LECTURE 8: COANALYTIC FUNCTORS

Throughout this lecture, we fix a prime number p and a positive integer n. In the previous lecture, we saw that the adjunction

$$\mathrm{Sp}_{T(n)} \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} \mathcal{S}_*^{v_n}$$

is monadic: that is, it exhibits $\mathcal{S}_*^{v_n}$ of v_n-periodic spaces as the ∞-category of algebras over the monad $\Phi \circ \Theta : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$. Our next goal is to understand what this monad looks like. In this lecture, we take the first steps by introducing the class of *coanalytic functors* from $\mathrm{Sp}_{T(n)}$ to itself (we will later show that $\Phi \circ \Theta$ belongs to this class).

Notation 1. Throughout this lecture, we regard $\mathrm{Sp}_{T(n)}$ as a symmetric monoidal ∞-category with respect to the $T(n)$*-local* smash product. We will denote this smash product by $(X, Y) \mapsto X \otimes Y$, so that we have $X \otimes Y = L_{T(n)}(X \wedge Y)$. We let $\mathcal{S}$ denote the ∞-category of spaces and $\mathcal{S}_*$ the ∞-category of pointed spaces.

Definition 2. A *symmetric sequence of* $T(n)$*-local spectra* is a collection of $T(n)$-local spectra $\{C(k)\}_{k \geq 0}$, where each $C(k)$ is equipped with an action of the symmetric group Σ_k. Equivalently, a symmetric sequence is a functor $\vec{C} : \mathrm{Fin}^{\simeq} \to \mathrm{Sp}_{T(n)}$, where $\mathrm{Fin}^{\simeq}$ denotes the category whose objects are finite sets and whose morphisms are bijections; in this case, we write $C(k)$ for the value of $\vec{C}$ on a standard k-element set $\{1, 2, \ldots, k\}$.

Remark 3. Here (and throughout this lecture) we interpret the notion of a spectrum with Σ_k-action in the "naive" sense, rather than the "genuine" sense of equivariant stable homotopy theory.

Construction 4. Let $\vec{C}$ be a symmetric sequence of $T(n)$-local spectra. We define a functor $F_{\vec{C}} : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ by the formula

$$F_{\vec{C}}(X) = \bigoplus_{k \geq 0} (C(k) \otimes X^{\otimes k})_{h\Sigma_k} \simeq \varinjlim_{I \in \mathrm{Fin}^{\simeq}} (\vec{C}(I) \otimes X^{\otimes I}).$$

Here the coproduct is formed in the ∞-category $\mathrm{Sp}_{T(n)}$.

The main goal of this lecture is to prove the following:

Theorem 5 (Mathew)**.** *The construction* $\vec{C} \mapsto F_{\vec{C}}$ *determines a fully faithful embedding from the* ∞*-category* $\mathrm{Fun}(\mathrm{Fin}^{\simeq}, \mathrm{Sp}_{T(n)})$ *of symmetric sequences to the* ∞*-category* $\mathrm{Fun}(\mathrm{Sp}_{T(n)}, \mathrm{Sp}_{T(n)})$.

Definition 6. We will say that a functor $F : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ is *coanalytic* if it has the form $F_{\vec{C}}$ for some symmetric sequence $\vec{C}$: that is, if it belongs to the essential image of the fully faithful embedding of Theorem 5. Then Theorem 5 asserts that the datum of a symmetric sequence is equivalent to the datum of a coanalytic functor.

Remark 7. The expression $F_{\vec{C}}(X) = \bigoplus_{k \geq 0}(C(k) \otimes X^{\otimes k})_{h\Sigma_k}$ should be viewed as a categorical analogue of a power series expansion

$$f(x) = \sum_{k \geq 0} \frac{c_k x^k}{k!}$$

of a function of a real variable. We refer to such functors as *coanalytic*, rather than *analytic*, to avoid conflict with the usual terminology in the calculus of functors (which will play an important role in this story, starting with the next lecture).

Warning 8. The analogue of Theorem 5 is false if we replace the ∞-category $\mathrm{Sp}_{T(n)}$ by the ∞-category of all spectra: in the ∞-category of spectra, there exist nontrivial natural transformations between homogeneous polynomial functors of different degrees. The fact that this phenomenon does not arise in the $T(n)$-local setting is a reflection of the vanishing of Tate cohomology, which we discussed in the previous lecture.

To prove Theorem 5, we wish to show that for every pair of symmetric sequences $\vec{C}$ and $\vec{C}'$, the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{Fin}^{\simeq}, \mathrm{Sp}_{T(n)})}(\vec{C}', \vec{C}) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)}, \mathrm{Sp}_{T(n)})}(F_{\vec{C}'}, F_{\vec{C}})$$

is a homotopy equivalence. Note that both sides carry homotopy colimits in $\vec{C}'$ to homotopy limits of spaces. We may therefore assume without loss of generality that $\vec{C}'$ is concentrated in a single degree: that is, there exists some integer k such that $C'(m)$ vanishes for $m \neq k$. Let us consider the restriction of Construction 4 to such symmetric sequences.

Construction 9. Let E be a $T(n)$-local spectrum with an action of the symmetric group Σ_k. We let $\lambda_k(E)$ denote the functor $\mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ given by the construction $\lambda_k(E)(X) = (E \otimes X^{\otimes k})_{h\Sigma_k}$. We regard the construction $E \mapsto \lambda_k(E)$ as a functor

$$\lambda_k : \{T(n)\text{-local spectra with } \Sigma_k\text{-action}\} \to \mathrm{Fun}(\mathrm{Sp}_{T(n)}, \mathrm{Sp}_{T(n)}).$$

It follows immediately from the construction that the functor λ_k preserves homotopy colimits. It follows formally (from a suitable version of the adjoint functor theorem) that the functor λ_k admits a right adjoint, which we will denote by

$$\partial^k : \mathrm{Fun}(\mathrm{Sp}_{T(n)}, \mathrm{Sp}_{T(n)}) \to \{T(n)\text{-local spectra with } \Sigma_k\text{-action}\}.$$

Given any functor $F : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$, we will refer to $\partial^k(F)$ as the *kth coderivative of F*.

Remark 10. Let S denote the $T(n)$-local sphere, and let $S[\Sigma_k] = L_{T(n)}(\Sigma^\infty_+ \Sigma_k)$ denote the free Σ_k-module in $\mathrm{Sp}_{T(n)}$ generated by S. Unwinding the definitions, we compute $\lambda_k(S[\Sigma_k])(X) = (S[\Sigma_k] \otimes X^{\otimes k})_{h\Sigma_k} \simeq X^{\otimes k}$. It follows that for any functor $F : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$, we have a canonical homotopy equivalence of spectra

$$\begin{aligned} \partial^k(F) &\simeq \underline{\mathrm{Map}}_{\mathrm{Sp}_{T(n)}}(S, \partial^k F) \\ &\simeq \underline{\mathrm{Map}}_{\mathrm{Fun}(B\Sigma_k, \mathrm{Sp}_{T(n)})}(S[\Sigma_k], \partial^k F) \\ &\simeq \underline{\mathrm{Map}}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)}, \mathrm{Sp}_{T(n)})}((\bullet)^{\otimes k}, F); \end{aligned}$$

here we use the notation $\underline{\mathcal{C}}(X, Y)$ to denote the *spectrum* of maps from X to Y in a stable ∞-category $\mathcal{C}$. In other words, we can identify $\partial^k(F)$ with the spectrum parametrizing maps $X^{\otimes k} \to F(X)$ which depend functorially on $X \in \mathrm{Sp}_{T(n)}$.

We can reformulate Theorem 5 as follows:

Proposition 11. *Let $\vec{C}$ be a symmetric sequence and let $k \geq 0$ be an integer. Then the canonical map $C(k) \to \partial^k F_{\vec{C}}$ is a homotopy equivalence of $T(n)$-local spectra.*

Example 12. Let $F : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ be the constant functor with value $C \in \mathrm{Sp}_{T(n)}$. Then F is a right Kan extension of its restriction to the zero spectrum $0 \in \mathrm{Sp}_{T(n)}$. It follows that for each $k \geq 0$, we have

$$\begin{aligned} \partial^k(F) &= \underline{\mathrm{Map}}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)}, \mathrm{Sp}_{T(n)})}((\bullet)^{\otimes k}, F) \\ &\simeq \underline{\mathrm{Map}}_{\mathrm{Sp}_{T(n)}}(0^{\otimes k}, F(0)) \\ &\simeq \begin{cases} C & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Assume first that Proposition 11 is known in the special case $k = 1$. We will show that it holds for any integer k. To simplify the discussion, we will assume that $k = 2$ (the general case follows by exactly the same reasoning, but requires more notation). Let $F : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ be any functor, and suppose that we want to compute the second coderivative $\partial^2(F)$. By virtue of Remark 10, we can identify $\partial^2(F)$ with the spectrum of natural transformations from the functor $X \mapsto X^2$ to the functor F. Let us decompose the first functor as a composition

$$\mathrm{Sp}_{T(n)} \xrightarrow{\delta} \mathrm{Sp}_{T(n)} \times \mathrm{Sp}_{T(n)} \xrightarrow{T} \mathrm{Sp}_{T(n)},$$

where δ denotes the diagonal map $\delta(X) = (X, X)$ and T denotes the tensor product functor $T(X, Y) = X \otimes Y$. Note that the functor δ admits a right adjoint

$\mathrm{Sp}_{T(n)} \times \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$, given by $\gamma(X,Y) = X \times Y = X \oplus Y$. The adjunction supplies a canonical homotopy equivalence

$$\underline{\mathrm{Map}}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathrm{Sp}_{T(n)})}(T \circ \delta, F) \simeq \underline{\mathrm{Map}}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)} \times \mathrm{Sp}_{T(n)},\mathrm{Sp}_{T(n)})}(T, F \circ \gamma).$$

Set $\mathcal{C} = \mathrm{Fun}(\mathrm{Sp}_{T(n)}, \mathrm{Sp}_{T(n)})$, so that we have a canonical equivalence

$$\mathrm{Fun}(\mathrm{Sp}_{T(n)} \times \mathrm{Sp}_{T(n)}, \mathrm{Sp}_{T(n)}) \simeq \mathrm{Fun}(\mathrm{Sp}_{T(n)}, \mathcal{C}).$$

Unwinding the definitions, we see that this equivalence carries the tensor product functor T to the map $\lambda_1 : \mathrm{Sp}_{T(n)} \to \mathcal{C}$ defined above, and carries $F \circ \gamma$ to the functor $X \mapsto F_X$, where $F_X : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ is given by the formula $F_X(Y) = F(X \oplus Y)$. We therefore have homotopy equivalences

$$\begin{aligned}
\underline{\mathrm{Map}}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)} \times \mathrm{Sp}_{T(n)},\mathrm{Sp}_{T(n)})}(T, F \circ \gamma) &\simeq \underline{\mathrm{Map}}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathcal{C})}(\lambda_1, X \mapsto F_X) \\
&\simeq \underline{\mathrm{Map}}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathrm{Sp}_{T(n)})}(\mathrm{id}, X \mapsto \partial^1 F_X) \\
&\simeq \partial^1(X \mapsto \partial^1 F_X).
\end{aligned}$$

Let us now unwind what this is saying in the special case where $F = F_{\vec{C}}$ for some symmetric sequence $\vec{C}$. In this case, we have

$$\begin{aligned}
F(X \oplus Y) &= \bigoplus_{m \geq 0} (C(m) \otimes (X \oplus Y)^m)_{h\Sigma_m} \\
&\simeq \bigoplus_{a,b \geq 0} (C(a+b) \otimes X^{\otimes a} \otimes Y^{\otimes b})_{h(\Sigma_a \times \Sigma_b)}
\end{aligned}$$

Consequently, if we fix a $T(n)$-local spectrum X, then F_X is also a coanalytic functor, with power series expansion

$$F_X(Y) = \bigoplus_{b \geq 0} ((\bigoplus_{a \geq 0} C(a+b) \otimes X^{\otimes a})_{h\Sigma_a} \otimes Y^{\otimes b})_{h\Sigma_b}.$$

Let's assume that Proposition 11 is known in the case $k = 1$. Then the first coderivative $\partial^1 F_X$ can be identified with the first coefficient in this expansion: that is, the spectrum

$$\bigoplus_{a \geq 0} (C(a+1) \otimes X^{\otimes a})_{h\Sigma_a}.$$

This is *also* a coanalytic functor of X, so we can apply Proposition 11 again to obtain a homotopy equivalence $\partial^2(F) \simeq \partial^1(X \mapsto \partial^1 F_X) \simeq C(1+1) = C(2)$. It is not hard to see that this homotopy equivalence is exactly the comparison map of Proposition 11, so that the latter map is a homotopy equivalence as desired.

It remains to prove Proposition 11 in the special case $k = 1$. In this case, we wish to compute the first coderivative

$$\partial^1(F_{\vec{C}}) = \underline{\mathrm{Map}}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathrm{Sp}_{T(n)})}(\mathrm{id}, F_{\vec{C}});$$

we wish to show that this is just given by the "linear term" $C(1)$.

Remark 13 (The First Coderivative)**.** Every spectrum X can be realized as the filtered direct limit $\varinjlim \Sigma^{\infty-d}\Omega^{\infty-d}X$. If X is $T(n)$-local, we can similarly realize X as a colimit

$$\varinjlim L_{T(n)}\Sigma^{\infty-d}\Omega^{\infty-d}X,$$

formed in the ∞-category of $T(n)$-local spectra. This realization depends functorially on X, and therefore presents the identity functor $\mathrm{id} : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ as a filtered colimit of the functors $L_{T(n)}\Sigma^{\infty-d}\Omega^{\infty-d}$. It follows that, for any functor $F : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$, we have a homotopy equivalence of spaces

$$\begin{aligned}
\Omega^\infty \partial^1(F) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathrm{Sp}_{T(n)})}(\mathrm{id}, F) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathrm{Sp}_{T(n)})}(\varinjlim L_{T(n)}\Sigma^{\infty-d}\Omega^{\infty-d}, F) \\
&\simeq \varprojlim \mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathrm{Sp}_{T(n)})}(L_{T(n)}\Sigma^{\infty-d}\Omega^{\infty-d}, F) \\
&\simeq \varprojlim \mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathcal{S}_*)}(\Omega^{\infty-d}, \Omega^{\infty-d}F).
\end{aligned}$$

For each integer d, we have a fiber sequence of mapping spaces

$$\begin{array}{c}
\mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathcal{S}_*)}(\Omega^{\infty-d}, \Omega^{\infty-d}F) \\
\downarrow{\scriptstyle \theta} \\
\mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathcal{S})}(\Omega^{\infty-d}, \Omega^{\infty-d}F) \\
\downarrow \\
\mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathcal{S})}(*, \Omega^{\infty-d}F)
\end{array}$$

where $*$ denotes the constant functor with value some one-point space. Note that the functors $\Omega^{\infty-d}, * : \mathrm{Sp}_{T(n)} \to \mathcal{S}$ are corepresentable: the first is corepresentable by the $T(n)$-local sphere S^{-d}, and the second is corepresentable by the zero spectrum. Applying Yoneda's lemma, we obtain homotopy equivalences

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathcal{S})}(\Omega^{\infty-d}, \Omega^{\infty-d}F) \simeq \Omega^{\infty-d}F(S^{-d})$$

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathcal{S})}(*, \Omega^{\infty-d}F) \simeq \Omega^{\infty-d}F(0).$$

Let's assume for simplicity that the functor F is *reduced*: that is, the spectrum $F(0)$ is contractible. We then obtain a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp}_{T(n)},\mathcal{S}_*)}(\Omega^{\infty-d}, \Omega^{\infty-d}F) \simeq \Omega^{\infty-d}F(S^{-d}) \simeq \Omega^\infty\Sigma^d F(S^{-d}).$$

Passing to the homotopy inverse limit over d, we obtain a homotopy equivalence of spaces

$$\Omega^\infty\partial^1(F) \simeq \varprojlim_d \Omega^\infty\Sigma^d F(S^{-d}).$$

Applying the same reasoning to all desuspensions of F, we can upgrade this to a homotopy equivalence of spectra

$$\partial^1(F) \simeq \varprojlim_d \Sigma^d F(S^{-d}).$$

Let us now return to the situation of Proposition 11. Let $\vec{C}$ be a symmetric sequence and set $F = F_{\vec{C}}$ be the associated coanalytic functor; we wish to show that the canonical map $C(1) \to \partial^1 F$ is a homotopy equivalence. Breaking up our symmetric sequence $\vec{C}$ as a finite sum, it suffices to treat the following three special cases:

(a) The *constant* case, where $C(m)$ vanishes for $m > 0$. In this case, $F_{\vec{C}}$ is the constant functor with value $C(0)$ and the desired result follows from Example 12.

(b) The *linear* case, where $C(m)$ vanishes for $m \neq 1$. In this case, the functor $F = F_{\vec{C}}$ is given by $X \mapsto C(1) \otimes X$. This is a reduced functor, so that Remark 13 allows us to identify $\partial^1 F$ with the homotopy limit of the tower

$$\cdots \to \Sigma^2 F(S^{-2}) \to \Sigma^1 F(S^{-1}) \to F(S^0).$$

The exactness of the functor F shows that this tower is constant with value $C(1)$.

(c) The case where $C(0) = C(1) = 0$. In this case, $F = F_{\vec{C}}$ is again reduced, so we can compute $\partial^1 F$ as the homotopy limit of the tower

$$\cdots \to \Sigma^2 F(S^{-2}) \to \Sigma^1 F(S^{-1}) \to F(S^0).$$

We wish to show that this homotopy limit vanishes. Since $\mathrm{Sp}_{T(n)}$ is generated under colimits by finite spectra of type n, it will suffice to show that the mapping spectrum $\partial^1(F)^K$ vanishes, whenever K is a finite spectrum of type n. We can write this mapping spectrum as the homotopy limit of a tower

$$\cdots \to \Sigma^2 F(S^{-2})^K \to \Sigma^1 F(S^{-1})^K \to F(S^0)^K.$$

The vanishing of this homotopy limit follows from the following much stronger nilpotence result:

Lemma 14. *Let K be a finite spectrum of type n. Then there exists an integer $d \gg 0$, depending only on the spectrum K, with the following property: for every symmetric sequence $\vec{C}$ with $C(0) = C(1) = 0$ and every integer t, the map*

$$\Sigma^{t+d} F_{\vec{C}}(S^{-t-d})^K \to \Sigma^t F_{\vec{C}}(S^{-t})^K$$

is nullhomotopic.

Note that in the statement of Lemma 14, we might as well assume that $t = 0$ (we can always reduce to this case by suspending the symmetric sequence $\vec{C}$). Moreover, the map of Lemma 14 decomposes as a sum of homogeneous pieces;

we can therefore reduce to the case where $\vec{C}$ is concentrated in a single degree $k \geq 2$. In this case, we must prove the following:

Lemma 15. *Let K be a finite spectrum of type n. There exists an integer $d \gg 0$, depending only on the spectrum K, with the following property: for every integer $k \geq 2$ and every $T(n)$-local spectrum C with an action of Σ_k, the canonical map*

$$\theta : \Sigma^d (C^K \otimes (S^{-d})^{\otimes k})_{h\Sigma_k} \to (C^K)_{h\Sigma_k}$$

is nullhomotopic.

In the situation of Lemma 15, the map θ is obtained from a map of $T(n)$-local with an action of Σ_k by passing to homotopy orbit spectra. It will therefore suffice to prove the following:

Lemma 16. *Let K be a finite spectrum of type n. There exists an integer $d \gg 0$, depending only on the spectrum K, with the following property: for every integer $k \geq 2$ and every $T(n)$-local spectrum C with an action of Σ_k, the canonical map*

$$\overline{\theta} : \Sigma^d (C^K \otimes (S^{-d})^{\otimes k}) \to C^K$$

is Σ_k-equivariantly nullhomotopic (here the equivariance is again in the "naive" sense, not in the sense of genuine equivariant stable homotopy theory).

For each integer k, let ρ_k denote the reduced standard representation of the symmetric group Σ_k. Unwinding the definitions, we see that the domain of the map $\overline{\theta}$ appearing in Lemma 15 can be identified (in a Σ_k-equivariant fashion) with the function spectrum $C^{S^{d\rho_k} \wedge K}$. Under this identification, $\overline{\theta}$ is obtained by precomposition with the Σ_k-equivariant map $S^0 \to S^{d\rho_k}$ obtained by the inclusion of $\{0\}$ into $\rho_k^{\oplus d}$. We can use this observation to reduce Lemma 16 to the following statement, in which the spectrum C plays no role:

Lemma 17. *Let K be a finite spectrum of type n. There exists an integer $d \gg 0$, depending only on the spectrum K, with the following property: for every integer $k \geq 2$, the canonical map*

$$L_{T(n)} K \to L_{T(n)} (S^{d\rho_k} \wedge K)$$

is nullhomotopic.

Let us first treat the case where the integer k is *not* a power of p. In this case, we can write $k = k' + k''$, where the $k', k'' > 0$ and the binomial coefficient $k!/(k')!(k'')!$ is not divisible by p. Let G be the product $\Sigma_{k'} \times \Sigma_{k''}$, which we regard as a subgroup of Σ_k. Since the index $|\Sigma_k : G|$ is relatively prime to p and all of the spectra under consideration are p-local, it will sufice to guarantee that the map

$$L_{T(n)} K \to L_{T(n)} (S^{d\rho_k} \wedge K)$$

is G-equivariantly nullhomotopic. This holds already when $d = 1$ (and before smashing with K): the inclusion of spheres $S^0 \hookrightarrow S^{\rho_k}$ is already G-equivariantly nullhomotopic as a map of spaces, because $\rho_k \simeq \rho_{k'} \oplus \rho_{k''} \oplus \mathbf{R}$ contains a nonzero summand on which G acts trivially.

We now consider the difficult case where $k = p^m$ for some integer $m > 1$. Here we use the same idea, but a more complicated subgroup of Σ_k. Let C_p denote the cyclic group of order p, and let G denote the wreath product $C_p \wr \Sigma_{p^{m-1}}$. Identifying Σ_{p^m} with the group of permutations of the product $C_p \times \{1, \ldots, p^{m-1}\}$, we can think of G as the subgroup of Σ_k spanned by those permutations σ which fit into a commutative diagram

$$\begin{array}{ccc} C_p \times \{1, \ldots, p^{m-1}\} & \xrightarrow{\sigma} & C_p \times \{1, \ldots, p^{m-1}\} \\ \downarrow & & \downarrow \\ C_p & \xrightarrow{t} & C_p, \end{array}$$

where $t : C_p \to C_p$ is translation by some element of C_p. A simple calculation shows that the index

$$|\Sigma_{p^m} : G| = \frac{(p^m)!}{p(p^{m-1}!)^p}$$

is relatively prime to p. Consequently, it will suffice to arrange that the map

$$L_{T(n)}K \to L_{T(n)}(S^{d\rho_{p^m}} \wedge K)$$

is G-equivariantly nullhomotopic. Note that, as a representation of G, the reduced standard representation ρ_{p^m} splits as a direct sum $\rho_{p^{m-1}}^{\oplus p} \oplus V$, where V is the reduced regular representation of C_p (regarded as a representation of G via the projection $G \to C_p$). It follows that the map appearing in Lemma 17 admits a G-equivariant factorization

$$L_{T(n)}K \xrightarrow{f} L_{T(n)}(S^{dV} \wedge K) \to L_{T(n)}(S^{d\rho_{p^m}} \wedge K).$$

We conclude that Lemma 17 is a consequence of the following:

Lemma 18. *Let K be a finite spectrum of type n. Then there exists an integer d for which the natural map*

$$e_d : L_{T(n)}K \to L_{T(n)}(S^{dV} \wedge K)$$

is C_p-equivariantly nullhomotopic. Here V denotes the reduced regular representation of C_p.

In the situation of Lemma 18, the action of C_p on $L_{T(n)}K$ is trivial. We can therefore identify the C_p-equivariant map e_d with a non-equivariant map

$$L_{T(n)}K \to (L_{T(n)}(S^{dV} \wedge K))^{hC_p}$$

in the ∞-category of $T(n)$-local spectra. We wish to show that this map is nullhomotopic for $d \gg 0$. Since K is a finite spectrum of type n, the spectrum

$L_{T(n)}K$ is compact in $\mathrm{Sp}_{T(n)}$. We are therefore reduced to showing that the induced map

$$L_{T(n)}K \to \varinjlim_{d} (L_{T(n)}(S^{dV} \wedge K))^{hC_p}$$

is nullhomotopic, where the colimit on the right hand side is formed in $\mathrm{Sp}_{T(n)}$. This colimit is the $T(n)$-local Tate construction $(L_{T(n)}K)^{tC_p}$ (here C_p acts trivially on K), which vanishes by the theorem of Kuhn proven in the previous Lecture. This completes the proof of Theorem 5.