\ \downarrow & & \downarrow \\ \mathrm{Spec_f}\, R_0 \coprod_{\mathrm{Spec_f}\, R_{01}} \mathrm{Spec_f}\, R_1 & \longrightarrow & \mathrm{Spec_f}\, R \end{array}$$

is an equivalence. To prove that the lower horizontal map is an equivalence, it will suffice to show that the vertical maps are equivalences. This is clear: if $B \in \mathrm{CAlg}^{\mathrm{sm}}_{/k}$ is m-truncated, then the maps

$$(\mathrm{Spec_f}\, \tau_{\leq n} R_0)(B) \to (\mathrm{Spec_f}\, R_0)(B) \qquad (\mathrm{Spec_f}\, \tau_{\leq n} R_1)(B) \to (\mathrm{Spec_f}\, R_1)(B)$$

$$(\mathrm{Spec_f}\, \tau_{\leq n} R_{01})(B) \to (\mathrm{Spec_f}\, R_{01})(B) \qquad (\mathrm{Spec_f}\, R(n))(B) \to (\mathrm{Spec_f}\, R)(B)$$

are homotopy equivalences provided that $n > m$.

- Now suppose that R is arbitrary. Let $I \subseteq \pi_0 R$ be the kernel of the map $\pi_0 R \to k$, and choose a tower of R-algebras

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma 5.1.5. The map

$$\theta : \mathrm{Spec_f}\, R_0 \coprod_{\mathrm{Spec_f}\, R_{01}} \mathrm{Spec_f}\, R_1 \to \mathrm{Spec_f}\, R$$

is a filtered colimit of maps

$$\mathrm{Spec_f}(A_i \otimes_R R_0) \coprod_{\mathrm{Spec_f}(A_i \otimes_R R_{01})} \mathrm{Spec_f}(A_i \otimes_R R_1) \to \mathrm{Spec_f}\, A_i.$$

It follows from the preceding case that each of these maps is an equivalence in $\mathrm{Moduli}_{/k}$, so that θ is an equivalence in $\mathrm{Moduli}_{/k}$.

□

6.3 Schlessinger's Criterion in Spectral Algebraic Geometry

Our goal in this section is to give the proof of Theorem 6.2.5. We begin by recalling some facts from commutative algebra.

Lemma 6.3.1. *Suppose we are given an inverse system*

$$\ldots \to B_3 \to B_2 \to B_1 \to B_0$$

be an inverse system of (ordinary) local Artinian rings with the same residue field k. Denote the maximal ideal of B_i by $\mathfrak{m}_i$, and suppose that the induced maps $\mathfrak{m}_{i+1}/\mathfrak{m}_{i+1}^2 \to \mathfrak{m}_i/\mathfrak{m}_i^2$ on Zariski cotangent spaces are isomorphisms. Then the inverse limit $B = \varprojlim B_i$ is a complete local Noetherian ring with residue field k and maximal ideal $\mathfrak{m} = \varprojlim \mathfrak{m}_i$. Moreover, there is a canonical isomorphism

$$\{B_i\}_{i>0} \simeq \{B/\mathfrak{m}^n\}_{n\geq 0}$$

of pro-objects in the category of commutative rings.

Proof. We first prove that B is Noetherian. Note that for each $n \geq 0$, each of the maps $B_{i+1} \to B_i$ induces surjections $\mathfrak{m}_{i+1}^n/\mathfrak{m}_{i+1}^{n+1} \to \mathfrak{m}_i^n/\mathfrak{m}_i^{n+1}$. Since $\mathfrak{m}_{i+1}$ and $\mathfrak{m}_i$ are nilpotent, we conclude that the map $\mathfrak{m}_{i+1} \to \mathfrak{m}_i$ is surjective: that is, $\mathfrak{m}_i$ is generated by the image of $\mathfrak{m}_{i+1}$. For each $n \geq 0$, let $\mathfrak{m}(n)$ denote the inverse limit $\varprojlim \mathfrak{m}_i^n$, regarded as an ideal in B. The inverse system $\{\mathfrak{m}_i^n\}_{i \geq 0}$ has surjective transition maps, so the exact sequences

$$0 \to \mathfrak{m}_i^n \to B_i \to B_i/\mathfrak{m}_i^n \to 0$$

determine an exact sequence

$$0 \to \mathfrak{m}(n) \to B \to \varprojlim\{B_i/\mathfrak{m}_i^n\} \to 0.$$

In other words, we have canonical isomorphsims $B/\mathfrak{m}(n) \simeq \varprojlim\{B_i/\mathfrak{m}_i^n\}$.

Since each B_i is local and Artinian, the ideals $\mathfrak{m}_i$ are nilpotent, so that the canonical maps

$$B_i \to \varprojlim_{n \geq 0} B_i/\mathfrak{m}_i^n$$

are isomorphisms. Taking the inverse limit over i, we obtain isomorphisms

$$\begin{aligned} B &\simeq \varprojlim_i B_i \\ &\simeq \varprojlim_i \varprojlim_n B_i/\mathfrak{m}_i^n \\ &\simeq \varprojlim_n \varprojlim_i B_i/\mathfrak{m}_i^n \\ &\simeq \varprojlim_n B/\mathfrak{m}(n). \end{aligned}$$

We will prove the following assertions:

(a) For each $n \geq 0$, we have an equality $\mathfrak{m}(n) = \mathfrak{m}^n$ of ideals of B.

(b) The ideal $\mathfrak{m}$ is finitely generated.

Assuming (a), we deduce that B is $\mathfrak{m}$-adically complete. Since $B/\mathfrak{m} \simeq k$ is Noetherian, it follows from (b) that the commutative ring B is also Noetherian (Proposition 4.3.12), hence a complete local ring with maximal ideal $\mathfrak{m}$ and residue field k.

Choose a finite set of generators $\overline{x}_1, \ldots, \overline{x}_p$ for the maximal ideal $\mathfrak{m}_0$ of B_0. Since the inverse system $\{\mathfrak{m}_i\}$ has surjective transition maps, we can lift each $\overline{x}_i$ to an element $x_i \in \mathfrak{m}$. Let $I \subseteq B$ denote the ideal generated by the x_i. Assertions (a) and (b) are immediate consequences of the following:

(c) For each $n \geq 0$, we have $\mathfrak{m}(n) = I^n$.

To prove (c), choose generators $y_1, \ldots, y_r$ for the ideal I^n. Note that since the associated graded rings

$$\bigoplus_{m \geq 0} \mathfrak{m}_i^m/\mathfrak{m}_i^{m+1}$$

are generated over k by the images of the x_i, the truncated ring

$$\bigoplus_{0 \leq m' \leq m} \mathfrak{m}_i^{m'}/\mathfrak{m}_i^{m'+1}$$

have dimension at most $\binom{p+m}{m}$ over k. It follows that the quotients $B_i/\mathfrak{m}_i^{m+1}$ have length at most $\binom{p+m}{m}$, so that the inverse systems $\{B_i/\mathfrak{m}_i^{m+1}\}_{i \geq 0}$ are eventually constant. It follows that for each $m \geq 0$, the map $B/\mathfrak{m}(m+1) \to B_i/\mathfrak{m}_i^{m+1}$ is an isomorphism for $i \gg 0$. Since the maps

$$I^m \to \mathfrak{m}_i^m \to \mathfrak{m}_i^m/\mathfrak{m}_i^{m+1}$$

are surjective, we conclude that each of the maps $I^m \to \mathfrak{m}(m)/\mathfrak{m}(m+1)$ is surjective.

Now let z be an arbitrary element of $\mathfrak{m}(n)$. We define a sequence of approximations $z_n, z_{n+1}, z_{n+2} \ldots \in I^n$ with

$$z \equiv z_m \pmod{)} m(m)$$

by induction as follows. Set $z_n = 0$. Assuming that z_m has been defined for $m \geq n$, we use the surjectivity of the map $I^m \to \mathfrak{m}(m)/\mathfrak{m}(m+1)$ to write

$$z - z_m \equiv \sum_{1 \leq i \leq r} c_{i,m} y_i \pmod{\mathfrak{m}(m+1)}$$

for some elements $c_{i,m} \in I^{m-n}$. For each $1 \leq i \leq r$, the sequence of finite sums $\{\sum_{m \leq m_0} c_{i,m}\}$ has eventually constant image in each B_i, and so converges uniquely to an element $c_i \in m$. It is now easy to check that $z = \sum_{1 \leq i \leq q} c_i y_i$ belongs to the ideal I^n, as desired. This completes the proof that B is Noetherian.

It remains to show that the pro-objects $\{B_i\}_{i \geq 0}$ and $\{B/\mathfrak{m}^n\}_{n \geq 0}$ are isomorphic. For each $i \geq 0$, let $I(i) \subseteq B$ denote the kernel of the map $B \to B_i$. It will suffice to show that the sequences of ideals

$$\cdots \subseteq I(3) \subseteq I(2) \subseteq I(1) \subseteq I(0) \subseteq B$$

$$\cdots \subseteq \mathfrak{m}^3 \subseteq \mathfrak{m}^2 \subseteq \mathfrak{m} \subseteq B$$

are mutually cofinal. In other words:

- For each $i \geq 0$, there exists an integer n such that $\mathfrak{m}^n \subseteq I(i)$. This follows immediately from the fact that the ideal $\mathfrak{m}_i$ is nilpotent.

- For each $n \geq 0$, there exists an integer i such that $I(i) \subseteq \mathfrak{m}^n$. This follows from the fact that the tower $\{B_i/\mathfrak{m}_i^n\}$ is eventually constant, which was established above.

□

Lemma 6.3.2. *Let R be an associative ring, and suppose we are given a tower*

$$\cdots \to M_2 \to M_1 \to M_0$$

consisting of discrete R-modules of finite length. Then:

(a) The group $\varprojlim^1 \{M_i\}$ is trivial.

(b) If the inverse limit $\varprojlim^0 \{M_i\} \simeq 0$, then the tower $\{M_i\}$ is isomorphic to zero as a pro-object of Mod_R.

Proof. For integers $i \leq j$, let $M_{i,j}$ denote the image of the map $M_j \to M_i$. We have a decreasing system of submodules

$$M_i = M_{i,i} \supseteq M_{i,i+1} \supseteq M_{i,i+2} \supseteq \cdots$$

Since M_i has finite length, this sequence eventually stabilizes to some submodule $M_{i,\infty} \subseteq M_i$. Let N_i denote the quotient $M_i/M_{i,\infty}$, so that we have a tower of short exact sequences

$$0 \to M_{i,\infty} \to M_i \to N_i \to 0.$$

For each $i \geq 0$, we can choose an integer $j \geq i$ such that the map $M_j \to M_i$ factors through $M_{i,\infty}$. It follows that the map $N_j \to N_i$ is zero. Consequently, the tower $\{N_i\}$ is a zero object in the category of pro-R-modules. It follows that the inclusion $\{M_{i,\infty}\} \hookrightarrow \{M_i\}$ is an isomorphism of pro-objects; we may therefore replace $\{M_i\}$ by $\{M_{i,\infty}\}$ and thereby reduce to the case where each of the maps $M_{i+1} \to M_i$ is surjective. In this case, assertion (a) is obvious. To prove (b), we note that if $M = \varprojlim^0 M_i$, then the map $M \to M_i$ is a surjection for each i. If M is zero, we conclude that each M_i is zero. □

Lemma 6.3.3. *Let $B = \varprojlim B_i$ be as in Lemma 6.3.1. Then the canonical map*

$$\varinjlim \operatorname{Spec}_{\mathrm{f}} B_i \to \operatorname{Spec}_{\mathrm{f}} B$$

is an equivalence in $\mathrm{Moduli}_{/k}$

Proof. Let $\mathfrak{m} \subseteq B$ be the maximal ideal, and choose a tower of B-algebras

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the hypotheses of Lemma 5.1.5. Let $R \in \mathrm{CAlg}^{\mathrm{sm}}_{/k}$, and assume that R is n-truncated. Since every map $B \to R$ annihilates some power of the maximal ideal $\mathfrak{m}$, the canonical map

$$\varinjlim \mathrm{Map}_{\mathrm{CAlg}_{/k}}(\tau_{\leq n} A_i, R) \simeq \varinjlim \mathrm{Map}_{\mathrm{CAlg}_{/k}}(A_i, R) \to \mathrm{Map}_{\mathrm{CAlg}_{/k}}(B, R)$$

is a homotopy equivalence. It will therefore suffice to prove the following:

($*$) The towers $\{\tau_{\leq n} A_i\}_{i \geq 0}$ and $\{B_i\}_{i \geq 0}$ are equivalent (when regarded as Pro-objects of $(\mathrm{CAlg}_B)_{/k}$)

We prove ($*$) by induction on n. In the case $n = 0$, it follows from Lemma 6.3.1 that both towers are Pro-equivalent to the tower of commutative rings

$$\cdots \to B/\mathfrak{m}^3 \to B/\mathfrak{m}^2 \to B/\mathfrak{m} \simeq k.$$

To carry out the inductive step, it will suffice to show that the towers $\{\tau_{\leq n-1} A_i\}_{i \geq 0}$ and $\{\tau_{\leq n} A_i\}_{i \geq 0}$ are Pro-equivalent for $n > 0$. Using Theorem A.7.4.1.26, we can construct a pullback square of towers

$$\begin{array}{ccc} \{\tau_{\leq n} A_i\}_{i \geq 0} & \longrightarrow & \{\tau_{\leq n-1} A_i\}_{i \geq 0} \\ \downarrow & & \downarrow \\ \{\tau_{\leq n-1} A_i\}_{i \geq 0} & \longrightarrow & \{(\tau_{\leq n-1} A_i) \oplus (\pi_n A_i)[n+1]\}_{i \geq 0} \end{array}$$

It will therefore suffice to show that the right horizontal map in this diagram is an equivalence of Pro-objects. In fact, we claim that the tower $\{\pi_n A_i\}$ is zero when regarded as a Pro-object of Mod_B. Each $\pi_m A_i$ is a finitely generated module over B which is annihilated by some power of $\mathfrak{m}$, and therefore has finite length. It follows that

$$\varprojlim{}^0 \{\pi_n A_i\} \simeq \pi_n \varprojlim A_i \simeq \pi_n B \simeq 0,$$

so that $\{\pi_n A_i\}$ is trivial as a pro-object of Mod_B by Lemma 6.3.2. □

Lemma 6.3.4. *Suppose that R is a complete local Noetherian ring with maximal ideal $\mathfrak{m}$, and that*

$$\ldots \to M_2 \to M_1 \to M_0$$

is an inverse system of finitely generated R-modules. Let $\mathfrak{m}$ be the maximal ideal of R, and suppose that each map $M_{j+1}/\mathfrak{m}M_{j+1} \to M_j/\mathfrak{m}M_j$ is an isomorphism. Then the inverse limit $M = \varprojlim^0\{M_j\}$ is a finitely generated R-module, and $\varprojlim^1\{M_j\} \simeq 0$.

Proof. Using Nakayama's lemma, we deduce that each of the maps $M_{i+1} \to M_i$. It follows that the the map $M \to M_0$ is surjective, so we can choose a finite collection of elements $x_1, \ldots, x_n \in M$ whose images form a basis for the vector space $M_0/\mathfrak{m}M_0$. Then the images of the x_i in each M_j form a basis for $M_j/\mathfrak{m}M_j$. It follows from Nakayama's lemma that each M_j is generated by the images of the x_i. For each integer $c \geq 0$, the quotient $M_j/\mathfrak{m}^c M_j$ has length at most nq, where q denotes the length of the Artinian ring $R/\mathfrak{m}^c R$. It follows that there exists an integer m_c such that the maps

$$M_{j+1}/\mathfrak{m}^c M_{j+1} \to M_j/\mathfrak{m}M_j$$

are bijective for $j \geq m_c$.

Fix an element $y \in M$. We will define a sequence of elements $y_0, y_1, \ldots \in M$ such that each of the differences $y - y_c$ has vanishing image in $M_j/\mathfrak{m}^c M_j$, for all j. Set $y_0 = 0$. Assume that $c \geq 0$ and that y_c has been defined, and let $m = m_{c+1}$. Then the image of $y - y_c$ in M_m belongs to $\mathfrak{m}^c M_m$. It follows that we can choose elements $\{\lambda_{i,c} \in \mathfrak{m}^c\}_{1 \leq i \leq n}$ such that y and $y_c + \sum_{1 \leq i \leq n} \lambda_{i,c} x_i$ have the same image in M_m. Set $y_{c+1} = y_c + \sum_{1 \leq i \leq n} \lambda_{i,c} x_i$; it follows from the choice of m that y_{c+1} has the required property.

For $1 \leq i \leq n$, the sum $\sum_{c \geq 0} \lambda_{i,c}$ converges $\mathfrak{m}$-adically to an element $\lambda_i \in R$ (since R is $\mathfrak{m}$-adically complete). It follows that the image of $z = y - \sum_{1 \leq i \leq n} \lambda_i x_i$ vanishes in each quotient $M_j/\mathfrak{m}^c M_j$. Since M_j is a finitely generated R-module, it is $\mathfrak{m}$-adically complete: we therefore deduce that the image of z in each M_j is zero. Since $M = \varprojlim^0 \{M_j\}$, we conclude that $z = 0$: that is, y belongs to the submodule of M generated by the elements x_i. □

Remark 6.3.5. In the situation of Lemma 6.3.4, suppose that each M_j is an R-module of finite length. Then there is a canonical isomorphism of Pro-systems

$$\{M_j\}_{j \geq 0} \simeq \{M/\mathfrak{m}^i M\}_{i \geq 0}.$$

To prove this, let $K_j \subseteq M$ be the kernel of the surjection $M \to M_j$; we claim that the descending chains of submodules

$$K_0 \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$$
$$M \supseteq \mathfrak{m}M \supseteq \mathfrak{m}^2 M \supseteq \mathfrak{m}^3 M \supseteq \cdots$$

are mutually cofinal. This is equivalent to the following pair of assertions:

(*a*) For every integer j, there exists an integer i such that $\mathfrak{m}^i M \subseteq K_i$. In other words, each M_j is annihilated by a sufficiently large power of the maximal ideal $\mathfrak{m}$: this follows from our assumption that M_j has finite length.

(*b*) For every integer c, there exists an integer m such that $K_m \subseteq \mathfrak{m}^c M$. In fact, we can take m to be the integer m_c appearing in the proof of Lemma 6.3.4. If $y \in K_m$, then the image of y in M_m is contained in $\mathfrak{m}^c M_m$. It follows that we can take $y_0 = y_1 = \cdots = y_c = 0$ in the proof of Lemma 6.3.4, so that the expression

$$y = \sum \lambda_i x_i$$

belongs to $\mathfrak{m}^c M$.

Lemma 6.3.6. *Let R and $\{M_j\}_{j \geq 0}$ be as in Lemma 6.3.4, let $M = \varprojlim M_j$, let A be a Noetherian $\mathbb{E}_\infty$-ring with $\pi_0 A = R$, and suppose we are given a map $\eta : L_A \to M[n+1]$ classifying a square-zero extension of $\widetilde{A}$ of A by $M[n]$. For each $j \geq 0$, let A_j denote the square-zero extension of A by M_j determined by the composite map*

$$A \xrightarrow{\eta} M[n+1] \to M_j[n+1].$$

Then the canonical map

$$\theta : \varinjlim \operatorname{Spec_f} A_j \to \operatorname{Spec_f} \widetilde{A}$$

is an equivalence in $\mathrm{Moduli}_{/k}$.

Proof. For each $m \geq 0$, let θ_m denote the canonical map

$$\theta_m : \varinjlim \operatorname{Spec_f} A \oplus M_j[m] \to \operatorname{Spec_f} A \oplus M[m].$$

Applying Proposition 6.2.7 to the pullback diagrams

$$\begin{array}{ccc} \widetilde{A} & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \oplus M[n+1] \end{array} \qquad \begin{array}{ccc} A_j & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \oplus M_j[n+1], \end{array}$$

we obtain a pushout diagram

$$\begin{array}{ccc} \theta & \leftarrow & \mathrm{id}_{\mathrm{Spec}_{\mathrm{f}}\, A} \\ \uparrow & & \uparrow \\ \mathrm{id}_{\mathrm{Spec}_{\mathrm{f}}\, R} & \leftarrow & \theta_{n+1} \end{array}$$

in the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{Moduli}_{/k})$. It will therefore suffice to show that θ_{n+1} is an equivalence.

We will prove that each θ_m is an equivalence. Choose a tower of A-algebras

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma 5.1.5, so that the canonical maps

$$\varinjlim \mathrm{Spec}_{\mathrm{f}}\, A_i \to \mathrm{Spec}_{\mathrm{f}}\, A \qquad \varinjlim \mathrm{Spec}_{\mathrm{f}}\, A_i \oplus (A_i \otimes_A M[m]) \to \mathrm{Spec}_{\mathrm{f}}\, A \oplus M[m]$$

are equivalences. Proposition 6.2.7 implies that the diagrams

$$\begin{array}{ccc} \mathrm{Spec}_{\mathrm{f}}\, A_i & \longrightarrow & \mathrm{Spec}_{\mathrm{f}}\, A \\ \downarrow & & \downarrow \\ \mathrm{Spec}_{\mathrm{f}}\, A_i \oplus (A_i \otimes_A M[m]) & \longrightarrow & \mathrm{Spec}_{\mathrm{f}}\, A \oplus (A_i \otimes_A M[m]) \end{array}$$

are pushout squares. Passing to the direct limit, we conclude that the map

$$\varinjlim \mathrm{Spec}_{\mathrm{f}}\, A_i \oplus (A_i \otimes_A M[m]) \to \varinjlim \mathrm{Spec}_{\mathrm{f}}\, A \oplus (A_i \otimes_A M[m])$$

is an equivalence, from which it follows that

$$\mathrm{Spec}_{\mathrm{f}}\, A \oplus M[m] \simeq \varinjlim_i \mathrm{Spec}_{\mathrm{f}}\, A \oplus (A_i \otimes_A M[m]).$$

Let $B \in \mathrm{CAlg}^{\mathrm{sm}}_{/k}$, and assume that B is q-truncated for some positive integer $q \geq m$. We wish to show that the map

$$\varinjlim \mathrm{Map}_{\mathrm{CAlg}_{/k}}(A \oplus M_j[m], B) \simeq \mathrm{Map}_{\mathrm{CAlg}_{/k}}(A \oplus M[m], B)$$

is a homotopy equivalence. The argument above shows that the map

$$\varinjlim \mathrm{Map}_{\mathrm{CAlg}_{/k}}(A \oplus \tau_{\leq q}(A_i \otimes_A M[m]), B) \simeq \varinjlim \mathrm{Map}_{\mathrm{CAlg}_{/k}}(A \oplus (A_i \otimes_A M[m]), B) \to \varinjlim \mathrm{Map}_{\mathrm{CAlg}_{/k}}(A \oplus M, B)$$

is a homotopy equivalence. It will therefore suffice to prove the following:

$(*)$ The towers $\{M_j\}_{j \geq 0}$ and $\{\tau_{\leq q-m}(A_i \otimes_A M)\}_{i \geq 0}$ are equivalent as Pro-objects of $(\mathrm{Mod}_A)_{M/}$.

The proof of $(*)$ proceeds by induction on q. We first treat the case where $q = m$. Since the homotopy groups $\pi_p(A_i \otimes_A M)$ are R-modules of finite length, Lemma 6.3.2 gives isomorphisms

$$\varprojlim{}^0 \pi_p(A_i \otimes_A M) \simeq \pi_p \varprojlim (A_i \otimes_A M) \simeq \begin{cases} M & \text{if } p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Applying Remark 6.3.5, we deduce that the towers $\{M_j\}_{j \geq 0}$ and $\{\tau_{\leq 0}(A_i \otimes_A M)\}_{i \geq 0}$ are both Pro-isomorphic to the tower $\{M/\mathfrak{m}^c M\}_{c \geq 0}$. To carry out the inductive step, let us suppose that $q \geq m$ and that the towers $\{M_j\}_{j \geq 0}$ and $\{\tau_{\leq q-m}(A_i \otimes_A M)\}_{i \geq 0}$ are equivalent. We have a fiber sequence of towers

$$\{\tau_{\leq q-m}(A_i \otimes_A M)\}_{i \geq 0} \to \{\tau_{\leq q+1-m}(A_i \otimes_A M)\}_{i \geq 0} \to \{(\pi_{q+1-m}(A_i \otimes_A M))[q+1-m]\}.$$

It will therefore suffice to show that the tower of discrete R-modules $\{\pi_{q+1-m}(A_i \otimes_A M)\}$ is trivial as Pro-object. This follows from Lemma 6.3.2 (since each $\pi_{q+1-m}(A_i \otimes_A M)$ is finitely generated as a module over $\pi_0 A_i$, and therefore of finite length as an R-module). $\square$

Proof of Theorem 6.2.5. We will construct a tower

$$\cdots \to A_2 \to A_1 \to A_0$$

in $\mathrm{CAlg}^{\mathrm{cg}}_{/k}$ such that each of the maps $A_{n+1} \to A_n$ exhibits A_n as an n-truncation of A_{n+1}, and a compatible sequence of natural transformations

$$\phi_n : \mathrm{Spec}_{\mathrm{f}}\, A_n \to X$$

with the following property: for each integer n, the induced map

$$\pi_i T^{\mathrm{red}}_{\mathrm{Spec}_{\mathrm{f}}\, A_n} \to \pi_i T^{\mathrm{red}}_X$$

is an isomorphism for $-n \leq i \leq 0$ and an injection for $i = -n-1$. Assuming that such a sequence can be found, we take $A = \varprojlim A_n$. Then $\mathrm{Spec}_{\mathrm{f}}\, A \simeq \varinjlim \mathrm{Spec}_{\mathrm{f}}\, A_n$, so we obtain a map $\mathrm{Spec}_{\mathrm{f}}\, A \to X$ which evidently has the desired properties.

Our construction proceeds by induction. We begin with the case $n = 0$, where we essentially reproduce the proof of the main theorem of [59]. We construct the ordinary Noetherian ring A_0 as the inverse limit of a sequence of local Artinian rings $\{B_j\}$ with residue field k, equipped with maps $\psi_j : \mathrm{Spec}_{\mathrm{f}}\, B_j \to X$, which we will identify with points of $X(B_j)$. We begin by setting $B_0 = k$, and take ψ_0 to be any point of the contractible space $X(k)$.

Assuming now that that B_j and ψ_j have already been constructed for some integer $j \geq 0$. Let F_j denote the fiber of the map of tangent complexes $T_{\mathrm{Spec}_{\mathrm{f}}\, B_j} \to T_X$, and let $V_j = \pi_{-1} F_j$. We have an exact sequence of vector spaces

$$\pi_0 T^{\mathrm{red}}_X \to V_j \to \pi_{-1} T^{\mathrm{red}}_{\mathrm{Spec}_{\mathrm{f}}\, B_j}$$

which shows that V_j is finite dimensional over k. Choose a map of k-module spectra $\eta : V_j[-1] \to F_j$ inducing an isomorphism on π_{-1}. Then η determines a map $\eta_0 : B_j \to k \oplus V_j^\vee[1]$ and a point of the fiber product

$$\eta_1 \in X(B_j) \times_{X(k \oplus V_j^\vee[1])} X(k)$$

lying over ψ_j. Let B_{j+1} denote the square-zero extension of B_j by V_j determined by η_0, so that η_1 determines a map $\psi_{j+1} : \mathrm{Spec}_{\mathrm{f}}\, B_{j+1} \to X$ extending $\phi(0)_j$.

For each $j \geq 0$, let $\mathfrak{m}_j$ denote the maximal ideal of B_j. We claim that for $j \geq 1$, the map $B_{j+1} \to B_j$ induces an isomorphism of Zariski cotangent spaces $\theta : (\mathfrak{m}_{j+1}/\mathfrak{m}_{j+1}^2) \to (\mathfrak{m}_j/\mathfrak{m}_j^2)$. The surjectivity of θ is clear. To prove injectivity, we note that Theorem A.7.4.3.1 supplies canonical isomorphisms

$$\mathfrak{m}_j/\mathfrak{m}_j^2 \simeq \pi_1 L_{k/B_j} \qquad V_j^\vee \simeq \pi_1(k \otimes_{B_{j+1}} L_{B_{j+1}/B_j}).$$

The fiber sequence of k-module spectra

$$k \otimes_{B_j} L_{B_{j+1}/B_j} \to L_{k/B_{j+1}} \to L_{k/B_j}$$

gives a long exact sequence of vector spaces

$$\pi_2 L_{k/B_j} \xrightarrow{\theta'} V_j^\vee \to \mathfrak{m}_{j+1}/\mathfrak{m}_{j+1}^2 \xrightarrow{\theta} \mathfrak{m}_j/\mathfrak{m}_j^2.$$

We therefore see that the injectivity of θ is equivalent to the surjectivity of θ'. The dual of θ' is the canonical map from $V_j \simeq \pi_{-1} F_j$ into $\pi_{-1} T^{\mathrm{red}}_{\mathrm{Spec}_{\mathrm{f}}\, B_j}$. Using the long exact sequence

$$\pi_0 T^{\mathrm{red}}_{\mathrm{Spec}_{\mathrm{f}}\, B_j} \xrightarrow{\theta''} \pi_0 T^{\mathrm{red}}_X \to \pi_{-1} F_j \xrightarrow{\theta'^\vee} \pi_{-1} T^{\mathrm{red}}_{\mathrm{Spec}_{\mathrm{f}}\, B_j},$$

we are reduced to proving that the map θ'' is surjective. This is clear, since the composite map

$$V_0 \simeq \pi_0 T^{\mathrm{red}}_{\mathrm{Spec}_{\mathrm{f}}\, B_1} \to \pi_0 T^{\mathrm{red}}_{\mathrm{Spec}_{\mathrm{f}}\, B_j} \xrightarrow{\theta''} T^{\mathrm{red}}_X$$

is an isomorphism by construction (note that $F_0 = T_X^{\mathrm{red}}[1]$).

Now we may apply Lemma 6.3.1 to conclude that the inverse limit A_0 of the tower

$$\ldots \to B_2 \to B_1 \to B_0 = k$$

is Noetherian, and Lemma 6.3.3 implies that the map

$$\varinjlim \operatorname{Spec_f} B_i \to \operatorname{Spec_f} A_0$$

is an equivalence. The compatible family of maps ψ_j induce a natural transformation $\phi_0 : \operatorname{Spec_f} A_0 \to X$. Let F denote the fiber of the canonical map $T_{\operatorname{Spec_f} A_0} \to T_{\operatorname{Spec_f} X}$. Then $F \simeq \varinjlim F_j$. By construction, each of the maps $\pi_{-1} F_j \to \pi_{-1} F_{j+1}$ is zero. It follows that $\pi_{-1} F \simeq 0$. Using the fiber sequence

$$F \to T^{\mathrm{red}}_{\operatorname{Spec_f} A_0} \to T^{\mathrm{red}}_X,$$

we conclude that the map $\pi_i T^{\mathrm{red}}_{\operatorname{Spec_f} A} \to T^{\mathrm{red}}_X$ is injective for $i = -1$ and surjective for $i = 0$. Note that $\pi_0 T^{\mathrm{red}}_{\operatorname{Spec_f} A_0}$ is the filtered colimit of the Zariski tangent spaces $(\mathfrak{m}_j / \mathfrak{m}_j^2)^\vee$. Since these tangent spaces are isomorphic for $j \geq 1$, we deduce that the canonical map

$$V_0 \simeq (\mathfrak{m}_1 / \mathfrak{m}_1^2) \to \pi_0 T^{\mathrm{red}}_{\operatorname{Spec_f} A}$$

is an isomorphism. Since the composite map

$$V_0 \to \pi_0 T^{\mathrm{red}}_{\operatorname{Spec_f} A_0} \to \pi_0 T^{\mathrm{red}}_X$$

is an isomorphism by construction, we deduce that the map $\pi_0 T^{\mathrm{red}}_{\operatorname{Spec_f} A_0} \to \pi_0 T^{\mathrm{red}}_X$ is an isomorphism. This completes the construction of the map $\phi_0 : \operatorname{Spec_f} A_0 \to X$.

Let us now suppose that $n \geq 0$ and the map $\phi_n : \operatorname{Spec_f} A_n \to X$ has been constructed. We will construct A_{n+1} as the limit of a tower of Noetherian $\mathbb{E}_\infty$-rings

$$\cdots \to C_3 \to C_2 \to C_1 \to C_0 = A_i,$$

equipped with compatible maps $\nu_j : \operatorname{Spec_f} C_j \to X$, starting with $\nu_0 = \phi_i$. Assume that ν_j has been defined for $j \geq 0$, let F_j denote the fiber of the induced map of tangent complexes $T_{\operatorname{Spec_f} C_j} \to T_X$, and set $V_j = \pi_{-n-2} F_j$. Choose a map $\eta : V_j[-n-2] \to F_j'$ which induces the identity on π_{-n-2}, so that η determines a map of $\mathbb{E}_\infty$-rings $\eta_0 : C_j \to k \oplus V_j^\vee[n+2]$ together with a nullhomotopy of the composite map

$$\operatorname{Spec_f}(k \oplus V_j^\vee[n+2]) \to \operatorname{Spec_f} C_j \to X.$$

We define C_{j+1} to be the square-zero extension of C_j determined by η_0. It follows from Proposition 6.2.7 that the diagram

$$\begin{array}{ccc} \operatorname{Spec_f} k \oplus V_j^\vee[n+2] & \longrightarrow & \operatorname{Spec_f} k \\ \downarrow & & \downarrow \\ \operatorname{Spec_f} C_j & \longrightarrow & \operatorname{Spec_f} C_{j+1} \end{array}$$

is a pushout square in $\mathrm{Moduli}_{/k}$, so that η determines a map $\nu_{j+1} : \operatorname{Spec_f} C_{j+1} \to X$ extending ν_j.

Since each of the maps $C_{j+1} \to C_j$ is $(n+2)$-connective, Theorem A.7.4.3.1 supplies $(2n+4)$-connective maps

$$k \otimes_{C_j} V_j^\vee[n+2] \to k \otimes_{C_j} L_{C_j / C_{j+1}}.$$

It follows that

$$\pi_m(k \otimes_{C_j} L_{C_j/C_{j+1}}) \simeq \begin{cases} V_j^\vee[n+2] & \text{if } m = n+2 \\ 0 & \text{if } m < n+2. \end{cases}$$

Combining this informatio with the fiber sequence of k-module spectra

$$(k \otimes L_{C_j/C_{j+1}})^\vee \to F_j \to F_{j+1},$$

we deduce that $\pi_m F_j \to \pi_m F_{j+1}$ is an isomorphism for $m > -n$, and that there is an exact sequence of k-vector spaces

$$0 \to \pi_{-n-1} F_j \to \pi_{-n-1} F_{j+1} \to V_j \xrightarrow{\beta} \pi_{-n-2} F_j \to \pi - n - 2 F_{j+1}.$$

By construction, the map β is an isomorphism. It follows that $\pi_{-n-1} F_j \simeq \pi_{-n-1} F_{j+1}$ and that the map $\pi_{-n-2} F_j \to \pi_{-n-2} F_{j+1}$ is zero. Since $\pi_m F_0 \simeq 0$ for $-n-2 < m < 0$, it follows by induction that $\pi_m F_j \simeq 0$ for $-n-2 < m < 0$ and every nonnegative integer j.

For each $j \geq 0$, we have a commutative diagram

$$\begin{array}{ccc} \pi_{-n-2}(k \otimes_{A_n} L_{A_n/C_{j+1}})^\vee & \longrightarrow & \pi_{-n-2} F_0 \\ \downarrow{\scriptstyle \theta_j} & & \downarrow \\ \pi_{-n-2}(k \otimes_{C_j} L_{C_j/C_{j+1}})^\vee & \longrightarrow & \pi_{-n-2} F_j. \end{array}$$

If $j > 0$, then the right vertical map is zero. Since the bottom horizontal map is an isomorphism, we conclude that the left vertical map is zero. Let M_j denote the discrete A_0-module given by $\pi_{n+1} C_j$. Since each of the maps $C_j \to A_n$ is $(n+2)$-connective, Theorem A.7.4.3.1 supplies $(2n+4)$-connective maps

$$k \otimes C_j M_j[n+2] \to k \otimes_{A_n} L_{A_n/C_j}.$$

Let $\mathfrak{m}$ denote the maximal ideal of A_0, so that we obtain an isomorphism $M_j/\mathfrak{m}M_j \simeq \pi_{n+2}(k \otimes_{A_n} L_{A_n/C_j})$. By construction, we have $M_0 = 0$ and for each $j \geq 0$ a short exact sequence

$$0 \to V_j^\vee \to M_{j+1} \to M_j \to 0.$$

In particular, we obtain exact seuqences of k-vector spaces

$$V_j^\vee \xrightarrow{\theta_j^\vee} M_{j+1}/\mathfrak{m}M_{j+1} \to M_j/\mathfrak{m}M_j \to 0.$$

Since $\theta_j = 0$ for $j > 0$, we deduce that the maps $M_{j+1}/\mathfrak{m}M_{j+1} \to M_j/\mathfrak{m}M_j$ are isomorphisms. Applying Lemma 6.3.4, we conclude that that $M = \varprojlim M_j$ is a finitely generated module over A_0 and that $\varprojlim^1 M_j \simeq 0$. Set $A_{n+1} = \varprojlim C_j$, so that A_{n+1} is an $(n+1)$-truncated $\mathbb{E}_\infty$-ring with $\tau_{\leq n} A_{n+1} \simeq A_n$ and $\pi_{n+1} A_{n+1} = M$. In particular, A_{n+1} is Noetherian.

Using Theorem A.7.4.1.26, we see that A_{n+1} is a square-zero extension of A_n by $M[n+1]$. Using Lemma 6.3.6, we see that the map $\varinjlim \operatorname{Spec_f} C_j \to \operatorname{Spec_f} A_{n+1}$ is an equivalence in $\operatorname{Moduli}_{/k}$. It follows that the compatible family of maps $\nu_j : \operatorname{Spec_f} C_j \to X$ determine a map $\phi_{n+1} : \operatorname{Spec_f} A_{n+1} \to X$. We claim that ϕ_{n+1} has the desired properties. Let U denote the fiber of the map of tangent complexes $T_{\operatorname{Spec_f} A_{n+1}} \to T_X$, so that $U \simeq \varinjlim F_j$. If $0 > m > -n-2$, then $\pi_m F_j \simeq 0$ for all j, so that $\pi_m U \simeq 0$. Since each of the maps $\pi_{-n-2} F_j \to \pi_{-n-2} F_{j+1}$ is zero, we also conclude that $\pi_{-n-2} U \simeq 0$. Using the fiber seqeunce

$$U \to T^{\text{red}}_{\operatorname{Spec_f} A_{n+1}} \to T^{\text{red}}_X,$$

we conclude that the map

$$\pi_m T^{\text{red}}_{\operatorname{Spec_f} A_{n+1}} \to \pi_m T^{\text{red}}_X$$

is an isomorphism for $0 > m > -n-2$, injective when $m = -n-2$, and surjective for $m = 0$. To prove the injectivity when $m = 0$, we note that the composite map

$$\pi_0 T^{\text{red}}_{\operatorname{Spec_f} A_0} \xrightarrow{\mu} \pi_0 T^{\text{red}}_{\operatorname{Spec_f} A_{n+1}} \to \pi_0 T^{\text{red}}_X$$

is an isomorphism. It will therefore suffice to show that the map μ is surjective. This map μ fits into an exact sequence of k-vector spaces

$$\pi_0 T^{\mathrm{red}}_{\mathrm{Spec_f}\, A_0} \to \pi_0 T^{\mathrm{red}}_{\mathrm{Spec_f}\, A_{n+1}} \to (\pi_1 k \otimes_{A_0} L_{A_0/A_{n+1}})^{\vee}$$

Consequently, to prove that μ is surjective, it suffices to verify that the relative cotangent complex $L_{A_0/A_{n+1}}$ is 2-connective. This follows from Corollary A.7.4.3.2, since the map $A_{n+1} \to A_0$ has 2-connective cofiber. □

A Stone Duality

Let X be a topological space. We say that X is a *Stone space* if it is compact, Hausdorff, and has no connected subsets consisting of more than one point. The category of Stone spaces has many different incarnations:

(*a*) According to the Stone duality theorem (Theorem A.3.26), a topological space X is a Stone space if and only if it is homeomorphic to the spectrum $\mathrm{Spt}(B)$ of a Boolean algebra B. Moreover, the construction $B \mapsto \mathrm{Spt}(B)$ determines a (contravariant) equivalence from the category of Boolean algebras to the category of Stone spaces.

(*b*) For any filtered inverse system of finite sets $\{S_\alpha\}$, the inverse limit $\varprojlim S_\alpha$ is a Stone space (when endowed with the inverse limit topology). This construction determines an equivalence from the category of profinite sets to the category of Stone spaces (Proposition A.1.6).

(*c*) Let p be a prime number and let R be a commutative ring in which $p = 0$ and every element $x \in R$ satisfies $x^p = x$ (in this case, we say that R is a *p-Boolean algebra*). Then the Zariski spectrum $\mathrm{Spec}^{\mathrm{Z}} R$ is a Stone space. Moreover, the construction $R \mapsto \mathrm{Spec}^{\mathrm{Z}} R$ determines a (contravariant) equivalence from the category of p-Boolean algebras to the category of Stone spaces (Proposition A.1.12; in the case $p = 2$, this reduces to the equivalence of (*a*)).

In this appendix, we will review the construction of the equivalences of categories described above. We begin in §A.1 with a quick exposition of the theory of Stone spaces and giving proofs of (*b*) and (*c*). In §A.2 we review the definition of the spectrum $\mathrm{Spt}(P)$ of an arbitrary distributive upper-semilattice P, and show that this construction determines a fully faithful embedding from the category of distributive upper-semilattices to a full subcategory of the category Top of topological spaces (Proposition A.2.14). In §A.3 we will specialize this result to the case where P is a Boolean algebra and use it to give a proof of (*a*). We will also record a few facts about the relationship between Boolean algebras and distributive lattices, which are useful in discussing constructible sets in algebraic geometry.

Remark A.0.7. The results described in this appendix are all well-known. We refer the reader to [28] for more details.

A.1 Stone Spaces

In this section, we review the theory of *Stone spaces*: that is, topological spaces which are compact, Hausdorff, and totally disconnected. We will see that the category of Stone spaces has many different descriptions: it can be obtained as a full subcategory of the category of topological spaces (Definition A.1.2), as the category of profinite sets (Proposition A.1.6), or as the category of p-Boolean algebras for any prime number p (Proposition A.1.12). We begin with a review of total disconnectedness.

Proposition A.1.1. *Let X be a compact Hausdorff space. The following conditions are equivalent:*

(*a*) *There exists a basis for the topology of X consisting of sets which are both closed and open.*

(*b*) *Every connected subset of X is a singleton.*

Proof. Suppose first that (a) is satisfied, and let $S \subseteq X$ be connected. Then S is nonempty; we wish to show that it contains only a single element. Suppose otherwise, and choose distinct points $x, y \in S$. Since X is Hausdorff, there exists an open set $U \subseteq X$ containing x but not y. Using condition (a), we can assume that the set U is also closed. Then $U \cap S$ and $(X - U) \cap S$ is a decomposition of S into nonempty closed and open subsets, contradicting the connectedness of S.

To prove the converse, we need the following fact:

$(*)$ Let $x, y \in X$. Assume that, for every closed and open subset $U \subseteq X$, if x belongs to U then y also belongs to U. Then there is a connected subset of X containing both x and y.

To prove $(*)$, consider the collection S of all closed subsets $Y \subseteq X$ which contain both x and y, having the property that any closed and open subset $U \subseteq Y$ containing x also contains y. Then S is nonempty (since $X \in S$). We claim that every linearly ordered subset of S has a lower bound in S. Suppose we are given such a linearly ordered set $\{Y_\alpha\}$, and let $Y = \bigcap Y_\alpha$. Then Y contains the points x and y. If $Y \notin S$, then we can decompose Y as the disjoint union of (closed and open) subsets $Y_-, Y_+ \subseteq Y$, with $x \in Y_-$ and $y \in Y_+$. Let us regard Y_- and Y_+ as compact subsets of X. Since X is Hausdorff, we can choose disjoint open sets $U_-, U_+ \subseteq X$ with $Y_- \subseteq U_-$ and $Y_+ \subseteq U_+$. The intersection

$$(X - U_-) \cap (X - U_+) \cap \bigcap_\alpha Y_\alpha$$

is empty. Since X is compact, we conclude that there exists an index α such that $(X - U_-) \cap (X - U_+) \cap Y_\alpha = \emptyset$. Then $Y_\alpha \cap U_-$ and $Y_\alpha \cap U_+$ are disjoint closed and open subsets of Y_α containing x and y respectively, contradicting our assumption that $Y_\alpha \in S$. This completes the proof that $Y \in S$, so that S satisfies the hypotheses of Zorn's lemma. We may therefore choose a minimal element $Z \in S$.

To complete the proof of $(*)$, it will suffice to show that Z is connected. Assume otherwise: then there exists a decomposition of Z into closed and open nonempty subsets $Z', Z'' \subseteq Z$. Since $Z \in S$, we have either $x, y \in Z'$ or $x, y \in Z''$; let us suppose that $x, y \in Z'$. The minimality of Z implies that $Z' \notin S$, so that Z' can be further decomposed into closed and open subsets $Z'_-, Z'_+ \subseteq Z'$ containing x and y, respectively. Then Z'_- and $Z'_+ \cup Z''$ are closed and open subsets of Z containing x and y, respectively, contradicting our assumption that $Z \in S$. This completes the proof of $(*)$.

Now suppose that (b) is satisfied; we wish to prove (a). It follows from condition $(*)$ that for every pair of distinct points $x, y \in X$, there exists a closed and open subset $V_{x,y}$ which contains y but does not contain x. Let $U \subseteq X$ be an open set; we wish to show that U contains a closed and open neighborhood of each point $x \in U$. Then $X - U$ is covered by the open sets $\{V_{x,y}\}_{y \in X-U}$. Since $X - U$ is compact, we can choose a finite subset $\{y_1, \ldots, y_n\} \subseteq X - U$ such that

$$X - U \subseteq \bigcup_{1 \leq i \leq n} V_{x,y_i}.$$

It follows that $X - \bigcup_{1 \leq i \leq n} V_{x,y_i}$ is a closed and open subset of X which contains x and is contained in U. □

Definition A.1.2. Let X be a topological space. We say that X is a *Stone space* if it is compact, Hausdorff, and satisfies the equivalent conditions of Proposition A.1.1. We let Top denote the category of topological spaces, and $\mathrm{Top}_{\mathrm{St}}$ the full subcategory of Top spanned by the Stone spaces.

Remark A.1.3. Let X be a compact Hausdorff space. The collection of closed and open subsets of X is closed under finite intersections. Consequently, to show that the X is a Stone space, it suffices to verify that the collection of closed and open sets forms a subbasis for the topology of X.

Remark A.1.4. Let X be a Stone space. Then every closed subset $Y \subseteq X$ is also a Stone space (with the induced topology).

Notation A.1.5. Let $\mathcal{S}\mathrm{et}$ denote the category of sets. We will abuse notation by identifying $\mathcal{S}\mathrm{et}$ with the full subcategory of Top spanned by those topological spaces which are endowed with the discrete topology. We let $\mathcal{S}\mathrm{et}^{\mathrm{fin}}$ denote the full subcategory of $\mathcal{S}\mathrm{et}$ spanned by the finite sets, and $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$ the category of Pro-objects of $\mathcal{S}\mathrm{et}^{\mathrm{fin}}$. We will refer to $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$ as the category of *profinite sets*.

Since the category Top admits filtered inverse limits, the inclusion $\mathcal{S}\mathrm{et} \subseteq \mathrm{Top}$ extends to a functor $\psi : \mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}}) \to \mathrm{Top}$ which preserves filtered inverse limits (moreover, this extension is unique up to unique isomorphism).

Proposition A.1.6. *The functor $\psi : \mathrm{Pro}(\mathcal{S}\mathrm{et}^{fin}) \to \mathrm{Top}$ of Notation A.1.5 is a fully faithful embedding, whose essential image is the full subcategory $\mathrm{Top}_{\mathrm{St}} \subseteq \mathrm{Top}$ spanned by the Stone spaces. In particular, the category of Stone spaces is equivalent to the category of profinite sets.*

The proof of Proposition A.1.6 will require some preliminaries.

Lemma A.1.7. *The category $\mathrm{Top}_{\mathrm{St}}$ of Stone spaces is closed under the formation of projective limits (in the larger category Top of topological spaces).*

Proof. Suppose we are given an arbitrary diagram $\{X_\alpha\}$ of Stone spaces; we wish to show that $\varprojlim X_\alpha$ is also a Stone space. Note that $\varprojlim_\alpha X_\alpha$ can be identified with a closed subspace of the product $\prod_\alpha X_\alpha$. It will therefore suffice to show that $\prod_\alpha X_\alpha$ is a Stone space (Remark A.1.4). This product is obviously Hausdorff, compact by virtue of Tychanoff's theorem, and has a subbasis consisting of inverse images of open subsets of the spaces X_α. Since each X_α has a basis of closed and open sets, we conclude that $\prod_\alpha X_\alpha$ has a subbasis consisting of closed and open sets, and is therefore a Stone space by Remark A.1.3. □

Lemma A.1.8. *Let A be a filtered partially ordered set, and suppose we are given a functor $X : A^{op} \to \mathcal{S}\mathrm{et}$. If the set $X(\alpha)$ is finite for each $\alpha \in A$, then the inverse limit $\varprojlim_{\alpha \in A} X(\alpha)$ is nonempty.*

Proof. Let S denote the collection of all subfunctors $X_0 \subseteq X$ such that the set $X_0(\alpha)$ is nonempty for each $\alpha \in A$. We regard S as a linearly ordered set with respect to inclusions. Note that any linearly ordered subset of S has an infimum in S, since the intersection of any chain of nonempty finite subsets of a finite set is again nonempty. It follows from Zorn's lemma that S has a minimal element $X_0 \subseteq X$. We will show that for each $\alpha \in A$, the set $X_0(\alpha)$ has a single element, so that $\varprojlim_{\alpha \in A} X_0(\alpha)$ consists of a single element. The desired result will then follow from the existence of a map $\varprojlim_{\alpha \in A} X_0(\alpha) \to \varprojlim_{\alpha \in A} X(\alpha)$.

Let $\alpha \in A$ and choose elements $x, y \in X_0(\alpha)$; we will prove that $x = y$. For $\beta \geq \alpha$, let $\phi_\beta : X_0(\beta) \to X_0(\alpha)$ be the corresponding map of finite sets, and define subfunctors $X_x, X_y \subseteq X_0$ by the formulae

$$X_x(\beta) = \begin{cases} \phi_\beta^{-1}(X_0(\alpha) - \{x\}) & \text{if } \beta \geq \alpha \\ X_0(\beta) & \text{otherwise.} \end{cases}$$

$$X_y(\beta) = \begin{cases} \phi_\beta^{-1}(X_0(\alpha) - \{y\}) & \text{if } \beta \geq \alpha \\ X_0(\beta) & \text{otherwise.} \end{cases}$$

Since X_0 was chosen to be a minimal element of S, we must have $X_x, X_y \notin S$. It follows that there exist elements $\beta, \gamma \in A$ such that the sets $X_x(\beta)$ and $X_y(\gamma)$ are empty. Since A is filtered, we may assume without loss of generality that $\beta = \gamma$. Note also that we must have $\beta \geq \alpha$, since otherwise $X_x(\beta) = X_0(\beta) \neq \emptyset$. Since $X_x(\beta) = \emptyset$, the map ϕ_β must be the constant map taking the value $x \in X_0(\alpha)$. The same argument shows that ϕ_β takes the constant value y. Since $X_0(\beta) \neq \emptyset$, this proves that $x = y$ as desired. □

Proof of Proposition A.1.6. Let us identify $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$ with the category of left exact functors $F : \mathcal{S}\mathrm{et}^{\mathrm{fin}} \to \mathcal{S}\mathrm{et}$. The functor ψ admits a left adjoint ϕ, which carries a topological space X to the left exact functor $\phi(X)$ given by the formula

$$\phi(X)(J) = \mathrm{Hom}_{\mathrm{Top}}(X, J).$$

We will prove the following:

(a) If S is a profinite set, then $X = \psi(S)$ is a Stone space. Moreover, the counit map $\phi(X) \to F$ is an isomorphism of profinite sets.

(b) If X is a Stone space and $S = \phi(X)$, then the unit map $X \to \psi(S)$ is a homeomorphism of topological spaces.

We begin by proving (a). Let S be a profinite set. Choose a filtered partially ordered set A and an isomorphism of profinite sets $S \simeq \varprojlim_{\alpha \in A} S_\alpha$ in $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$, where each S_α is a finite set. Then $X = \psi(S)$ can be identified with the inverse limit of the diagram $\{S_\alpha\}$ in the category Top of topological spaces, which is a Stone space by virtue of Lemma A.1.7. We now show that the map $\phi(X) \to S$ is an isomorphism of profinite sets. Unwinding the definitions, we must show that for every finite set T, the natural map

$$\theta : \varinjlim \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(S_\alpha, T) \to \mathrm{Hom}_{\mathrm{Top}}(X, T)$$

is a bijection. We first show that θ is injective. Suppose we are given a pair of maps $f_0, f_1 : S_\alpha \to T$ such that the composite maps $X \xrightarrow{\phi_\alpha} S_\alpha \to T$ coincide. We wish to show that there exists $\beta \geq \alpha$ such that the composite maps $S_\beta \to S_\alpha \to T$ coincide. Let $S' = \{s \in S_\alpha : f_0(s) \neq f_1(s)\}$. For each $s \in S'$, the inverse image $\phi_\alpha^{-1}\{s\} \subseteq X$ is empty. Using Lemma A.1.8, we deduce that the inverse image of $\{s\}$ in S_{β_s} is empty for some $\beta_s \geq \alpha$. Since S' is finite, we may choose $\beta \in A$ such that $\beta \geq \beta_s$ for all $s \in S'$. Then the inverse image of S' in S_β is empty, so that β has the desired property.

We now show that θ is surjective. Suppose we are given a continuous map $f : X \to T$. We wish to show that f factors through $\phi_\alpha : X \to S_\alpha$ for some index $\alpha \in A$. If T is empty, then X is empty and so (by Lemma A.1.8) the set S_α is empty for some $\alpha \in A$, and therefore f factors through S_α. Let us therefore assume that T is nonempty. Fix $t \in T$ and let $X_t = f^{-1}\{t\}$. Note that X_t is both open and closed in X. Since X is compact, X_t is also compact. The topological space X has a basis consisting of sets of the form $\phi_\alpha^{-1}\{s\}$, where $s \in S_\alpha$. In particular, for every point $x \in X_t$, we can choose a $\alpha_x \in A$ and a point $s_x \in S_{\alpha_x}$ such that $x \in \phi_{\alpha_x}^{-1}\{s_x\} \subseteq X_t$. The sets $U_x = \phi_{\alpha_x}^{-1}\{s_x\}$ form an open covering of X_t. Since X_t is compact, there exist finitely many points $x_1, \ldots, x_n \in X_t$ such that $X_t = \bigcup_{1 \leq i \leq n} U_{x_i}$. Since A is filtered, we can choose an index $\alpha_t \in A$ such that $\alpha_t \geq \alpha_{x_i}$ for $1 \leq i \leq n$. Because T is finite, we may further choose α such that $\alpha \geq \alpha_t$ for all $t \in T$. Let $S_t = \{s \in S_\alpha : \emptyset \neq \phi_\alpha^{-1}\{s\} \subseteq X_t\}$. Then

$$X_t \subseteq \bigcup_{1 \leq i \leq n} U_{x_i} \subseteq \phi_\alpha^{-1} S_t \subseteq X_t.$$

Note that the subsets $S_t \subseteq S_\alpha$ are disjoint. Since T is nonempty, there exists a map of finite sets $f' : S_\alpha \to T$ such that $S_t \subseteq {f'}^{-1}\{t\}$ for each $t \in T$. Then $f = f' \circ \phi_\alpha$ as desired. This completes the proof of (a).

We now prove (b). Fix a Stone space X, and let $\mathcal{C}$ be the category whose objects are pairs (T, f), where T is a finite set (which we regard as a discrete topological space) and $f : X \to T$ is a continuous map. Unwinding the definitions, we see that $S = \phi(X) \in \mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$ is given by the filtered limit of finite sets $\varprojlim_{(T,f) \in \mathcal{C}} T$. Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ spanned by those pairs (T, f) such that f is surjective. We observe that the inclusion $\mathcal{C}_0^{op} \hookrightarrow \mathcal{C}^{op}$ is cofinal (it admits a left adjoint), so that $S \simeq \varprojlim_{(T,f) \in \mathcal{C}_0} T$. We wish to prove that the unit map $u : X \to \psi(S)$ is a homeomorphism. Since $\psi(S)$ has a basis of open sets consisting of inverse images of points under the maps $\psi(S) \to T$ for $(T, f) \in \mathcal{C}_0$, we deduce that the map u has dense image. Since X is compact and $\psi(S)$ is Hausdorff, the map u is automatically closed and therefore surjective. To complete the proof, it will suffice to show that u is injective. To this end, suppose we are given two distinct points $x, y \in X$. Since X is a Stone space, we can choose a continuous map $f : X \to \{0, 1\}$ such that $f(x) = 0$ and $f(y) = 1$. By construction, this map factors through $\psi(S)$, so that $u(x) \neq u(y)$. □

We now describe a more algebraic incarnation of the category of Stone spaces.

Definition A.1.9. Let p be a prime number. We say that a commutative ring R is a *p-Boolean algebra* if the following conditions are satisfied:

(i) We have $p = 0$ in R: that is, R is an algebra over the finite field $\mathbf{F}_p$.

(ii) For each $x \in R$, we have $x^p = x$.

We let BAlg_p denote the category whose objects are p-Boolean algebras and whose morphisms are ring homomorphisms.

Remark A.1.10. In the special case $p = 2$, condition (ii) of Definition A.1.9 implies that $2^2 = 2$, which implies (i). If p is an odd prime, these conditions are independent (for example, the ring $\mathbf{Z}/2\mathbf{Z}$ satisfies condition (ii) but does not satisfy (i)).

Example A.1.11. Let X be a topological space, and let $C(X; \mathbf{F}_p)$ denote the ring of locally constant functions from X to $\mathbf{F}_p$. Then $C(X; \mathbf{F}_p)$ is a p-Boolean algebra.

The proof of our next result will show that every p-Boolean algebra is of the form given in Example A.1.11.

Proposition A.1.12. *Let p be a prime number. The construction $R \mapsto \mathrm{Spec}^{\mathrm{Z}} R$ induces a fully faithful embedding $\mathrm{BAlg}_p^{op} \to \mathrm{Top}$, whose essential image is the full subcategory $\mathrm{Top}_{\mathrm{St}} \subseteq \mathrm{Top}$ spanned by the Stone spaces. In particular, the category of Stone spaces is equivalent to (the opposite of) the category of p-Boolean algebras.*

Proof. For every topological space X, we let $C(X; \mathbf{F}_p)$ denote the ring of locally constant functions from X into $\mathbf{F}_p$. For every point $x \in X$, let $\mathfrak{p}_x \subseteq C(X; \mathbf{F}_p)$ be the prime ideal consisting of those functions which vanish at the point x. For every locally constant function $f : X \to \mathbf{F}_p$, the set $\{x \in X : f(x) \neq 0\}$ is open (and closed) in X, so the construction $x \mapsto \mathfrak{p}_x$ determines a continuous map $u_X : X \to \mathrm{Spec}^{\mathrm{Z}} C(X; \mathbf{F}_p)$. This construction is functorial in X and gives a natural transformation of functors $u : \mathrm{id}_{\mathrm{Top}} \to \mathrm{Spec}^{\mathrm{Z}} C(\bullet; \mathbf{F}_p)$. We first prove:

(∗) The natural transformation u is the unit of an adjunction between the functors $\mathrm{Spec}^{\mathrm{Z}} : \mathrm{BAlg}_p^{op} \to \mathrm{Top}$ and $C(\bullet; \mathbf{F}_p) : \mathrm{Top} \to \mathrm{BAlg}_p^{op}$.

More concretely, (∗) asserts that for every topological space X and every p-Boolean algebra R, the composite map

$$\theta : \mathrm{Hom}_{\mathrm{BAlg}_p}(R, C(X; \mathbf{F}_p)) \to \mathrm{Hom}_{\mathrm{Top}}(\mathrm{Spec}^{\mathrm{Z}} C(X; \mathbf{F}_p), \mathrm{Spec}^{\mathrm{Z}} R) \xrightarrow{\circ u_X} \mathrm{Hom}_{\mathrm{Top}}(X, \mathrm{Spec}^{\mathrm{Z}} R)$$

is a bijection. Let $f : R \to C(X; \mathbf{F}_p)$ be an arbitrary ring homomorphism. For $a \in R$, $x \in X$, and $\lambda \in \mathbf{F}_p$, we have $f(a)(x) = \lambda$ if and only if $a - \lambda$ belongs to the prime ideal $\theta(f)(x)$; this proves that θ is injective. To prove surjectivity, consider an arbitrary continuous map $X \to \mathrm{Spec}^{\mathrm{Z}} R$, which we will denote by $x \mapsto \mathfrak{p}_x$. For each $a \in R$, we have $a^p = a$, so the product $a(a-1)(a-2)\cdots(a-p+1)$ vanishes. It follows that every prime ideal of R contains $a - \lambda$ for some $\lambda \in \mathbf{F}_p$; note that λ is necessarily unique. There is therefore a unique function $f_a : X \to \mathbf{F}_p$, characterized by the property that $a = f_a(x)$ in the quotient ring $R/\mathfrak{p}_x$. From the uniqueness, we immediately deduce that $a \mapsto f_a$ determines a ring homomorphism from R to the ring of all functions from X to $\mathbf{F}_p$. To complete the proof of (∗), it will suffice to show that each of the functions f_a is locally constant. For $\lambda \in \mathbf{F}_p$, we have

$$f_a^{-1}\{\lambda\} = h^{-1}\{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}} R : a - \lambda \in \mathfrak{p}\}.$$

Since h is continuous, this is a closed subset of X. We then deduce that

$$f_a^{-1}\{\lambda\} = X - \bigcup_{\lambda' \neq \lambda} f_a^{-1}\{\lambda'\}$$

is open (since the field $\mathbf{F}_p$ is finite), so that f_a is locally constant as desired.

To complete the proof, we will verify the following:

(a) For every p-Boolean algebra R, the spectrum $X = \operatorname{Spec}^{\mathrm{Z}} R$ is a Stone space, and the canonical map $v_R : R \mapsto C(X; \mathbf{F}_p)$ is an isomorphism.

(b) If X is a Stone space, then the unit map $u_X : X \to \operatorname{Spec}^{\mathrm{Z}} C(X; \mathbf{F}_p)$ is a homeomorphism.

We first prove (a). Let R be a p-Boolean algebra. Then $X = \operatorname{Spec}^{\mathrm{Z}} R$ is a compact topological space. Then the Zariski spectrum has a basis for its topology given by $\{\mathfrak{p} \in X : a \notin \mathfrak{p}\}$, where $a \in R$. Note that a prime ideal $\mathfrak{p}$ of R contains a if and only if it does not contain $b = a^{p-1} - 1$ (since $ab = 0$ and (a, b) is the unit ideal in R). It follows that X has a basis of closed and open sets. Since the open sets separated points in X, we conclude that X is Hausdorff, and therefore a Stone space.

To complete the proof of (a), we must show that the map $v_R : R \to C(X; \mathbf{F}_p)$ is an isomorphism of commutative rings. If $a \in \ker(v_R)$, then a belongs to every prime ideal in R. It follows that a is nilpotent, so that $a^{p^k} = 0$ for $k \gg 0$. Using the fact that R is a p-boolean algebra, we deduce that $a = 0$. This proves that v_R is injective. To prove the surjectivity, we note that every element of $C(X; \mathbf{F}_p)$ can be written as an $\mathbf{F}_p$-linear combination of functions of the form

$$f(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y. \end{cases}$$

where Y is a closed and open subset of $X = \operatorname{Spec}^{\mathrm{Z}} R$. Then Y determine an idempotent $e \in R$, and we observe that $f = v_R(e)$.

We now prove (b). Let X be a Stone space; we wish to show that $u_X : X \to \operatorname{Spec}^{\mathrm{Z}} C(X; \mathbf{F}_p)$ is a homeomorphism. We first claim that u_X has dense image. Assume otherwise; then there exists a nonempty open subset of $\operatorname{Spec}^{\mathrm{Z}} C(X; \mathbf{F}_p)$ which does not intersection $u_X(X)$. Without loss of generality, we may assume that this open subset has the form

$$U = \{\mathfrak{p} \subseteq C(X; \mathbf{F}_p) : f \notin \mathfrak{p}\}$$

for some $f \in C(X; \mathbf{F}_p)$. Since this set does not intersect $u_X(X)$, the function f must vanish at every point of X. It follows that $f = 0$, so that $U = \emptyset$ contrary to our assumption. Since X is compact and $\operatorname{Spec}^{\mathrm{Z}} C(X; \mathbf{F}_p)$ is Hausdorff, the map u_X is automatically closed and therefore surjective. To complete the proof that u_X is a homeomorphism, it will suffice to show that u_X is injective. For this, it suffices to show that if $x, y \in X$ are distinct points, then there exists a locally constant functon $f : X \to \mathbf{F}_p$ which vanishes at x and does not vanish at y. This follows from our assumption that X is a Stone space. □

Remark A.1.13. Let R be a p-Boolean algebra. The following conditions on R are equivalent:

(a) As a set, R is finite.

(b) As an algebra over $\mathbf{F}_p$, R is finitely generated.

(c) As an algebra over $\mathbf{F}_p$, R is finitely presented.

(d) The space $\operatorname{Spec}^{\mathrm{Z}} R$ is finite.

Under the equivalence of categories supplied by Proposition A.1.12, the p-Boolean algebras satisfying these conditions correspond to the finite sets (regarded as Stone spaces with the discrete topology). We may therefore use Proposition A.1.12 to reformulate Proposition A.1.6 as follows:

(∗) The category BAlg_p of p-Boolean algebras is compactly generated, and its compact objects are precisely those p-Boolean algebras which satisfy any (and therefore all) of the conditions (a) through (d).

It is fairly easy to prove (∗) directly, thereby obtaining a slightly different proof of Proposition A.1.6.

A.2 Upper Semilattices

Let X be a Stone space, and let $B = C(X; \mathbf{F}_2)$ be the ring of locally constant functions from X to the field $\mathbf{F}_2 = \{0, 1\}$. It follows from Proposition A.1.12 that we can reconstruct X functorially from B, which we can identify with the collection of all closed and open subsets of X. In fact, this reconstruction is possible for a much larger class of topological spaces X, which includes (for example) the Zariski spectrum of any commutative ring (see Proposition A.2.14). The basic point is that if X is not Hausdorff, we should emphasize the *compact* open subsets of X, rather than the closed and open subsets of X.

Definition A.2.1. An *upper semilattice* is a partially ordered P set such that every finite subset $S \subseteq P$ has a supremum $\bigvee S$.

For partially ordered set P to be an upper semilattice, it is necessary and sufficient that P has least element $\perp$ and every pair of elements $x, y \in P$ has a least upper bound. We denote this least upper bound by $x \vee y$, and refer to it as the *join* of x and y.

Remark A.2.2. Let P be an upper semilattice. Then the join operation $\vee : P \times P \to P$ endows P with the structure of a commutative monoid. Moreover, every element $x \in P$ is idempotent: that is, we have $x = x \vee x$. Conversely, if M is a commutative monoid in which every element is idempotent, then we can introduce a partial ordering of M by writing $x \leq y$ if and only if $xy = y$. This partial ordering exhibits M as an upper semilattice.

Definition A.2.3. Let P be an upper semilattice. We say that a subset $I \subseteq P$ is an *ideal* if it is closed downwards and closed under finite joins. We say that a subset $F \subseteq P$ is a *filter* if it is closed upwards and every finite subset $S \subseteq F$ has a lower bound in F. We say that an ideal I is *prime* if $P - I$ is a filter.

Remark A.2.4. Any ideal $I \subseteq P$ contains the least element $\perp \in P$. Note that I is prime if and only if the following pair of conditions holds:

(*i*) The empty set $\emptyset \subseteq P - I$ has a lower bound in $P - I$: that is, $I \neq P$.

(*ii*) For every pair of elements $x, y \in P$ such that $x, y \notin I$, there exists $z \leq x, y$ such that $z \notin I$.

Definition A.2.5. Let P and P' be upper semilattices. A *distributor* from P to P' is a subset $D \subseteq P \times P'$ satisfying the following conditions:

(*i*) If $(x, x') \in D$, $y \leq x$, and $x' \leq y'$, then $(y, y') \in D$.

(*ii*) Let $S = \{y_i\}$ be a finite subset of P', let $y = \bigvee S$, and let $x \in P$. Then $(x, y) \in D$ if and only if we can write $x = \bigvee \{x_i\}$ for some finite collection of elements $\{x_i\} \subseteq P$ such that $(x_i, y_i) \in D$ for every index i.

(*iii*) Let $S = \{y_i\}$ be a finite subset of P' and let $x \in P$ be such that $(x, y_i) \in D$ for every index i. Then there exists an element $y \in P'$ such that $(x, y) \in D$, and $y \leq y_i$ for every index i.

We say that an upper semilattice P is *distributive* if the set $\{(x, y) \in P \times P : x \leq y\}$ is a distributor from P to itself.

Remark A.2.6. Let P be an upper semilattice. The set $\{(x, y) \in P \times P : x \leq y\}$ automatically satisfies conditions (*i*) and (*iii*) of Definition A.2.5. Consequently, P is distributive if and only if for every inequality $x \leq \bigvee \{y_i\}$, we can write $x = \bigvee \{x_i\}$ for some collection of elements x_i satisfying $x_i \leq y_i$. This is obvious if the set $\{y_i\}$ is empty. Using induction on the size of the set $\{y_i\}$, we see that P is distributive if and only if the following condition is satisfied:

$(*)$ For every inequality $x \leq y \vee z$ in P, we can write $x = y_0 \vee z_0$, where $y_0 \leq y$ and $z_0 \leq z$.

Remark A.2.7. Let P, P', and P'' be upper semilattices, and suppose we are given distributors $D \subseteq P \times P'$ and $D' \subseteq P' \times P''$. We define the *composition* D' with D to be the relation

$$D'D = \{(x,z) \in P \times P'' : (\exists y \in P')[((x,y) \in D) \wedge ((y,z) \in D')]\}.$$

Then $D'D$ is a distributor from P to P''. The composition of distributors is associative. Moreover, if P is a distributive upper semilattice and we let id_P denote the distributor $\{(x,y) \in P \times P : x \leq y\}$, then $\mathrm{id}_P R = R$ for any distributor R from P' to P, and $R' \mathrm{id}_P = R'$ for any distributor R' from P to P''. We therefore obtain a category $\mathrm{Lat}_{\mathrm{s}}$ whose objects are distributive upper-semilattices, where the morphisms from P to P' are given by distributors from P to P'.

Construction A.2.8. Let P be a distributive upper semilattice. We let $\mathrm{Spt}(P)$ denote the collection of all prime ideals of P. We will refer to $\mathrm{Spt}(P)$ as the *spectrum* of P.

Notation A.2.9. Let P be a distributive upper semilattice. If $I \subseteq P$ is an ideal, we let $\mathrm{Spt}(P)_I$ denote the collection of those prime ideals $\mathfrak{p} \subseteq P$ such that $I \nsubseteq \mathfrak{p}$. If $x \in P$, we let $\mathrm{Spt}(P)_x = \{\mathfrak{p} \in \mathrm{Spt}(P) : x \notin \mathfrak{p}\}$.

Proposition A.2.10. *Let P be a distributive upper semilattice and let $\mathrm{Spt}(P)$ be the spectrum of P. Then:*

(1) *There exists a topology on the set $\mathrm{Spt}(P)$, for which the open sets are those of the form $\mathrm{Spt}(P)_I$, where I ranges over the ideals of P.*

(2) *The construction $I \mapsto \mathrm{Spt}(P)_I$ determines an isomorphism from the partially ordered set of ideals of P and the partially ordered set of open subsets of $\mathrm{Spt}(P)$.*

(3) *For each $x \in P$, the subset $\mathrm{Spt}(P)_x \subseteq \mathrm{Spt}(P)$ is open. Moreover, the collection of sets of the form $\mathrm{Spt}(P)_x$ form a basis for the topology of $\mathrm{Spt}(P)$.*

(4) *For every finite subset $S \subseteq P$ having join $\bigvee S = x$, the open set $\mathrm{St}(P)_x$ is given by the union $\bigcup_{y \in S} \mathrm{St}(P)_y$.*

(5) *Each of the open sets $\mathrm{Spt}(P)_x$ is quasi-compact. Conversely, every quasi-compact open subset of $\mathrm{Spt}(P)$ has the form $\mathrm{Spt}(P)_x$ for some uniquely determined $x \in P$.*

(6) *The topological space $\mathrm{Spt}(P)$ is sober: that is, every irreducible closed subset of $\mathrm{Spt}(P)$ has a unique generic point.*

Lemma A.2.11. *Let P be a distributive upper semilattice containing an element x. For every ideal $I \subseteq P$ which does not contain x, there exists a prime ideal $\mathfrak{p} \subseteq P$ which contains I but does not contain x.*

Proof. Using Zorn's lemma, we can choose an ideal $\mathfrak{p} \subseteq P$ which is maximal among those ideals which contain I and do not contain x. We will complete the proof by showing that $\mathfrak{p}$ is prime. Since $x \notin \mathfrak{p}$, it is clear that $P - \mathfrak{p}$ is nonempty. It will therefore suffice to show that every pair of elements $y, z \in P - \mathfrak{p}$ have a lower bound in $P - \mathfrak{p}$. The maximality of $\mathfrak{p}$ implies that x belongs to the ideal generated by $\mathfrak{p}$ and y. It follows that $x \leq y \vee y'$ for some $y' \in \mathfrak{p}$. Since P is distributive, we can write $x = y_0 \vee y'_0$ for some $y_0 \leq y$ and some $y'_0 \in \mathfrak{p}$. The same argument shows that $x \leq z \vee z'$ for some $z' \in \mathfrak{p}$. Then $y_0 \leq z \vee z'$, so that $y_0 = z_0 \vee z'_0$ for some $z_0 \leq z$ and some $z'_0 \in \mathfrak{p}$. Then z_0 is a lower bound for y and z. We claim that $z_0 \notin \mathfrak{p}$: otherwise, we deduce that $y_0 = z_0 \vee z'_0 \in \mathfrak{p}$, so that $x = y_0 \vee y'_0 \in \mathfrak{p}$, a contradiction. □

Proof of Proposition A.2.10. We first prove (1). Suppose first that we are given a finite collection of open subsets $\mathrm{Spt}(P)_{I_\alpha}$ of $\mathrm{Spt}(P)$, and let $I = \bigcap_\alpha I_\alpha$. To prove that $\bigcap_\alpha \mathrm{Spt}(P)_{I_\alpha}$ is open, it will suffice to show that $\bigcap_\alpha \mathrm{Spt}(P)_{I_\alpha} = \mathrm{Spt}(P)_I$. That is, we must show that a prime ideal $\mathfrak{p} \subseteq P$ contains I if and only if it contains some I_α. The "if" direction is obvious. For the converse, suppose that each I_α contains an element $x_\alpha \in P - \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, the finite collection of elements $\{x_\alpha\}$ have a lower bound $x \in P - \mathfrak{p}$. Since each I_α is closed downwards, we deduce that $x \in I = \bigcap_\alpha I_\alpha$.

Now suppose we are given an arbitrary collection of open subsets $\mathrm{Spt}(P)_{I_\beta}$ of $\mathrm{Spt}(P)$; we wish to show that $\bigcup_\beta \mathrm{Spt}(P)_{I_\beta}$ is open. Let I smallest ideal containing each I_β. Then a prime ideal $\mathfrak{p}$ contains I if and only if it contains each I_β; so that $\bigcup_\beta \mathrm{Spt}(P)_{I_\beta} = \mathrm{Spt}(P)_I$. This completes the proof of (1).

We now prove (2). Consider two ideals $I, J \subseteq P$; we wish to show that $I \subseteq J$ if and only if $\mathrm{Spt}(P)_I \subseteq \mathrm{Spt}(P)_J$. Let $K = I \cap J$, so that $\mathrm{Spt}(P)_K = \mathrm{Spt}(P)_I \cap \mathrm{Spt}(P)_J$ (by the argument given above). Then $K \subseteq I$. We wish to show that $K = I$ if and only if $\mathrm{Spt}(P)_K = \mathrm{Spt}(P)_I$. The "only if" direction is obvious. For the converse, we must show that if $I \neq K$, then there is a prime ideal $\mathfrak{p}$ such that $K \subseteq \mathfrak{p}$ but $I \not\subseteq \mathfrak{p}$.

To prove (3), we note that $\mathrm{Spt}(P)_x = \mathrm{Spt}(P)_I$ where I is the ideal $\{y \in P : y \leq x\}$; this proves that $\mathrm{Spt}(P)_x$ is open. For any ideal $J \subseteq P$, we have $\mathrm{Spt}(P)_J = \bigcup_{x \in J} \mathrm{Spt}(P)_x$, so that the open sets of the form $\mathrm{Spt}(P)_x$ form a basis for the topology of $\mathrm{Spt}(P)$. Assertion (4) follows immediately from the definition of a prime ideal.

We now prove (5). Let $x \in P$, and suppose that $\mathrm{Spt}(P)_x$ admits a covering by open sets of the form $\mathrm{Spt}(P)_{I_\alpha} \subseteq \mathrm{Spt}(P)_x$. Let J be the smallest ideal containing each I_α. It follows from the proof of (1) that $\mathrm{Spt}(P)_J = \bigcup_\alpha \mathrm{Spt}(P)_{I_\alpha} = \mathrm{Spt}(P)_x$. Invoking (2), we deduce that $J = \{y \in P : y \leq x\}$. In particular, $x \in J$. It follows that $x \leq x_1 \vee \ldots \vee x_n$ for some elements $x_i \in I_{\alpha(i)}$, from which we deduce that $\mathrm{Spt}(P)_x = \bigcup_{1 \leq i \leq n} \mathrm{Spt}(P)_{I_{\alpha(i)}}$. This proves that $\mathrm{Spt}(P)_x$ is quasi-compact. Conversely, suppose that $U \subseteq \mathrm{Spt}(P)$ is any quasi-compact open set. Then U has a finite covering by basic open sets of the form $\mathrm{Spt}(P)_{y_1}, \ldots, \mathrm{Spt}(P)_{y_n}$. It follows from (4) that $U = \mathrm{Spt}(P)_y$, where $y = y_1 \vee \ldots \vee y_n$.

We now prove (6). Suppose that $K \subseteq \mathrm{Spt}(P)$ is an irreducible closed subset. Then $K = \mathrm{Spt}(P) - \mathrm{Spt}(P)_I$ for some ideal $I \subseteq P$, which is uniquely determined by condition (2). By definition, a prime ideal $\mathfrak{p} \in \mathrm{Spt}(P)$ is a generic point for K if K is the smallest closed subset containing $\mathfrak{p}$. According to condition (2), this is equivalent to the requirement that I be the largest ideal such that $I \subseteq \mathfrak{p}$. That is, $\mathfrak{p}$ is a generic point for K if and only if $\mathfrak{p} = I$. This proves the uniqueness of $\mathfrak{p}$. For existence, it suffices to show that I is a prime ideal. Since K is nonempty, $I \neq P$. It will therefore suffice to show that every pair of elements $x, y \in P - I$ have a lower bound in $P - I$. Since $x, y \notin I$, the open sets $\mathrm{Spt}(P)_x$ and $\mathrm{Spt}(P)_y$ have nonempty intersection with K. Because K is irreducible, we conclude that $\mathrm{Spt}(P)_x \cap \mathrm{Spt}(P)_y \cap K \neq \emptyset$. That is, there exists a prime ideal $\mathfrak{q}$ such that $x, y \notin \mathfrak{q}$ while $I \subseteq \mathfrak{q}$. Since $\mathfrak{q}$ is prime, x and y have a lower bound $z \in P - \mathfrak{q}$. Then z is a lower bound for x and y in $P - I$. □

Construction A.2.12. Let P and P' be distributive upper semilattices, and let $D \subseteq P \times P'$ be a distributor from P to P'. We define a map $\mathrm{Spt}(D) : \mathrm{Spt}(P) \to \mathrm{Spt}(P')$ by the formula

$$\mathrm{Spt}(D)(\mathfrak{p}) = \{y \in P' : (\forall x \in P)[(x, y) \in D \Rightarrow x \in \mathfrak{p}]\}.$$

We claim that, for every prime ideal $\mathfrak{p} \subseteq P$, the subset $\mathrm{Spt}(D)(\mathfrak{p})$ is a prime ideal in P'. It is clear that $\mathrm{Spt}(D)(\mathfrak{p})$ is closed downwards. If $\{y_1, \ldots, y_n\} \subseteq P' - \mathrm{Spt}(D)(\mathfrak{p})$ is a finite subset, then we can choose a finite subset $\{x_1, \ldots, x_n\} \subseteq P - \mathfrak{p}$ such that $(x_i, y_i) \in D$ for $1 \leq i \leq n$. Since $\mathfrak{p}$ is prime, the elements x_i have a lower bound $x \in P - \mathfrak{p}$. Then $(x, y_i) \in D$ for $1 \leq i \leq n$. Since D is a distributor, we deduce that $(x, y) \in D$ for some lower bound y for $\{y_1, \ldots, y_n\}$. Noting that $y \notin \mathrm{Spt}(D)(\mathfrak{p})$, we see that $P' - \mathrm{Spt}(D)(\mathfrak{p})$ is a filter. To show that $\mathrm{Spt}(D)(\mathfrak{p})$ is an ideal, suppose we are given a finite collection of elements $\{y'_1, \ldots, y'_m\} \subseteq \mathrm{Spt}(D)(\mathfrak{p})$. If the join $y'_1 \vee \cdots \vee y'_m$ does not belong to $\mathrm{Spt}(D)(\mathfrak{p})$, then $(x', y'_1 \vee \cdots \vee y'_m) \in D$ for some $x \in \mathfrak{p}$. We can therefore write $x' = x'_1 \vee \cdots \vee x'_m$ where $(x'_i, y'_i) \in D$ for every index i. Since each $y'_i \in \mathrm{Spt}(D)(\mathfrak{p})$, we conclude that $x'_i \in \mathfrak{p}$, so that $x' = x'_1 \vee \cdots x'_n \in \mathfrak{p}$, a contradiction.

Remark A.2.13. In the situation of Construction A.2.12, the map $\mathrm{Spt}(D) : \mathrm{Spt}(P) \to \mathrm{Spt}(P')$ is continuous. To prove this, we note that if $I \subseteq P'$ is an ideal, then $\mathrm{Spt}(D)^{-1} \mathrm{Spt}(P')_I = \mathrm{Spt}(P)_J$, where J is the ideal $\{x \in P : (\exists y \in I)[(x, y) \in D]\}$.

It follows from Remark A.2.13 that we can view Spt as a functor from the category $\mathrm{Lat_s}$ of distributive upper semilattices (with morphisms given by distributors) to the category Top of topological spaces.

We can now state the main result of this section.

Proposition A.2.14 (Duality for Distributive Upper Semilattices)**.** *The functor* Spt : $\mathrm{Lat_s} \to \mathrm{Top}$ *is fully faithful. Moreover, a topological space X belongs to the essential image of* Spt *if and only if it is sober and has a basis consisting of quasi-compact open sets.*

Proof. Let P and P' be distributive upper semilattices, let $f : \mathrm{Spt}(P) \to \mathrm{Spt}(P')$ be a continuous map, and let $D \subseteq P \times P'$ be a distributor. We first prove the following:

($*$) We have $f = \mathrm{Spt}(D)$ if and only if $D = \{(x, y) \in P \times P' : \mathrm{Spt}(P)_x \subseteq f^{-1} \mathrm{Spt}(P')_y\}$.

This shows in particular that D is uniquely determined by f, so that the functor Spt is faithful. We begin by proving the "only if" direction of ($*$). Suppose that $f = \mathrm{Spt}(D)$. If $(x, y) \in D$, then for every prime ideal $\mathfrak{p} \subseteq P$ not containing x, we have $y \in \mathrm{Spt}(D)(\mathfrak{p}) = f(\mathfrak{p})$, so that $\mathrm{Spt}(P)_x \subseteq f^{-1} \mathrm{Spt}(P')_y$. Conversely, suppose $(x, y) \notin D$. Then $I = \{x' \in P : (x', y) \in D\}$ is an ideal of P which does not contain the element x. Using Lemma A.2.11, we can choose a prime ideal $\mathfrak{p}$ containing I and not containing x. Then $\mathfrak{p} \in \mathrm{Spt}(P)_x$ but $f(F) = \mathrm{Spt}(D)(\mathfrak{p}) \notin \mathrm{Spt}(P')_y$, so that $\mathrm{Spt}(P)_x \nsubseteq f^{-1} \mathrm{Spt}(P')_y$.

We next prove the "if" direction of ($*$). Assume that $D = \{(x, y) \in P \times P' : \mathrm{Spt}(P)_x \subseteq f^{-1} \mathrm{Spt}(P')_y\}$, and let $\mathfrak{p} \subseteq P$ be a prime ideal. We wish to show that $f(F) = \mathrm{Spt}(D)(\mathfrak{p})$. We have

$$\begin{aligned} y \notin f(\mathfrak{p}) &\Leftrightarrow f(F) \in \mathrm{Spt}(P')_y \\ &\Leftrightarrow F \in f^{-1} \mathrm{Spt}(P')_y \\ &\Leftrightarrow (\exists x \in P)[F \in \mathrm{Spt}(P)_x \subseteq f^{-1} \mathrm{Spt}(P')_y] \\ &\Leftrightarrow (\exists x \in P)[(x \in F) \wedge (x, y) \in D] \\ &\Leftrightarrow y \notin \mathrm{Spt}(D)(\mathfrak{p}). \end{aligned}$$

We now prove that the functor Spt is full. Let $f : \mathrm{Spt}(P) \to \mathrm{Spt}(P')$ be a continuous map, and set $D = \{(x, y) \in P \times P' : \mathrm{Spt}(P)_x \subseteq f^{-1} \mathrm{Spt}(P')_y\}$. We will show that D is a distributor; then assertion ($*$) immediately implies that $f = \mathrm{Spt}(D)$. Let us verify the conditions of Definition A.2.5:

(i) It is clear that if $(x, y) \in D$, $x' \leq x$, and $y \leq y'$, then $(x', y') \in D$.

(ii) Let $S = \{y_i\}$ be a finite subset of P', let $y = \bigvee S$, and let $x \in P$. Then $(x, y) \in D$ if and only if $\mathrm{Spt}(P)_x \subseteq \bigcup_i f^{-1} \mathrm{Spt}(P')_{y_i}$. In this case, $\mathrm{Spt}(P)_x$ admits a covering by quasi-compact open sets $U_{i,j}$ such that $U_{i,j} \subseteq f^{-1} \mathrm{Spt}(P')_{y_i}$. Since $\mathrm{Spt}(P)_x$ is quasi-compact, we can assume that this covering is finite. Let $U_i = \bigcup_j U_{i,j}$. Then each U_i is a quasi-compact open subset of $\mathrm{Spt}(P)$, and is therefore of the form $\mathrm{Spt}(P)_{x_i}$ for some $x_i \in P$. Since $\mathrm{Spt}(P)_x = \bigcup U_i$, we have $x = x_1 \vee \cdots \vee x_n$. Moreover, the containment $U_i \subseteq f^{-1} \mathrm{Spt}(P')_{y_i}$ implies that $(x_i, y_i) \in D$ for $1 \leq i \leq n$.

(iii) Let $S = \{y_i\}$ be a finite subset of P' and let $x \in P$ be such that $(x, y_i) \in D$ for every index i. Then $U = \bigcap \mathrm{Spt}(P')_{y_i}$ is an open subset of $\mathrm{Spt}(P')$ containing $f(\mathrm{Spt}(P)_x)$. Since f is continuous, $\mathrm{Spt}(P)_x$ is quasi-compact. We may therefore choose a finite covering of $f(\mathrm{Spt}(P)_x)$ by quasi-compact open subsets of $\mathrm{Spt}(P')$ which are contained in U. Let V be the union of these quasi-compact open sets, so that $V = \mathrm{Spt}(P)_y$ for some $y \in Y$. Then $\mathrm{Spt}(P)_x \subseteq f^{-1}V$, so that $(x, y) \in D$ and $y \leq y_i$ for each i.

We now describe the essential image of the functor Spt. Proposition A.2.10 implies that for every distributive upper semilattice P, the spectrum $\mathrm{Spt}(P)$ is a sober topological space having a basis of quasi-compact open sets. Conversely, suppose that X is any sober topological space having a basis of quasi-compact open sets. Let P be the collection of all quasi-compact open subsets of X, partially ordered by inclusion. Since the collection of quasi-compact open subsets of X is closed under finite unions, we see that P is an upper semilattice. We next claim that P is distributive. Let U, V, and W be quasi-compact open subsets of X such that $U \subseteq V \cup W$. Then $U \cap V$ and $U \cap W$ is an open covering of U. Since X has a basis of quasi-compact open sets, this covering admits a refinement $\{U_\alpha\}$ where each U_α is quasi-compact. Since U is quasi-compact, we may assume that the set of indices α is finite. Then $U = V_1 \cup \cdots \cup V_m \cup W_1 \cup \cdots \cup W_{m'}$, where $V_i \subseteq V$, $W_i \subseteq W$, and each of the open sets V_i and W_i is quasi-compact. Let $V' = \bigcup_i V_i$ and

$W' = \bigcup_i W_i$. Then V' and W' are quasi-compact open subsets of X satisfying $U = V' \cup W'$, $V' \subseteq V$, and $W' \subseteq W$.

We now define a map $\Phi : X \to \operatorname{Spt}(P)$ by the formula $\Phi(x) = \{U \in P : x \notin U\}$. To prove that ϕ is well-defined, we must show that for every point $x \in X$, the subset $\Phi(x) \subseteq P$ is a prime ideal. It is easy to see that $\Phi(x)$ is an ideal. If we are given a finite collection of elements $U_1, \ldots, U_n \in P - \Phi(x)$, then $x \in \bigcap_i U_i$. Since X has a basis of quasi-compact open sets, we can choose a quasi-compact open set $V \subseteq \bigcap_i U_i$ containing x, so that V is a lower bound for the subset $\{U_i\} \subseteq \Phi(x)$. This proves that $P - \Phi(x)$ is a filter, so that $\Phi(x)$ is prime.

For each $U \in P$, we have

$$\Phi^{-1} \operatorname{Spt}(P)_U = \{x \in X : \Phi(x) \in \operatorname{Spt}(P)_U\} = \{x \in X : x \in U\} = U.$$

Since the open sets of the form $\operatorname{Spt}(P)_U$ form a basis for the topology of $\operatorname{Spt}(P)$ (Proposition A.2.10), we deduce that Φ is continuous. We next show that Φ is bijective. Let $\mathfrak{p} \subseteq P$ be a prime ideal; we wish to show that there is a unique point $x \in X$ such that $F = \Phi(x)$. Let $V = \bigcup_{U \in \mathfrak{p}} U$. Note that if $\mathfrak{p} = \Phi(x)$, then V is the union of all those quasi-compact open subsets of X which do not contain the point x. It follows that x is a generic point of $X - V$. Since X is sober, we conclude that the point x is unique if it exists. To prove the existence, we will show that the closed set $K = X - V$ is irreducible. Since $\mathfrak{p}$ is prime, there exists a quasi-compact open set $W \subseteq X$ which is not contained in $\mathfrak{p}$. We claim that $W \cap K \neq \emptyset$. Assume otherwise; then $W \subseteq V = \bigcup_{U \notin F} U$. Since W is quasi-compact, it is contained in a finite union $U_1 \cup \cdots \cup U_n$ where each U_i belongs to $\mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, we conclude that $U_1 \cup \cdots \cup U_n \in \mathfrak{p}$, contradicting our assumption that $U \notin \mathfrak{p}$. To complete the proof that K is irreducible, it will suffice to show that if W and W' are open subsets of X such that $W \cap K \neq \emptyset$ and $W' \cap K \neq \emptyset$, then $W \cap W' \cap K \neq \emptyset$. Since X has a basis of quasi-compact open sets, we may assume without loss of generality that W and W' are quasi-compact. The definition of V then guarantees that W and W' belong to the filter $P - \mathfrak{p}$. It follows that W and W' have a lower bound $W'' \in P - \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, W'' is not contained in any finite union of open sets belonging to $\mathfrak{p}$. The quasi-compactness of W'' then implies that W'' is not contained in $\bigcup_{U \in \mathfrak{p}} U = V$, so that $\emptyset \neq W'' \cap K \subseteq W \cap W' \cap K$.

To complete the proof, it will suffice to show that the continuous bijection $\Phi : X \to \operatorname{Spt}(P)$ is an open map. Since X has a basis consisting of quasi-compact open sets, it will suffice to show that for every quasi-compact open set $U \subseteq X$, the set $\Phi(U) \subseteq \operatorname{Spt}(P)$ is open. In fact, we claim that $\Phi(U) = \operatorname{Spt}(P)_U$. The containment $\Phi(U) \subseteq \operatorname{Spt}(P)_U$ was established above. To verify the reverse inclusion, let $\mathfrak{p} \subseteq P$ be a prime ideal not containing U. The bijectivity of Φ implies that $\mathfrak{p} = \Phi(x)$ for some point $x \in X$. It now suffices to observe that $x \in U$ (since this is equivalent to the condition that $U \notin \mathfrak{p} = \Phi(x)$). □

A.3 Lattices and Boolean Algebras

In §A.2, we introduced the category $\mathrm{Lat_s}$ of distributive upper semilattices (with morphisms given by distributors), and showed that the spectrum construction $P \mapsto \operatorname{Spt}(P)$ determines a fully faithful embedding from $\mathrm{Lat_s}$ to the category of topological spaces (Proposition A.2.14). In this section, we will study this equivalence in the more restrictive setting of distributive lattices.

Definition A.3.1. Let P be a partially ordered set. We say that P is a *lattice* if both P and P^{op} are upper semilattices: that is, if every finite subset $S \subseteq P$ has both a greatest lower bound and a least upper bound.

Notation A.3.2. If P is a lattice and we are given a finite subset $S \subseteq P$, we will denote its greatest lower bound by $\bigwedge S$. In the special case where $S = \{x, y\}$, we will denote this greatest lower bound by $x \wedge y$.

Proposition A.3.3. *Let P be a lattice. The following conditions are equivalent:*

(1) *The lattice P is distributive when regarded as an upper semilattice (see Definition A.2.5).*

(2) *For every triple of elements $x, y, z \in P$, we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.*

(3) *The lattice P^{op} is distributive when regarded as an upper semilattice.*

(4) *For every triple of elements $x, y, z \in P$, we have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.*

Proof. We first prove that (1) $\Rightarrow$ (2). Let x, y, and z be elements of a lattice P. Since $x \wedge y \leq x \wedge (y \vee z) \geq x \wedge z$, we automatically have

$$x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z).$$

Suppose that P is distributive as an upper semilattice. Then the inequality $x \wedge (y \vee z) \leq y \vee z$ implies that we can write $x \wedge (y \vee z) = y' \vee z'$, where $y' \leq y$ and $z' \leq z$. Then $y', z' \leq x$, so that $y' \leq x \wedge y$ and $z' \leq x \wedge z$. It follows that

$$x \wedge (y \vee z) = y' \vee z' \leq (x \wedge y) \vee (x \wedge z).$$

Conversely, suppose that (2) holds. We will prove that P is distributive as an upper semilattice. Suppose we have an inequality $x \leq y \vee z$ in P. Then

$$x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

where $x \wedge y \leq y$ and $x \wedge z \leq z$. This completes the proof that (1) $\Leftrightarrow$ (2), and the equivalence (3) $\Leftrightarrow$ (4) follows by the same argument.

We now prove that (2) $\Leftrightarrow$ (4). By symmetry, it will suffice to show that (2) $\Rightarrow$ (4). Let $x, y, z \in P$. If assumption (2) is satisfied, we have

$$\begin{aligned} (x \vee y) \wedge (x \vee z) &= (x \wedge (x \vee z)) \vee (y \wedge (x \vee z)) \\ &= x \vee ((y \wedge x) \vee (y \wedge z)) \\ &= (x \vee (y \wedge x)) \vee (y \wedge z) \\ &= x \vee (y \wedge z). \end{aligned}$$

□

Definition A.3.4. We say that a lattice P is *distributive* if it satisfies the equivalent conditions of Proposition A.3.3.

Definition A.3.5. Let X be a topological space having a basis of quasi-compact open sets. We say that X is *quasi-separated* if, for every pair of quasi-compact open sets $U, V \subseteq X$, the intersection $U \cap V$ is quasi-compact.

Proposition A.3.6. *Let P be a distributive upper semilattice. The following conditions are equivalent:*

(1) *The partially ordered set P is a distributive lattice.*

(2) *The topological space* $\mathrm{Spt}(P)$ *is quasi-compact and quasi-separated.*

Proof. Suppose first that condition (2) is satisfied. Let us identify P with the collection of quasi-compact open subsets of $\mathrm{Spt}(P)$. For any finite collection $\{U_i\}$ of such subsets, condition (2) guarantees that $U = \bigcup U_i$ is quasi-compact, so that U is a greatest lower bound for $\{U_i\}$ in P. Conversely, suppose that (1) is satisfied. Let $\{U_i\}_{1 \leq i \leq n}$ be a finite collection of quasi-compact open subsets of $\mathrm{Spt}(P)$, and let U be their greatest lower bound in P. Then U is the largest quasi-compact open subset contained in $\bigcap U_i$. Since $\mathrm{Spt}(P)$ has a basis of quasi-compact open sets, we must have $U = \bigcap U_i$, so that $\bigcap U_i$ is quasi-compact. Taking $n = 0$, we learn that $\mathrm{Spt}(P)$ is quasi-compact; taking $n = 2$, we learn that $\mathrm{Spt}(P)$ is quasi-separated. □

Definition A.3.7. Let P and P' be lattices. A *lattice homomorphism* from P to P' is a map $\lambda : P \to P'$ such that, for every finite subset $S \subseteq P$, we have

$$\lambda(\bigvee S) = \bigvee \lambda(S) \qquad \lambda(\bigwedge S) = \bigwedge \lambda(S).$$

We let Lat denote the category whose objects are distributive lattices and whose morphisms are lattice homomorphisms.

Remark A.3.8. A map of lattices $\lambda : P \to P'$ is a lattice homomorphism if and only if λ satisfies

$$\lambda(\bot) = \bot \qquad \lambda(x \vee y) = \lambda(x) \vee \lambda(y)$$

$$\lambda(\top) = \top \qquad \lambda(x \wedge y) = \lambda(x) \wedge \lambda(y).$$

Here $\bot$ and $\top$ denote the least and greatest elements of P and P'.

Construction A.3.9. Let P and P' be distributive lattices, and let $\lambda : P' \to P$ be a lattice homomorphism. We let $D_\lambda \subseteq P \times P'$ denote the subset $\{(x, y) \in P \times P' : x \leq \lambda(y)\}$. Then D_λ is a distributor from P to P'. The construction $\lambda \mapsto R_\lambda$ determines a functor $\mathrm{Lat}^{op} \to \mathrm{Lat}_{\mathrm{s}}$.

Remark A.3.10. The functor $\mathrm{Lat}^{op} \to \mathrm{Lat}_{\mathrm{s}}$ of Construction A.3.9 is faithful. That is, we can recover a lattice homomorphism $\lambda : P' \to P$ from the underlying distributor D_λ. For each $y \in P'$, $\lambda(y)$ can be characterized as the largest element of x such that $(x, y) \in R_\lambda$.

Notation A.3.11. Let P be a distributive lattice. We let $\mathrm{Spt}(P)$ denote the spectrum of P, regarded as an upper semilattice (Construction A.2.8). If $\lambda : P \to P'$ is a lattice homomorphism, we let $\mathrm{Spt}(\lambda) : \mathrm{Spt}(P') \to \mathrm{Spt}(P)$ denote the map associated to the distributor D_λ of Construction A.3.9. The construction $P \mapsto \mathrm{Spt}(P)$ determines a functor $\mathrm{Lat}^{op} \to \mathrm{Top}$. We will abuse notation by denoting this functor by Spt.

Remark A.3.12. Let P be a distributive lattice which is given as a filtered colimit of distributive lattices P_α. Then the canonical map

$$\mathrm{Spt}(P) \simeq \varprojlim \mathrm{Spt}(P_\alpha)$$

is a homeomorphism.

Remark A.3.13. The definition of the spectrum $\mathrm{Spt}(P)$ can be simplified a bit if we work in the setting of distributive lattices. Note than an ideal $\mathfrak{p} \subseteq P$ is prime if and only if it satisfies the following pair of conditions:

(*i*) The greatest element $\top \in P$ is not contained in $\mathfrak{p}$.

(*ii*) If $x \wedge y \in \mathfrak{p}$, then either x or y belongs to $\mathfrak{p}$.

Proposition A.3.14. *The functor* $\mathrm{Spt} : \mathrm{Lat}^{op} \to \mathrm{Top}$ *is faithful. Moreover:*

(1) *A topological space X lies in the essential image of* Spt *if and only if it is sober, quasi-compact, quasi-separated, and has a basis of quasi-compact open sets.*

(2) *Let P and P' be distributive lattices. Then a continuous map $f : \mathrm{Spt}(P) \to \mathrm{Spt}(P')$ arises from a lattice homomorphism $\lambda : P' \to P$ (necessarily unique) if and only if, for every quasi-compact open subset $U \subseteq \mathrm{Spt}(P')$, the inverse image $f^{-1}U \subseteq \mathrm{Spt}(P)$ is also quasi-compact.*

Proof. The faithfulness follows from Proposition A.2.14 and Remark A.3.10. Assertion (1) follows from Propositions A.2.14 and A.3.6. We now prove (2). Suppose first that $\lambda : P' \to P$ is a lattice homomorphism, let D_λ be the corresponding distributor, and $f : \mathrm{Spt}(P) \to \mathrm{Spt}(P')$ the induced map. For each $y \in P'$, we have

$$f^{-1}\,\mathrm{Spt}(P')_y = \bigcup_{\mathrm{Spt}(P)_x \subseteq f^{-1}\mathrm{Spt}(P')_y} \mathrm{Spt}(P)_x = \bigcup_{(x,y) \in D_\lambda} \mathrm{Spt}(P)_x = \bigcup_{x \leq \lambda(y)} \mathrm{Spt}(P)_x = \mathrm{Spt}(P)_{\lambda(y)},$$

so that f^{-1} carries quasi-compact open subsets of $\mathrm{Spt}(P')$ to quasi-compact open subsets of $\mathrm{Spt}(P)$. Conversely, suppose that $f : \mathrm{Spt}(P) \to \mathrm{Spt}(P')$ is a continuous map such that $f^{-1}U$ is quasi-compact whenever $U \subseteq \mathrm{Spt}(P')$ is quasi-compact. We then have $f^{-1}\,\mathrm{Spt}(P')_y = \mathrm{Spt}(P)_{\lambda(y)}$ for some map $\lambda : P' \to P$. Since the formation of inverse images commutes with unions and intersections, we conclude that λ is a lattice homomorphism. Note that $\mathrm{Spt}(P)_x \subseteq f^{-1}\,\mathrm{Spt}(P')_y$ if and only if $x \leq \lambda(y)$, so that the underlying distributor of the continuous map f is given by D_λ. □

Definition A.3.15. Let P be a distributive lattice containing a least element $\perp$ and a greatest element $\top$. Let $x \in P$ be an element. A *complement* of x is an element $x^c \in P$ such that

$$x \wedge x^c = \perp \qquad x \vee x^c = \top.$$

We will say that x is *complemented* if there exists a complement for x. We say that P is a *Boolean algebra* if every element of P is complemented. We let BAlg denote the full subcategory of Lat spanned by the Boolean algebras.

Remark A.3.16. Let P be a distributive lattice containing an element x. If x^c and $x^{c'}$ are complements of x, then $x^c = x^{c'}$. To see this, we note that

$$x^c = x^c \wedge \top = x^c \wedge (x \vee x^{c'}) = (x^c \wedge x) \vee (x^c \wedge x^{c'}) = \perp \vee (x^c \wedge x^{c'}) = x^c \wedge x^{c'}$$

so that $x^c \leq x^{c'}$. The same argument shows that $x^{c'} \leq x^c$.

Remark A.3.17. Let P be a distributive lattice containing elements x, x^c. Then x^c is a complement of x if and only if x is a complement of x^c. In this case, we will simply say that x and x^c are complementary.

Remark A.3.18. Let $\lambda : P' \to P$ be a homomorphism of distributive lattices. Suppose that $y, y^c \in P'$ are complementary. Then $\lambda(y), \lambda(y^c) \in P$ are complementary.

The following result makes explicit the relationship between Definitions A.1.9 and A.3.15.

Proposition A.3.19. *Let B be a commutative ring in which every element $x \in B$ satisfies $x^2 = x$ (that is, a 2-Boolean algebra, in the sense of Definition A.1.9). For $x, y \in B$, write $x \leq y$ if $xy = x$. Then $\leq$ defines a partial ordering on B, which makes B into a Boolean algebra (in the sense of Definition A.3.15). Moreover, the construction*

$$(B, +, \times) \mapsto (B, \leq)$$

determines a bijection between Boolean 2-algebra structures on B and Boolean algebra structures on B.

Proof. We first show that $\leq$ is a partial ordering on B. For every element $x \in B$, we have $x^2 = x$ so that $x \leq x$. If $x \leq y$ and $y \leq z$, then we have $xz = (xy)z = x(yz) = xy = x$ so that $x \leq z$. Finally, if $x \leq y$ and $y \leq x$, then $x = xy = yx = y$.

For every pair of elements $x, y \in B$, we have $(xy)x = x^2y = xy = xy^2 = (xy)y$, so that $xy \leq x, y$. Moreover, xy is a greatest lower bound for x and y: if $z \leq x, y$, then $z(xy) = (zx)y = zy = z$ so that $z \leq xy$. Moreover, the unit $1 \in B$ satisfies $1x = x$ for all x, and is therefore a largest element of B. This proves that B is a lower semilattice, with $x \wedge y = xy \qquad \top = 1$.

Note that the map $x \mapsto 1 - x$ is an order-reversing bijection from B to itself: if $x \leq y$, then we have

$$(1-y)(1-x) = 1 - y - x + yx = 1 - y - x + x = 1 - y,$$

so that $1 - y \leq 1 - x$. It follows by duality that B is also an upper semilattice, with join given by $x \vee y = 1 - ((1-x) \wedge (1-y)) = 1 - (1-x)(1-y) = x + y - xy$ and least element given by $1 - 1 = 0$.

We next show that B is a distributive lattice by verifying condition (2) of Proposition A.3.3. Given $x, y, z \in B$, we have

$$\begin{aligned} x \wedge (y \vee z) &= x(y + z - yz) \\ &= xy + xz - xyz \\ &= xy + xz - (xy)(xz) \\ &= (x \wedge y) \vee (x \wedge z). \end{aligned}$$

We now claim that, as a distributive lattice, B is complemented. In fact, the complement of any element $x \in B$ is given by $1 - x$: we have

$$x \vee (1-x) = x + (1-x) - x(1-x) = x + (1-x) + (x^2 - x) = 1 = \top$$

$$x \wedge (1-x) = x(1-x) = x - x^2 = 0 = \perp .$$

This completes the proof that B is a Boolean algebra.

We next prove that the construction $(B, +, \times) \mapsto (B, \leq)$ determines an injective map from Boolean 2-algebra structures on B to Boolean algebra structures on B. In other words, we prove that the addition and multiplication on B are uniquely determined by the induced ordering on B. For multiplication, this follows from the formula $xy = x \wedge y$. For addition, we have

$$\begin{aligned} x + y &= x(1-y) + (1-x)y - x(1-y)(1-x)y \\ &= (x \wedge y^c) \vee (x^c \vee y). \end{aligned}$$

To complete the proof, suppose that B is an arbitrary lower semilattice. Define a multiplication on B by the formula $xy = x \wedge y$. This multiplication is commutative, associative, unital, and we have $x^2 = x$ for all $x \in B$ (see Remark A.2.2). Moreover, it is clear that $x \leq y$ if and only if $x = xy$. We wish to show that if B is a Boolean algebra, then there exists an addition $+ : B \times B \to B$ which makes B into a commutative ring. It follows from the analysis above that the addition on B is uniquely determined: it is necessarily given by the formula $x + y = (x \wedge y^c) \vee (x^c \wedge y)$. This addition is obviously commutative, and there is an additive identity given by the least element of B (since $(\perp \wedge y^c) \vee (\top \wedge y) = \perp \vee y = y$). We have

$$x + x = (x \wedge x^c) \vee (x^c \wedge x) = \perp \vee \perp = 0$$

so that every element is its own additive inverse. Note that for $x, y \in R$, we have

$$x \wedge (xy)^c = x \wedge (x^c \vee y^c) = (x \wedge x^c) \vee (x \wedge y^c) = x \wedge y^c.$$

Using this, we compute

$$\begin{aligned} x(y+z) &= x \wedge ((y \wedge z^c) \vee (y^c \wedge z)) \\ &= (x \wedge y \wedge z^c) \vee (x \wedge y^c \wedge z) \\ &= (x \wedge y \wedge (xz)^c) \vee (x \wedge (xy)^c \wedge z) \\ &= (xy \wedge (xz)^c) \vee ((xy)^c \wedge xz) \\ &= xy + xz \end{aligned}$$

so that multiplication distributes over addition. It remains only to verify that addition is associative. We have

$$\begin{aligned} x + (y+z) &= x + ((y \wedge z^c) \vee (y^c \wedge z)) \\ &= (x \wedge ((y \wedge z^c) \vee (y^c \wedge z))^c) \vee (x^c \wedge ((y \wedge z^c) \vee (y^c \wedge z))) \\ &= (x \wedge (y^c \vee z) \wedge (y \vee z^c)) \vee (x^c \wedge y \wedge z^c) \vee (x^c \wedge y^c \wedge z) \\ &= (x \wedge y \wedge z) \vee (x \wedge y^c \wedge z^c) \vee (x^c \wedge y \wedge z^c) \vee (x^c \wedge y^c \wedge z) \end{aligned}$$

and a similar calculation gives

$$(x+y) + z = (x \wedge y \wedge z) \vee (x \wedge y^c \wedge z^c) \vee (x^c \wedge y \wedge z^c) \vee (x^c \wedge y^c \wedge z).$$

□

Corollary A.3.20. *The construction of Proposition A.3.19 determines an equivalence of categories* $\mathrm{BAlg} \simeq \mathrm{BAlg}_2$ *(which is the identity on the level of the underlying sets).*

Remark A.3.21. Let B be a Boolean algebra, and regard B as a commutative ring as in Proposition A.3.19. Then:

(1) A subset $I \subseteq B$ is an ideal in the sense of Definition A.2.3 if and only if it is an ideal in the sense of commutative algebra.

(2) An subset $I \subseteq B$ is a prime ideal in the sense of Definition A.2.3 if and only if it is a prime ideal in the sense of commutative algebra.

To prove (1), assume first that I is closed under addition and under multiplication by elements of B. If $x \leq y \in I$, then $x = xy \in I$, so that I is closed downwards. It is clear that I contains the least element $0 \in B$. If $x, y \in I$, then $x \vee y = x + y - xy \in I$, completing the proof that I is an ideal in the sense of Definition A.2.3. Conversely, suppose that I satisfies the conditions of Definition A.2.3. Then I is a downward-closed subset of B containing 0, and is therefore closed under multiplication by elements of B. For $x, y \in I$, we have $x + y = (x^c \wedge y) \vee (x \wedge y^c) \in I \vee I \subseteq I$. This proves (1); assertion (2) follows from (1) and Remark A.3.13.

Remark A.3.22. Let B be a Boolean algebra. It follows from Remark A.3.21 that there is a canonical bijection (in fact equality) between the sets $\mathrm{Spt}(B)$ and $\mathrm{Spec}^{\mathrm{Z}} B$, where we regard B as a commutative ring via Proposition A.3.19. By inspection, this bijection is a homeomorphism.

Proposition A.3.23. *Let P be a distributive upper semilattice. The following conditions are equivalent:*

(1) *The partially ordered set P is a Boolean algebra.*

(2) *The topological space* $\mathrm{Spt}(P)$ *is a Stone space.*

(3) *The spectrum* $\mathrm{Spt}(P)$ *is compact and Hausdorff.*

Proof. The implication (1) ⇒ (2) follows from Remark A.3.22 and Proposition A.1.12, and the implication (2) ⇒ (3) is a tautology. We will show that (3) ⇒ (1). Let $x \in P$, so that $\mathrm{Spt}(P)_x$ is a quasi-compact open subset of $\mathrm{Spt}(P)$. Since $\mathrm{Spt}(P)$ is Hausdorff, the subset $\mathrm{Spt}(P)_x$ is also closed. Let U be the complement of $\mathrm{Spt}(P)_x$. Then U is a closed subset of $\mathrm{Spt}(P)$ and therefore compact. Since it is also an open subset of $\mathrm{Spt}(P)$, it has the form $\mathrm{Spt}(P)_y$ for some $y \in P$. We now observe that x' is a complement to x. □

Corollary A.3.24. *Let P be a distributive lattice, let B be a Boolean algebra, and let $D \subseteq P \times B$ be a distributor. Then $D = D_\lambda$ for some lattice homomorphism $\lambda : B \to P$ (necessarily unique, by Remark A.3.10).*

Remark A.3.25. Corollary A.3.24 implies that we can regard the category BAlg of Boolean algebras as a full subcategory of *both* the category of Lat of distributive lattices (and lattice homomorphisms) and the category ${\mathrm{Lat_s}}^{op}$ of distributive upper semilattices (and distributors), despite the fact that the embedding $\mathrm{Lat} \hookrightarrow {\mathrm{Lat_s}}^{op}$ is not full.

Proof of Corollary A.3.24. Using Propositions A.2.14 and A.3.14, we are reduced to proving that if $f : \mathrm{Spt}(P) \to \mathrm{Spt}(B)$ is a continuous map and $U \subseteq \mathrm{Spt}(B)$ is a quasi-compact open subset, then $f^{-1}U$ is a quasi-compact open subset of $\mathrm{Spt}(P)$. Since $\mathrm{Spt}(B)$ is Hausdorff (Proposition A.3.23), the subset $U \subseteq \mathrm{Spt}(B)$ is closed. The continuity of f guarantees that $f^{-1}U$ is a closed subset of $\mathrm{Spt}(P)$, hence quasi-compact (since $\mathrm{Spt}(P)$ is quasi-compact). □

Theorem A.3.26 (Stone Duality)**.** *The construction $B \mapsto \mathrm{Spt}(B)$ induces a fully faithful embedding* $\mathrm{Spt} : \mathrm{BAlg}^{op} \to \mathrm{Top}$*, whose essential image is the category* $\mathrm{Top_{St}}$ *of Stone spaces.*

Proof. Combine Proposition A.3.23, Proposition A.3.14, and Corollary A.3.24. Alternatively, combine Proposition A.3.19, Proposition A.1.12, and Remark A.3.21. □

We next study the relationship between distributive lattices and Boolean algebras in more detail.

Proposition A.3.27. *The categories* Lat *and* BAlg *are presentable. Moreover, the inclusion functor* BAlg $\hookrightarrow$ Lat *preserves small limits and filtered colimits, and therefore admits a left adjoint* $U : \text{Lat} \to \text{BAlg}$ *(Corollary T.5.5.2.9).*

Proof. The first assertion is easy. If we are given a filtered diagram $\{B_\alpha\}$ of Boolean algebras having colimit $P \in \text{Lat}$, then every element $x \in P$ is the image of an element x_α of some B_α. Since x_α complemented, x is complemented (Remark A.3.18). Suppose that $\{B'_\beta\}$ is an arbitrary diagram of Boolean algebras having limit $P' \in \text{Lat}$. Let y be an arbitrary element of P'. For each index β, let y_β denote its image in B'_β. Since B'_β is a Boolean algebra, y_β admits a complement $y^c_\beta \in B'_\beta$ (uniquely determined Remark A.3.16). Using Remark A.3.18, we deduce that the complements $\{y^c_\beta\}$ determine an element in P', which is easily seen to be a complement to y. ☐

Notation A.3.28. Let Top denote the category of topological spaces. We let Top_{coh} denote the subcategory of Top whose objects are sober, quasi-compact, quasi-separated topological spaces with a basis of quasi-compact open sets, and whose morphisms are continuous maps $f : X \to Y$ such that for every quasi-compact open subset $U \subseteq Y$, the inverse image $f^{-1}U \subseteq X$ is quasi-compact.

Combining Theorem A.3.26 and Proposition A.3.14, we obtain the following consequence of Proposition A.3.27:

Proposition A.3.29. *The inclusion functor* $\text{Top}_{\text{St}} \hookrightarrow \text{Top}_{\text{coh}}$ *admits a right adjoint.*

Notation A.3.30. We will denote the right adjoint to the inclusion functor $\text{Top}_{\text{St}} \hookrightarrow \text{Top}_{\text{coh}}$ by $X \mapsto X_{\text{c}}$.

Proposition A.3.31. *Let* $X \in \text{Top}^{\text{coh}}$*. Then the canonical map* $\phi : X_{\text{c}} \to X$ *is bijective.*

Proof. Let $*$ denote the topological space consisting of a single point, so that $* \in \text{Top}^{\text{St}}$. As a map of sets, ϕ is given by the composition of bijections $X_{\text{c}} \simeq \text{Hom}_{\text{Top}_{\text{St}}}(*, X_{\text{c}}) \simeq \text{Hom}_{\text{Top}_{\text{coh}}}(*, X) \simeq X$. ☐

Remark A.3.32. Let $X \in \text{Top}^{\text{coh}}$. We will use Proposition A.3.31 to identify the underlying sets of the topological spaces X and X_{c}. We may therefore view X_{c} as the space X endowed with a new topology, which we refer to as the *constructible topology*. We say that a subset $K \subseteq X$ is *constructible* if it is compact and open when regarded as a subset of X_{c}. We can characterize the constructible sets as the smallest Boolean algebra of subsets of X which contains every quasi-compact open subset of X. More concretely, the constructible sets are those which are given by finite unions of sets of the form $U - V$, where $V \subseteq U$ are quasi-compact open subsets of X.

Example A.3.33. Given a commutative ring R, we say that a subset $S \subseteq \text{Spec}^{\text{Z}} R$ is *constructible* if it is a quasi-compact open subset of $(\text{Spec}^{\text{Z}} R)_{\text{c}}$: that is, if it belongs to the Boolean algebra of subsets of $\text{Spec}^{\text{Z}} R$ generated by the quasi-compact open sets.

We close this section with a few observations related to Example A.3.33.

Proposition A.3.34. *For every commutative ring* R*, let* $\mathcal{U}(R)$ *denote the distributive lattice of quasi-compact open subsets of the Zariski spectrum* $\text{Spec}^{\text{Z}}(R)$*. Then the functor* $R \mapsto \mathcal{U}(R)$ *commutes with filtered colimits.*

Proof. The partially ordered set of *all* open subsets of $\text{Spec}^{\text{Z}}(R)$ is isomorphic to the partially ordered set of radical ideals $I \subseteq R$. Under this isomorphism, $\mathcal{U}(R)$ corresponds to the collection of radical ideals I such that $I = \sqrt{J}$ for some finitely generated ideal $J \subseteq R$.

Let $\{R_\alpha\}_{\alpha \in A}$ be a diagram of commutative rings indexed by a filtered partially ordered set A, and let R be a colimit of this diagram. We wish to show that the canonical map $\phi : \varinjlim \mathcal{U}(R_\alpha) \to \mathcal{U}(R)$ is surjective. The surjectivity of ϕ follows from the observation that every finitely generated ideal $J \subseteq R$ has the form $J_\alpha R$, where J_α is a finitely generated ideal in R_α for some $\alpha \in A$. To prove the injectivity, we must show that if $J, J' \subseteq R_\alpha$ are two finitely generated ideals such that JR and $J'R$ have the same radical, then JR_β and $J'R_\beta$

have the same radical for some $\beta \geq \alpha$. Choose generators $x_1, \ldots, x_n \in R_\alpha$ for the ideal J, and generators $y_1, \ldots, y_m \in R_\alpha$ for the ideal J'. Let $\psi : R_\alpha \to R$ be the canonical map. The equality $\sqrt{JR} = \sqrt{J'R}$ implies that there are equations of the form

$$\psi(x_i)^{c_i} = \sum_j \lambda_{i,j}\psi(y_j) \qquad \psi(y_j)^{d_j} = \sum_i \mu_{i,j}\psi(x_i)$$

in the commutative ring R, where c_i and d_j are positive integers. Choose $\beta \geq \alpha$ such that the coefficients $\lambda_{i,j}$ and $\mu_{i,j}$ can be lifted to elements $\overline{\lambda}_{i,j}, \overline{\mu}_{i,j} \in R_\beta$. Let $\psi_\beta : R_\alpha \to R_\beta$ be the canonical map. Enlarging β if necessary, we may assume that the equations

$$\psi_\beta(x_i)^{c_i} = \sum_j \overline{\lambda}_{i,j}\psi_\beta(y_j) \qquad \psi_\beta(y_j)^{d_j} = \sum_i \overline{\mu}_{i,j}\psi_\beta(x_i)$$

hold in the commutative ring R_β, so that $\sqrt{JR_\beta} = \sqrt{J'R_\beta}$ as desired. □

Corollary A.3.35. *For every commutative ring R, let $\mathcal{B}(R)$ denote the Boolean algebra consisting of constructible subsets of $\operatorname{Spec}^{\mathrm{Z}} R$. Then the functor $R \mapsto \mathcal{B}(R)$ commutes with filtered colimits.*

Proof. Let $R \mapsto \mathcal{U}(R)$ be the functor of Proposition A.3.34. Using Proposition A.3.31, we see that $\mathcal{B}$ is given by the composition

$$\mathrm{Ring} \xrightarrow{\mathcal{U}} \mathrm{Lat} \xrightarrow{U} \mathrm{BAlg},$$

where U is as in Proposition A.3.27. The functor U commutes with all colimits (since it is a left adjoint), and the functor $\mathcal{U}$ commutes with filtered colimits by Proposition A.3.34. □

Corollary A.3.36. *Let $\{R_\alpha\}_{\alpha \in A}$ be a diagram of commutative rings having colimit R, indexed by a filtered partially ordered set A Let $\alpha \in A$ and let $K \subseteq \operatorname{Spec}^{\mathrm{Z}} R_\alpha$ be a constructible subset. Suppose that the inverse image of K in $\operatorname{Spec}^{\mathrm{Z}} R$ is empty. Then there exists $\beta \geq \alpha$ in A such that the inverse image of K in $\operatorname{Spec}^{\mathrm{Z}} R_\beta$ is empty.*

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