LECTURE 9: CLASSIFICATION OF COANALYTIC FUNCTORS

In Lecture 7, we saw that the adjunction

$$\mathrm{Sp}_{T(n)} \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} \mathcal{S}_*^{v_n}$$

is monadic: that is, we can identify the ∞-category $\mathcal{S}_*^{v_n}$ of v_n-periodic spaces with the ∞-category of algebras over the monad $(\Phi \circ \Theta) : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$. In Lecture 8, we introduced the class of *coanalytic functors* from $\mathrm{Sp}_{T(n)}$ to itself: that is, the class of functors that are defined by a symmetric sequence. Our goal in this lecture is to prove the following:

Theorem 1 (Heuts)**.** *The functor* $(\Phi \circ \Theta) : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ *is coanalytic.*

We will deduce Theorem 1 from the following more general result:

Theorem 2. *Let* $F : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ *be a functor. Then* F *is coanalytic if and only if* F *preserves sifted homotopy colimits: that is, if and only if it preserves filtered homotopy colimits and geometric realizations of simplicial objects.*

Remark 3. The Bousfield-Kuhn functor $\Phi : \mathcal{S}_*^{v_n} \to \mathrm{Sp}_{T(n)}$ preserves sifted homotopy colimits (this was the main step in the proof that Φ is monadic), and the functor $\Theta : \mathrm{Sp}_{T(n)} \to \mathcal{S}_*^{v_n}$ preserves all homotopy colimits (since it is a left adjoint). Consequently, the composite functor $\Phi \circ \Theta$ preserves sifted homotopy colimits. Consequently, Theorem 1 is an immediate consequence of Theorem 2.

Remark 4. The "only if" direction of Theorem 2 is immediate: if $F : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ is a coanalytic functor, then it is given by the formula

$$F(X) = \bigoplus_{m \geq 0} (C(m) \otimes X^{\otimes m})_{h\Sigma_m}$$

for some symmetric sequence $\{C(m)\}_{m \geq 0}$, from which is it clear that F commutes with sifted homotopy colimits.

The remainder of this lecture is devoted to the proof of Theorem 2. We begin with a long digression. Throughout this lecture, we fix a prime number p and a finite suspension space A of type $(n+1)$; let d be the smallest integer for which the homology group $\mathrm{H}_d(A; \mathbf{F}_p)$ is nonzero. Let us begin by studying the relationship between two notions of "local" homotopy theory:

- The theory of L_n^f-local *spaces*: by definition, these are *pointed* spaces X which are p-local, d-connected, and P_A-local: the last condition means that the mapping space $\mathrm{Map}_*(A, X)$ is contractible.

- The theory of L_n^f-local *spectra*: by definition, these are spectra E which are p-local for which the *spectrum* of maps $\underline{\mathrm{Map}}(\Sigma^\infty(A), E)$ is contractible.

These conditions are closely related to one another. Note that, for any spectrum E, we have a canonical homotopy equivalence

$$\mathrm{Map}_*(A, \Omega^\infty E) \simeq \Omega^\infty \underline{\mathrm{Map}}(\Sigma^\infty(A), E).$$

Consequently, if E is L_n^f-local, then the space $\Omega^\infty(E)$ is P_A-local. The converse is not quite true: for example, if E is an Eilenberg-MacLane spectrum $\Sigma^k H\,\mathbf{F}_p$, then the 0th space $\Omega^\infty(E) \simeq K(\mathbf{F}_p, k)$ is P_A-local for $k < d$, but E is not L_n^f-local. However, we have the following weaker assertion:

Proposition 5. *Let E be a p-local spectrum. Then E is L_n^f-local if and only if each of the spaces $\Omega^{\infty-m}(E)$ is P_A-local.*

Proof. Use the homotopy equivalences

$$\mathrm{Map}_*(A, \Omega^{\infty-m} E) \simeq \Omega^{\infty-m} \underline{\mathrm{Map}}(\Sigma^\infty(A), E).$$

□

Now suppose that E is an arbitrary p-local spectrum, given by the sequence of pointed spaces $E(m) = \Omega^{\infty-m}(E)$ and homotopy equivalences $E(m) \simeq \Omega E(m+1)$. We can try to form an L_n^f-local spectrum by applying the functor P_A termwise: that is, by considering the sequence of spaces $\{P_A E(m)\}$. Since the functor P_A does not commute with the formation of loop spaces, this sequence is no longer a spectrum. However, it is still a *prespectrum*: that is, we have canonical maps

$$P_A E(m) \simeq P_A(\Omega E(m+1)) \xrightarrow{\alpha_m} \Omega P_A E(m+1).$$

However, these maps are not too far from being homotopy equivalences: recall that we have a canonical homotopy equivalence $P_A(\Omega E(m+1)) \simeq \Omega P_{\Sigma A} E(m+1)$ (see Proposition 5.2 of Lecture 3). Under this equivalence, we can identify α_m with the loop space of the natural map $P_{\Sigma A} E(m+1) \to P_A E(m+1)$, whose homotopy fibers are Eilenberg-MacLane space $K(G, d)$ where G is p-power torsion (Lecture 4, Proposition 13). It follows that each of the transition maps

$$P_A E(m) \to \Omega P_A E(m+1)$$

has homotopy fibers of the form $K(G, d-1)$, where G is p-power torsion.

The prespectrum $\{P_A E(m)\}_{m \geq 0}$ has an associated spectrum, given by $E' = \{\varinjlim_k \Omega^k P_A E(m+k)\}_{m \geq 0}$. Since the collection of P_A-local spaces is closed under homotopy limits and filtered homotopy colimits, each of the spaces $\varinjlim_k \Omega^k P_A E(m+k)$ is P_A-local. Applying Proposition 5, we deduce that the spectrum E' is L_n^f-local. Moreover, it follows immediately from the construction that E' is *universal* among L_n^f-local spectra receiving a map from E: that is, it is the localization $L_n^f E$. We therefore obtain the formula

$$\Omega^{\infty-m} L_n^f E \simeq \varinjlim_k \Omega^k P_A E(m+k)$$

$$\Omega^\infty L_n^f E = \varinjlim (P_A E(0) \to \Omega P_A E(1) \to \Omega^2 P_A E(2) \to \cdots).$$

Note that each of the transition maps in this sequence has homotopy fibers of the form $K(G,t)$, where G is a p-power torsion group and $t < d$. This proves the following:

Proposition 6. *Let E be a p-local spectrum. Then the canonical map*

$$P_A(\Omega^\infty E) \to \Omega^\infty(L_n^f E)$$

has $(d-1)$-truncated homotopy fibers (whose homotopy groups are p-power torsion).

Let us now study an application of Proposition 6. Let R be an E_∞-ring spectrum and let $\mathrm{GL}_1(R) \subseteq \Omega^\infty(R)$ denote the union of those connected components which are invertible in $\pi_0 R$. The multiplicative structure on R endows $\mathrm{GL}_1(R)$ with the structure of an infinite loop space; we let $\mathfrak{gl}_1(R)$ denote the associated connective spectrum. Beware that if R is L_n^f-local, then $\mathfrak{gl}_1(R)$ need not be. However, it is not far off:

Proposition 7. *Let R be an E_∞-ring spectrum which is L_n^f-local, and let K be the homotopy fiber of the canonical map $\mathfrak{gl}_1(R) \to L_n^f \mathfrak{gl}_1(R)$. Then K is $(d-1)$-truncated (that is, the homotopy groups $\pi_m K$ vanish for $m \geq d$).*

Proof. Let $\mathfrak{gl}_1^\circ(R)$ denote the connected cover of $\mathfrak{gl}_1(R)$. We have a fiber sequence

$$\mathrm{fib}(\mathfrak{gl}_1^\circ(R) \to L_n^f \mathfrak{gl}_1^\circ(R)) \to K \to \mathrm{fib}(H(\pi_0 R)^\times \to H(\pi_0 R^\times)_{\mathbf{Q}},$$

where the third term is 0-truncated. It will therefore suffice to show that the fiber of the map $\mathfrak{gl}_1^\circ(R) \to L_n^f \mathfrak{gl}_1^\circ(R)$ is $(d-1)$-truncated. This follows from Proposition 6, since the spectrum $\mathfrak{gl}_1^\circ(R)$ is p-local and the 0th space $\Omega^\infty(\mathfrak{gl}_1^\circ(R)) \simeq (\Omega^\infty R)^\circ$ is P_A-local. □

Remark 8. Note that the map $\mathfrak{gl}_1(R) \to L_n^f \mathfrak{gl}_1(R)$ appearing in Proposition 7 has nothing to do with the space A. It is therefore natural to ask if we can do better: what is the best upper bound on the homotopy of the spectrum K appearing in Proposition 7? It seems reasonable to conjecture that the homotopy groups of K are concentrated in degrees $\leq n$; one can show (by similar arguments) that this conjecture holds if we replace the localization functor L_n^f by the related functor L_n.

At this point, it will be useful to recall an important application of the Bousfield-Kuhn functor:

Construction 9 (Rezk)**.** Let R be an E_∞-ring spectrum and let $n > 0$ be an integer. Consider the pointed map $\mathrm{GL}_1(R) \xrightarrow{x \mapsto x-1} \Omega^\infty(R)$. This map induces a homotopy equivalence of connected components and is therefore a v_n-periodic homotopy equivalence: that is, it induces a homotopy equivalence of spectra

$$L_{T(n)}\, \mathfrak{gl}_1(R) = \Phi\Omega^\infty\, \mathfrak{gl}_1(R) = \Phi\,\mathrm{GL}_1(R) \to \Phi\Omega^\infty(R) \simeq L_{T(n)}(R).$$

Composing with the natural map $\mathfrak{gl}_1(R) \to L_{T(n)}\,\mathfrak{gl}_1(R)$, we obtain a map of spectra

$$\log_{T(n)} : \mathfrak{gl}_1(R) \to L_{T(n)}(R),$$

which exhibits $L_{T(n)}(R)$ as the $T(n)$-localization of $\mathfrak{gl}_1(R)$.

Remark 10. One virtue of Construction 9 is that it allows us to "compute" the localizations $L_n^f\,\mathfrak{gl}_1(R)$. We first recall that for any spectrum E, we have a canonical pullback square

$$\begin{array}{ccc} L_n^f E & \longrightarrow & L_{T(n)}E \\ \downarrow & & \downarrow \\ L_{n-1}^f E & \longrightarrow & L_{n-1}^f L_{T(n)}E. \end{array}$$

Specializing to the case $E = \mathfrak{gl}_1(R)$ any applying Construction 9, we obtain a pullback square

$$\begin{array}{ccc} L_n^f\,\mathfrak{gl}_1(R) & \longrightarrow & L_{T(n)}R \\ \downarrow & & \downarrow \\ L_{n-1}^f\,\mathfrak{gl}_1(R) & \xrightarrow{\beta} & L_{n-1}^f L_{T(n)}R. \end{array}$$

In principle, this allows for an inductive "calculation" of $L_n^f\,\mathfrak{gl}_1(R)$ (provided that one can somehow understand the map β).

Let us now describe a different application of Construction 9. We first note that the functor

$$\mathrm{GL}_1 : \{E_\infty \text{ ring spectra}\} \to \{E_\infty -\text{ spaces}\}$$

has a left adjoint, given by the construction $X \mapsto \Sigma^\infty_+(X)$. Put another way, the functor

$$\mathfrak{gl}_1 : \{E_\infty \text{ ring spectra}\} \to \{\text{connective spectra}\}$$

has a left adjoint, given by $\Sigma^\infty_+ \circ \Omega^\infty$. It follows that, for any connective spectrum E, we can regard $\Sigma^\infty_+\Omega^\infty E$ as an E_∞-ring spectrum equipped with a canonical map $E \to \mathfrak{gl}_1(\Sigma^\infty_+\Omega^\infty E)$. Composing with the $T(n)$-local logarithm, we obtain a map of spectra

$$\alpha : E \to \mathfrak{gl}_1(\Sigma^\infty_+\Omega^\infty E) \xrightarrow{\log_{T(n)}} L_{T(n)}\Sigma^\infty_+\Omega^\infty E.$$

Let $\mathrm{Sym}^*(E) = \bigvee_{n\geq 0}(E^{\wedge n})_{h\Sigma_n}$ denote the free E_∞-ring spectrum generated by E. Then α extends uniquely to a map of E_∞-ring spectra $\mathrm{Sym}^*(E) \to L_{T(n)}\Sigma^\infty_+\Omega^\infty E$. Since the target of this map is $T(n)$-local, it factors through a map of $T(n)$-local E_∞-ring spectra

$$\alpha' : L_{T(n)}\,\mathrm{Sym}^*(E) \to L_{T(n)}\Sigma^\infty_+\Omega^\infty E.$$

Theorem 11 (Kuhn)**.** *Let E be a spectrum which is p-local and d-connected. If $L^f_{n-1}(E) \simeq 0$, then the map $\alpha' : L_{T(n)}\,\mathrm{Sym}^*(E) \to L_{T(n)}\Sigma^\infty_+\Omega^\infty E$ is an equivalence.*

Proof. Note that the domain and codomain of α' can be characterized by universal properties in the setting of $T(n)$-local E_∞-ring spectra. If R is any $T(n)$-local E_∞-ring spectrum, then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{E_\infty}(L_{T(n)}\Sigma^\infty_+\Omega^\infty E, R) & \longrightarrow & \mathrm{Map}_{E_\infty}(L_{T(n)}\,\mathrm{Sym}^*(E), R) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{Sp}}(E, \mathfrak{gl}_1(R)) & \xrightarrow{\theta} & \mathrm{Map}_{\mathrm{Sp}}(E, R) \end{array}$$

where the vertical maps are homotopy equivalences and θ is given by composition with the map $\log_{T(n)} : \mathfrak{gl}_1(R) \to R$. It will therefore suffice to show that θ is a homotopy equivalence. We note that θ factors as a composition

$$\mathrm{Map}_{\mathrm{Sp}}(E, \mathfrak{gl}_1(R)) \xrightarrow{\theta'} \mathrm{Map}_{\mathrm{Sp}}(E, L^n_f\,\mathfrak{gl}_1(R)) \xrightarrow{\theta''} \mathrm{Map}_{\mathrm{Sp}}(E, R).$$

Here θ' is a homotopy equivalence by virtue of Proposition 6 (since we have assumed that the spectrum E is d-connected). We are therefore reduced to showing that the map θ'' is a homotopy equivalence. Let Y denote the cofiber of the map $L^f_n\,\mathfrak{gl}_1(R) \xrightarrow{\log_{T(n)}} R$; since the logarithm of Construction 9 is a $T(n)$-local equivalence, it follows that Y is L^f_{n-1}-local. We have a fiber sequence

$$\mathrm{Map}_{\mathrm{Sp}}(E, L^n_f\,\mathfrak{gl}_1(R)) \xrightarrow{\theta''} \mathrm{Map}_{\mathrm{Sp}}(E, R) \to \mathrm{Map}_{\mathrm{Sp}}(E, Y).$$

We are now reduced to showing that the mapping space $\mathrm{Map}_{\mathrm{Sp}}(E, Y)$ is contractible, which follows from our assumption $L^f_{n-1}(E) \simeq 0$. □

We now return to the proof of Theorem 2.

Notation 12 (Monochromatic Spectra)**.** We let $\mathrm{Sp}_{M(n)}$ denote the ∞-category of spectra which are L^f_n-local and L^f_{n-1}-acyclic. Note that the $T(n)$-localization functor $L_{T(n)}$ induces an equivalence of ∞-categories $\mathrm{Sp}_{M(n)} \to \mathrm{Sp}_{T(n)}$.

We will say that a functor $F : \mathrm{Sp}_{M(n)} \to \mathrm{Sp}_{T(n)}$ is *coanalytic* if it has the form $F' \circ L_{T(n)}$, where $F' : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ is coanalytic.

We begin with a simple observation:

Lemma 13. *The collection of coanalytic functors from $\mathrm{Sp}_{M(n)}$ to $\mathrm{Sp}_{T(n)}$ is closed under homotopy colimits.*

Proof. It will suffice to show that the collection of coanalytic functors from $\mathrm{Sp}_{T(n)}$ to itself is closed under homotopy colimits. By definition, this collection is the essential image of the functor

$$\mathrm{Fun}(\mathrm{Fin}^{\simeq}, \mathrm{Sp}_{T(n)}) \to \mathrm{Fun}(\mathrm{Sp}_{T(n)}, \mathrm{Sp}_{T(n)}) \qquad \vec{C} \mapsto F_{\vec{C}}$$

which preserves homotopy colimits. We proved in Lecture 8 that this functor is fully faithful, so its essential image is closed under homotopy colimits. □

We now recall that every spectrum E has a canonical presentation as a homotopy colimit of (shifted) suspension spectra: more precisely, we have a canonical equivalence $\varinjlim \Sigma^{\infty-m}\Omega^{\infty-m}E \to E$. We will need the following slight variant:

Lemma 14. *For every spectrum E, the canonical map*

$$\varinjlim \Sigma^{\infty-m}((\Omega^{\infty-m}E)\langle d\rangle) \to \varinjlim \Sigma^{\infty-m}\Omega^{\infty-m}E \to E$$

is an equivalence.

Proof. When E is k-connected for some integer k, this is clear (the first map is a filtered colimit of maps $\Sigma^{\infty-m}((\Omega^{\infty-m}E)\langle d\rangle) \to \Sigma^{\infty-m}\Omega^{\infty-m}E$ which are homotopy equivalences for $m \gg 0$). The general case follows by writing E as a filtered colimit of its connected covers $E\langle k\rangle$. □

Now suppose that $F : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ is a functor which preserves sifted homotopy colimits. We wish to show that F is coanalytic or, equivalently, that the composite functor $(F \circ L_{T(n)}) : \mathrm{Sp}_{M(n)} \to \mathrm{Sp}_{T(n)}$ is coanalytic. Using Lemma 14, we can write $F \circ L_{T(n)}$ as a filtered colimit of functors $E \mapsto (F \circ L_{T(n)} \circ \Sigma^{\infty-m})(\Omega^{\infty-m}(E)\langle d\rangle)$. By virtue of Lemma 13, it will suffice to show that each of these functors is coanalytic (as a functor from $\mathrm{Sp}_{M(n)}$ to $\mathrm{Sp}_{T(n)}$). Replacing F by $F \circ \Sigma^{-m}$ (and precomposing with Σ^m), we can reduce to the case $m = 0$: that is, we wish to prove the following:

Theorem 15. *Let $F : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ be a functor which preserves sifted homotopy colimits. Then the functor $E \mapsto (F \circ L_{T(n)} \circ \Sigma^{\infty})(\Omega^{\infty}(E)\langle d\rangle)$ is coanalytic (as a functor from $\mathrm{Sp}_{M(n)}$ to $\mathrm{Sp}_{T(n)}$).*

To prove Theorem 15, let us first consider the functor $f = F \circ L_{T(n)} \circ \Sigma^{\infty}$, which we regard as a functor from the ∞-category $\mathcal{S}_*$ of pointed spaces to the ∞-category $\mathrm{Sp}_{T(n)}$ of $T(n)$-local spectra. Since F commutes with sifted homotopy colimits, so does f. It will therefore suffice to prove the following:

Theorem 16. *Let $f : \mathcal{S}_* \to \mathrm{Sp}_{T(n)}$ be a functor which preserves sifted homotopy colimits. Then the functor $E \mapsto f(\Omega^{\infty}(E)\langle d\rangle)$ is coanalytic (as a functor from $\mathrm{Sp}_{M(n)}$ to $\mathrm{Sp}_{T(n)}$).*

We now invoke a bit of category theory: the ∞-category $\mathcal{S}_*$ of pointed spaces is freely generated, under sifted homotopy colimits, by the full subcategory consisting of finite pointed sets. In particular, a functor $f : \mathcal{S}_* \to \mathrm{Sp}_{T(n)}$ which preserves sifted homotopy colimits is determined (up to equivalence) by its restriction to the category Fin_* of finite pointed sets. Concretely, the functor f can be computed by a homotopy coend

$$f(X) = \int^{\mathrm{Fin}_*} f(\bullet) \wedge \mathrm{Map}_*(\bullet, X)_+.$$

In particular, it can be written as a homotopy colimit (indexed by the twisted arrow category of Fin_*) of functors of the form

$$X \mapsto f(I) \wedge \mathrm{Map}_*(J, X)_+,$$

where I and J are pointed finite sets (and the smash product is computed in the $T(n)$-local category). Using Lemma 13, we are reduced to proving that Theorem 16 holds for functors of this form: that is, we may assume that f is given by the formula $f(X) = C \otimes (L_{T(n)}\Sigma^\infty_+ X)^{\otimes m}$ for some $T(n)$-local spectrum C and some $m \geq 0$. Since the collection of coanalytic functors contains all constant functors and is closed under smash products, we can further reduce to the case where $C = L_{T(n)}(S)$ and $m = 1$, so that $f(X) = L_{T(n)}\Sigma^\infty_+(X)$. In other words, we are reduced to the following:

Theorem 17. *The functor $E \mapsto L_{T(n)}\Sigma^\infty_+(\Omega^\infty(E)\langle d \rangle)$ is coanalytic (when regarded as a functor from $\mathrm{Sp}_{M(n)}$ to $\mathrm{Sp}_{T(n)}$).*

To prove Theorem 17, we observe that if $E \in \mathrm{Sp}_{M(n)}$, then the d-connected cover $E\langle d \rangle$ is d-connected and L^f_{n-1}-acyclic. Applying Theorem 11 then gives

$$\begin{aligned} L_{T(n)}\Sigma^\infty_+(\Omega^\infty(E)\langle d \rangle) &= L_{T(n)}\Sigma^\infty_+\Omega^\infty(E\langle d \rangle) \\ &\simeq L_{T(n)} \mathrm{Sym}^*(E\langle d \rangle) \\ &\simeq \bigoplus_{m \geq 0} (L_{T(n)}(E\langle d \rangle))^{\otimes m}_{h\Sigma_m} \\ &\simeq \bigoplus_{m \geq 0} (L_{T(n)} E)^{\otimes m}_{h\Sigma_m} \end{aligned}$$

where the coproduct and homotopy coinvariants are computed in the ∞-category $\mathrm{Sp}_{T(n)}$. This functor is clearly coanalytic (associated to the constant symmetric sequence $\{L_{T(n)}(S)\}_{m \geq 0}$).