LECTURE 6: THE BOUSFIELD-KUHN FUNCTOR

Let V be a finite space of type $\geq n$, equipped with a v_n-self map $v : \Sigma^t V \to V$. In the previous lecture, we defined the spectrum $\Phi_v(X)$, where X is a pointed space. By construction, $\Phi_v(X)$ is a t-periodic spectrum whose 0th space is given by the direct limit of the sequence

$$\mathrm{Map}_*(V, X) \xrightarrow{\circ v} \mathrm{Map}_*(\Sigma^t V, X) \xrightarrow{\circ v} \mathrm{Map}_*(\Sigma^{2t} V, X) \to \cdots$$

It is clear that the construction $X \mapsto \Phi_v(X)$ determines a functor of ∞-categories

$$\Phi_v : \mathcal{S}_* \to \mathrm{Sp},$$

where $\mathcal{S}_*$ denotes the ∞-category of pointed spaces and Sp denotes the ∞-category of spectra. Our first goal in this lecture is to study the extent to which $\Phi_v(X)$ is also functorial in V.

Notation 1. Let t be a positive integer. We let $\mathcal{C}_t$ denote the ∞-category whose objects are finite pointed spaces V equipped with a v_n-self map $v : \Sigma^t V \to V$. More precisely, we form a pullback diagram of ∞-categories

$$\begin{array}{ccc} \overline{\mathcal{C}}_t & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{S}_*) \\ \downarrow & & \downarrow \\ \mathcal{S}_* & \xrightarrow{\Sigma^t \times \mathrm{id}} & \mathcal{S}_* \times \mathcal{S}_*, \end{array}$$

so that the objects of $\overline{\mathcal{C}}_t$ can be identified with pointed spaces V equipped with an arbitrary pointed map $v : \Sigma^t V \to V$; we then take $\mathcal{C}_t$ to be the full subcategory of $\overline{\mathcal{C}}_t$ spanned by those pairs (V, v) where V is finite and p-local and v is a v_n-self map (which forces V to be of type $\geq n$).

For each integer $t > 0$, the construction $(V, v) \mapsto \Phi_V$ determines a functor of ∞-categories

$$\Phi_\bullet : \mathcal{C}_t^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{S}_*, \mathrm{Sp}).$$

In the last lecture, we made two observations about these functors:

Remark 2. For every pair of positive integers s, t, there is a functor $\mathcal{C}_t \to \mathcal{C}_{st}$, which sends a pair (V, v) to (V, v^s); here v^s denotes the composite map

$$\Sigma^{st} V \xrightarrow{\Sigma^{(s-1)t}(v)} \Sigma^{(s-1)t} V \to \cdots \to \Sigma^t V \xrightarrow{v} V.$$

This functor fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_t & \longrightarrow & \mathcal{C}_{st} \\ & {\scriptstyle\Phi_\bullet}\searrow \quad \swarrow{\scriptstyle\Phi_\bullet} & \\ & \mathrm{Fun}(\mathcal{S}_*, \mathrm{Sp}). & \end{array}$$

Remark 3. For every positive integer t, the construction $(V, v) \mapsto (\Sigma V, \Sigma(v))$ determines a functor from $\mathcal{C}_t$ to itself. This functor fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_t & \xrightarrow{\Sigma} & \mathcal{C}_t \\ & {\scriptstyle\Phi_\bullet}\searrow \quad \swarrow{\scriptstyle\Sigma\Phi_\bullet} & \\ & \mathrm{Fun}(\mathcal{S}_*, \mathrm{Sp}). & \end{array}$$

Using these observations, we see that the functors $\Phi_\bullet$ can be amalgamated to a single functor $\mathcal{C}' \to \mathrm{Fun}(\mathcal{S}_*, \mathrm{Sp})$, where $\mathcal{C}'$ is obtained from the ∞-categories $\mathcal{C}_t$ by taking a direct limit along the transition functors given by suspension and raising self-maps to powers; more precisely, we take $\mathcal{C}'$ to be the direct limit of the sequence

$$\mathcal{C}_{1!} \to \mathcal{C}_{2!} \to \mathcal{C}_{3!} \to \cdots,$$

where the map from $\mathcal{C}_{(m-1)!}$ to $\mathcal{C}_{m!}$ is given by $(V, v) \mapsto (\Sigma V, \Sigma(v^m))$. We will abuse notation by denoting this functor also by $\Phi_\bullet : \mathcal{C}' \to \mathrm{Fun}(\mathcal{S}_*, \mathrm{Sp})$.

Lemma 4. *The ∞-category $\mathcal{C}'$ can be identified with the full subcategory of* Sp *spanned by the finite spectra of type $\geq n$.*

Proof. The functors $\mathcal{C}_{m!} \xrightarrow{\Sigma^{\infty - m}} \mathrm{Sp}$ can be amalgamated to a single functor $F : \mathcal{C}' \to \mathrm{Sp}$. The essential image of F consists of those spectra X having the property that some suspension of X has the form $\Sigma^\infty V$, where V is a finite pointed space equipped with a v_n-self map. The existence theorem for v_n-self maps guarantees that this is precisely the collection of finite spectra of type $\geq n$.

We now complete the proof by showing that F is fully faithful. For each integer $t > 0$, let $\mathcal{C}'_t$ denote the direct limit of the sequence

$$\mathcal{C}_t \xrightarrow{\Sigma} \mathcal{C}_t \xrightarrow{\Sigma} \mathcal{C}_t \to \cdots.$$

Then we can identify the objects of $\mathcal{C}'_t$ with pairs (X, v), where X is a finite spectrum and $v : \Sigma^t X \to X$ is a v_n-self map, and $\mathcal{C}'$

$$\mathcal{C}'_{1!} \to \mathcal{C}'_{2!} \to \mathcal{C}'_{3!} \to \cdots$$

where the transition maps are given by $(X, v) \mapsto (X, v^m)$. Fix a pair of objects (X, v) and (Y, w) in $\mathcal{C}'_t$. Unwinding the definition, we have a homotopy fiber sequence

$$\mathrm{Map}_{\mathcal{C}'_t}((X,v),(Y,w)) \to \mathrm{Map}_{\mathrm{Sp}}(X,Y) \xrightarrow{\rho} \mathrm{Map}_{\mathrm{Sp}}(\Sigma^t X, Y),$$

where the map ρ is given informally by the formula $\rho(f) = w \circ f - f \circ v$. Taking $t = m!$ and identifying (X,v) and (Y,w) with their images in $\mathcal{C}'$, we obtain a fiber sequence

$$\mathrm{Map}_{\mathcal{C}'}((X,v),(Y,w)) \to \mathrm{Map}_{\mathrm{Sp}}(X,Y) \to B,$$

where B is the direct limit of mapping spaces $\mathrm{Map}_{\mathrm{Sp}}(\Sigma^{st}X,Y)$, with transition maps $\mathrm{Map}_{\mathrm{Sp}}(\Sigma^{st}X,Y) \to \mathrm{Map}_{\mathrm{Sp}}(\Sigma^{stu}X,Y)$ given by

$$f \mapsto \sum_{i+j=u-1} w^{is} \circ f \circ v^{js}.$$

To complete the proof, it will suffice to show that B is contractible. Unwinding the definitions, this translates tot he condition that for any map of spectra $f : \Sigma^{st}X \to Y$, there exists some integer $u > 0$ for which the sum $\sum_{i+j=u-1} w^{is} \circ f \circ v^{js}$ is nullhomotopic. It follows from the theory of v_n-self maps that we have $w^a \circ f = f \circ v^a$ for some $a \gg 0$. Replacing s by sa, we can reduce to the case $w \circ f = f \circ v$, in which case we have

$$\sum_{i+j=u-1} w^{is} \circ f \circ v^{js} = u w^{(u-1)s} \circ f$$

which vanishes for u sufficiently divisible (since the group $\pi_0 \mathrm{Map}_{\mathrm{Sp}}(\Sigma^{stu}X,Y)$ is finite). □

Let $\mathrm{Sp}^{\mathrm{fin}}_{\geq n}$ denote the full subcategory of Sp spanned by the finite spectra of type $\geq n$. Using the identification of Lemma 4, we obtain a functor

$$\Phi_\bullet : (\mathrm{Sp}^{\mathrm{fin}}_{\geq n})^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{S}_*, \mathrm{Sp})$$

which can be described informally as follows: if E is a finite spectrum of type n, then we can choose some integer k such that $\Sigma^k E \simeq \Sigma^\infty V$, where V is a finite space of type n which admits a v_n-self map. In this case, we have $\Phi_E = \Sigma^k \circ \Phi_V$.

Remark 5. We noted in the last lecture that the construction $V \mapsto \Phi_V$ carries cofiber sequences of type n spaces to fiber sequences of functors. It follows from this observation that the functor $E \mapsto \Phi_E$ is exact.

Proposition 6 (Bousfield-Kuhn)**.** *There is a unique functor $\Phi : \mathcal{S}_* \to \mathrm{Sp}$ (up to equivalence) with the following properties:*

(a) *For every pointed space X, the spectrum $\Phi(X)$ is $T(n)$-local.*
(b) *There are equivalences $\Phi_E(X) \simeq \Phi(X)^E$, depending functorially on $E \in \mathrm{Sp}^{\mathrm{fin}}_{\geq n}$ and $X \in \mathcal{S}_*$.*

Proof. Let $F : (\mathrm{Sp}^{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{S}_*, \mathrm{Sp})$ be a right Kan extension of the functor $E \mapsto \Phi_E$. Set $\Phi = F(S)$, where S is the sphere spectrum. More concretely, we have

$$\Phi(X) = \varprojlim_{E \to S} \Phi_E(X)$$

where the limit is taken over the ∞-category of all finite type n spectra E equipped with a map $E \to S$. We saw in the previous lecture that each $\Phi_E(X)$ is $T(n)$-local, so $\Phi(X)$ is also $T(n)$-local for each $X \in \mathcal{S}_*$. Using the exactness of the functor $E \mapsto \Phi_E$, it is easy to see that F is also an exact functor, and is therefore determined by its value $F(S) = \Phi$ on the sphere spectrum by the formula $F(E) = \Phi^E$. It follows that Φ satisfies conditions (a) and (b).

Now suppose that $\Phi' : \mathcal{S}_* \to \mathrm{Sp}$ is any functor satisfying (a) and (b). Define $F' : (\mathrm{Sp}^{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{S}_*, \mathrm{Sp})$ by the formula $F'(E) = \Phi'^E$. It follows from (b) that the functors F and F' agree on finite spectra of type $\geq n$. The universal property of right Kan extensions then guarantees a natural transformation $F' \to F$. Evaluating on the sphere spectrum, we obtain a natural transformation $\Phi' \to \Phi$. By construction, this natural transformation induces an equivalence $\Phi'(X)^E \to \Phi(X)^E$ for any pointed space X and any finite spectrum E of type $\geq n$. In particular, the map $\Phi'(X) \to \Phi(X)$ becomes an equivalence after smashing with some finite spectrum of type $\geq n$, and is therefore a $T(n)$-equivalence. Assumption (a) guarantees that $\Phi'(X)$ and $\Phi(X)$ are both $T(n)$-local, so the map $\Phi'(X) \to \Phi(X)$ is a homotopy equivalence. □

The functor Φ of Proposition 6 is called the *Bousfield-Kuhn functor*. The proof of Proposition 6 shows that it is given by the formula

$$\Phi(X) = \varprojlim_{E \to S} \Phi_E(X),$$

where over the ∞-category of all finite type n spectra E equipped with a map $E \to S$. In practice, it is often more convenient to describe $\Phi(X)$ as the homotopy limit $\varprojlim \Phi_{E_k}(X)$, where

$$E_0 \to E_1 \to E_2 \to \cdots$$

is a direct system of type n spectra which is cofinal among all finite type n spectra with a map to S. Such a cofinal system can always be found: for example, in the case $n = 1$, we can take the system of Moore spectra

$$\Sigma^{-1} S/p \to \Sigma^{-1} S/p^2 \to \Sigma^{-1} S/p^3 \to \cdots$$

Let us now summarize some of the key properties of the Bousfield-Kuhn functor.

Proposition 7. *The functor $\Phi : \mathcal{S}_* \to \mathrm{Sp}$ is left exact: that is, it preserves finite homotopy limits.*

Proof. Since the collection of left exact functors is closed under homotopy limits, it will suffice to show that each $\Phi_E : \mathcal{S}_* \to \mathrm{Sp}$ is left exact. Replacing E by a suspension, we can assume $E = \Sigma^\infty V$ for some finite pointed space V equipped with a v_n-self map $v : \Sigma^t V \to V$.

To show that a spectrum-valued functor F is left exact, it suffices to show that $\Omega^{\infty+kt} \circ F$ is left exact for every integer k. Using the periodicity of Φ_V, we are reduced to showing that the functor $\Omega^\infty \Phi_V$ is left exact. This functor is given by the construction

$$X \mapsto \varinjlim(\mathrm{Map}_*(V, X) \to \mathrm{Map}_*(\Sigma^t V, X) \to \cdots).$$

Since the collection of left exact functors is closed under filtered colimits, we are reduced to showing that each of the functors $X \mapsto \mathrm{Map}_*(\Sigma^{ct} V, X)$ is left exact, which is clear (these functors preserve *all* homotopy limits). □

Proposition 8. *Let $f : X \to Y$ be a map of pointed spaces. The following conditions are equivalent:*

(1) *The map f is a v_n-periodic homotopy equivalence, in the sense of the previous lecture.*
(2) *The map f induces a homotopy equivalence of spectra $\Phi(X) \to \Phi(Y)$.*

Proof. Condition (1) is the assertion that f induces a homotopy equivalence $\Phi_V(X) \to \Phi_V(Y)$, whenever V is a finite pointed space equipped with a v_n-self map. It then follows immediately (by taking a suitable suspension) that $\Phi_E(X) \simeq \Phi_E(Y)$ for every finite spectrum E of type $\geq n$. Passing to the inverse limit, we conclude that $\Phi(X) \simeq \Phi(Y)$; this proves that (1) $\Rightarrow$ (2). The converse follows from property (b) of Proposition 6. □

Corollary 9. *The functor $\Phi : \mathcal{S}_* \to \mathrm{Sp}$ admits an essentially unique factorization as a composition*

$$\mathcal{S}_* \xrightarrow{M_n^f} \mathcal{S}_*^{v_n} \to \mathrm{Sp},$$

where $\mathcal{S}_*^{v_n}$ is defined as in the previous lecture.

Proof. Combine Proposition 8 with the universal property of $\mathcal{S}_*^{v_n}$. □

In what follows, we will abuse notation by identifying the Bousfield-Kuhn functor Φ with the functor $\mathcal{S}_*^{v_n} \to \mathrm{Sp}$ appearing in the statement of Corollary 9. This abuse is fairly mild: note that the functor $\mathcal{S}_*^{v_n} \to \mathrm{Sp}$ can be identified with the restriction of the functor $\Phi : \mathcal{S}_* \to \mathrm{Sp}$, if we regard $\mathcal{S}_*^{v_n}$ as a full subcategory of $\mathcal{S}_*$ (as in the previous lecture).

The Bousfield-Kuhn functor Φ is a bit better behaved if we regard it as a functor with domain $\mathcal{S}_*^{v_n}$, by virtue of the following:

Proposition 10 (Bousfield)**.** *The functor $\Phi : \mathcal{S}_*^{v_n} \to \mathrm{Sp}$ admits a left adjoint $\Theta : \mathrm{Sp} \to \mathcal{S}_*^{v_n}$.*

Warning 11. The Bousfield-Kuhn functor $\Phi : \mathcal{S}_* \to \mathrm{Sp}$ does not admit a left adjoint. It preserves finite homotopy limits (Proposition 7), but does not preserve infinite products.

Proof of Proposition 10. Writing Φ as a homotopy limit of functors of the form Φ_E (where E ranges over finite spectra of type $\geq n$), we are reduced to showing that each Φ_E admits a left adjoint Θ_E (we can then recover Θ as a homotopy *colimit* of the functors Θ_E). Replacing E by a suitable suspension, we can assume that $E = \Sigma^\infty(V)$, where V is a finite pointed space of type n equipped with a v_n-self map $v : \Sigma^t V \to V$. To prove the existence of Θ_E, we must show that for each spectrum Z, the functor

$$X \mapsto \mathrm{Map}_{\mathrm{Sp}}(Z, \Phi_E(X))$$

is corepresented by some object $\Theta_E(Z) \in \mathcal{S}_*^{v_n}$. Since the ∞-category $\mathcal{S}_*^{v_n}$ admits homotopy colimits, the collection of those spectra Z for which $\Theta_E(Z)$ exists is closed under homotopy colimits. Note that the ∞-category Sp is generated under homotopy colimits by objects of the form S^{kt}, where k is a (possibly negative) integer. We are therefore reduced to proving that each of the functors

$$X \mapsto \mathrm{Map}_{\mathrm{Sp}}(S^{kt}, \Phi_E(X))$$

is corepresentable. Using the periodicity of the functor $\Phi_E = \Phi_V$, we can reduce to the case $k = 0$. In this case, we wish to prove that the functor $X \mapsto \Omega^\infty \Phi_E(X)$ is corepresentable by an object of $\mathcal{S}_*^{v_n}$; note that this functor carries X to the colimit of the diagram

$$\mathrm{Map}_*(V, X) \to \mathrm{Map}_*(\Sigma^t V, X) \to \mathrm{Map}_*(\Sigma^{2t} V, X) \to \cdots.$$

Fix finite pointed spaces A and B of types $(n+1)$ and n, respectively, having the same connectivity $d = \mathrm{cn}(A) + 1 = \mathrm{cn}(B) + 1$. Let us identify $\mathcal{S}_*^{v_n}$ with the full subcategory of $\mathcal{S}_*$ spanned by those spaces which are p-local, d-connected, P_A-local, and P_B-acyclic. In the previous lecture, we saw that if X is P_A-local, then the sequence

$$\mathrm{Map}_*(V, X) \to \mathrm{Map}_*(\Sigma^t V, X) \to \mathrm{Map}_*(\Sigma^{2t} V, X) \to \cdots.$$

is eventually constant: it homotopy equivalent to $\mathrm{Map}_*(\Sigma^{ct} V, X)$ for c sufficiently large (roughly speaking, we need to take k large enough that $\Sigma^{kt} V$ is more connected than A). We are therefore reduced to showing that on the ∞-category $\mathcal{S}_*^{v_n}$, the functor $X \mapsto \mathrm{Map}_*(\Sigma^{ct} V, X)$ is corepresentable. In fact, it is corepresented by $P_A(\Sigma^{ct} V)$: note that this space is obviously P_A-local, p-local (since V is p-local), and d-connected (provided that c is sufficiently large). It is also P_B-acyclic for c sufficiently large: we have $P_B(P_A(\Sigma^{ct} V)) = P_B(\Sigma^{ct} V) \simeq *$, since $\mathrm{tp}(\Sigma^{ct} V) = n = \mathrm{tp}(B)$ and the connectivity of $\Sigma^{ct} V$ is larger than d (again for c large). $\square$