# LECTURE 4: $L_n^f$-LOCAL SPACES

Let us begin by reviewing some definitions from the preceding lectures.

**Definition 1.** Let $A$ be a space. A space $X$ is $P_A$*-local* if the diagonal map $X \to X^A$ is a homotopy equivalence. The inclusion functor

$$\{P_A\text{-local spaces}\} \hookrightarrow \{\text{Spaces}\}$$

admits a left adjoint, which we will denote by $X \mapsto P_A(X)$. We will say that a space $X$ is $P_A$*-acyclic* if $P_A(X)$ is contractible.

**Example 2.** If $A = \varnothing$, then a space $X$ is $P_A$-local if and only if $X$ is either empty or contractible, and

$$P_A(X) \simeq \begin{cases} \varnothing & \text{if } X = \varnothing \\ * & \text{if } X \neq \varnothing. \end{cases}$$

Consequently, $X$ is $P_A$-acyclic if and only if $X$ is nonempty.

**Remark 3.** By virtue of Example 2, we might as well assume that the space $A$ is nonempty. If we fix a base point $a \in A$, then evaluation at $a$ induces a map $e_a : X^A \to X$ which is left inverse to the diagonal map $X \to X^A$. Consequently, $X$ is $P_A$-local if and only if $e_a$ is a homotopy equivalence. Note that the homotopy fiber of $e_a$ over a point $x \in X$ is the space $\mathrm{Map}_*(A, X)$ of pointed maps from $(A, a)$ to $(X, x)$. It follows that $X$ is a $P_A$-local if and only if $\mathrm{Map}_*(A, X)$ is contractible, for *every* choice of base point $x \in X$ (of course, if $X$ is connected, it suffices to check this at *any* base point $x \in X$).

**Remark 4.** In the situation of Definition 1, the homotopy fibers of the canonical map $X \to P_A(X)$ are $P_A$-acyclic (Theorem 4.8 from the previous lecture).

Let us now try to get a feeling for Definition 1 by studying some examples.

**Example 5.** Let $A = S^n$ be a sphere of dimension $n$. Then a space $X$ is $P_A$-local if and only if it is $(n-1)$-truncated: that is, if and only if the homotopy groups $\pi_m(X, x)$ vanish for $m \geq n$ (and any choice of base point $x \in X$). In this case, the space $P_A(X)$ is the Postnikov truncation $\tau_{\leq n-1} X$ (obtained by killing homotopy groups in degrees $n$ and above). Moreover, a space $X$ is $P_A$-acyclic if and only if it is $(n-1)$-connected.

**Remark 6.** Example 2 can be regarded as a degenerate special case of Example 5 (by regarding the empty space $\varnothing$ as a sphere of dimension $(-1)$).

For the rest of this lecture, we fix a prime number $p$.

**Example 7.** Let $A = M(p)$ be the "mod $p$" Moore space, given by the cofiber of the $p$-fold covering map $p : S^1 \to S^1$ (for example, if $p = 2$, then $A$ is a real projective space of dimension 2). Let $X$ be a simply connected $p$-local space, and choose base points for $A$ and $X$. We have a homotopy fiber sequence

$$\mathrm{Map}_*(M(p), X) \to \mathrm{Map}_*(S^1, X) \to \mathrm{Map}_*(S^1, X)$$

which gives rise to a long exact sequence of homotopy groups

$$\cdots \to \pi_3 X \xrightarrow{p} \pi_3 X \to \pi_1 \mathrm{Map}_*(A, X) \to \pi_2 X \xrightarrow{p} \pi_2 X \to \pi_0 \mathrm{Map}_*(A, X) \to 0.$$

It follows that $X$ is $P_A$-local if and only if it is *rational*: that is, the homotopy groups of $X$ are vector spaces over $\mathbf{Q}$. In general, the space $P_A(X)$ is the rationalization $X_{\mathbf{Q}}$, and $X$ is $P_A$-acyclic if and only if the homotopy groups $\pi_* X$ are torsion groups.

**Variant 8.** Let $A = \Sigma^{d-1} M(p)$ be the $(d-1)$-fold suspension of the space considered in Example 7. Then, for any simply connected $p$-local space $X$, we have a long exact sequence

$$\cdots \to \pi_1 \mathrm{Map}_*(A, X) \to \pi_{d+1} X \xrightarrow{p} \pi_{d+1} X \to \pi_0 \mathrm{Map}_*(A, X) \to \pi_d(X) \xrightarrow{p} \pi_d(X).$$

It follows that:

- The space $X$ is $P_A$-local if and only if the homotopy groups $\pi_n X$ are rational vector spaces for $n > d$, and $\pi_d X$ is torsion-free.
- The map $X \to P_A(X)$ exhibits $\pi_n P_A(X)$ as a rationalization of $\pi_n X$ for $n > d$, induces an isomorphism on $\pi_n$ for $n < d$, and induces an isomorphism $\pi_d(X)/\mathrm{torsion} \simeq \pi_d P_A(X)$.
- The space $X$ is $P_A$-acyclic if and only if the homotopy groups $\pi_n X$ are torsion for $n \geq d$ and vanish for $n < d$.

We will be interested in comparing different localization functors.

**Proposition 9.** *Let $A$ and $A'$ be spaces. The following conditions are equivalent:*

(a) *Every $P_{A'}$-acyclic space is also $P_A$-acyclic.*
(b) *The space $A'$ is $P_A$-acyclic.*
(c) *Every $P_A$-local space is also $P_{A'}$-local.*

*Proof.* The implication $(a) \Rightarrow (b)$ is obvious. Assume that $(b)$ is satisfied and that $X$ is $P_A$-local. To verify $(c)$, we must show that the diagonal map $X \to \mathrm{Map}(A', X)$ is a homotopy equivalence. Using our assumption that $X$ is $P_A$-local, we can identify $\mathrm{Map}(A', X)$ with $\mathrm{Map}(P_A(A'), X)$, and the desired result now follows from the contractibility of $P_A(A')$.

Now assume that $(c)$ is satisfied. Then, for any space $X$, the tautological map $f : X \to P_A(X)$ factors through $P_{A'}(X)$. If $X$ is $P_{A'}$-acyclic, then $f$ is nullhomotopic, so that $P_A(X)$ is contractible and $X$ is also $P_A$-acyclic. □

**Notation 10.** For the rest of this lecture, we will be interested in studying the localization functor $P_A$ in the case where $A$ satisfies the following assumptions:

- The space $A$ can be written as a suspension $\Sigma B$, where $B$ is a finite pointed space.
- The reduced homology groups $\mathrm{H}_*^{\mathrm{red}}(A; \mathbf{Z})$ are annihilated by $p^k$ for $k \gg 0$ (it follows from this that $B$ must be connected, so that $A$ is simply connected).
- The space $A$ is not contractible (otherwise, every space is $P_A$-local).

In this case, there are two integers which will be relevant to us:

- The *type* $\mathrm{tp}(A)$, defined as the smallest integer $n$ such that $K(n)_*^{\mathrm{red}}(A)$ is nonzero.
- The *connectivity* of $A$: that is, the largest integer $\mathrm{cn}(A)$ such that $A$ is $\mathrm{cn}(A)$-connected. In what follows, it will be more useful to emphasize the integer $d = \mathrm{cn}(A) + 1$: that is, the *smallest* integer for which the homology group $\mathrm{H}_d(A; \mathbf{Z}/p\mathbf{Z})$ is nonzero.

**Theorem 11** (Bousfield)**.** *Let $A$ and $A'$ be spaces satisfying the requirements of Notation 10, having types $n = \mathrm{tp}(A)$ and $n' = \mathrm{tp}(A')$ and $d = \mathrm{cn}(A) + 1$ and $d' = \mathrm{cn}(A) + 1$. The following conditions are equivalent:*

(*a*) *The spaces $A$ and $A'$ satisfy the equivalent conditions of Proposition 9: that is, every $P_A$-local space is also $P_{A'}$-local.*
(*b*) *We have $n \leq n'$ and $d \leq d'$.*

The proof of Theorem 11 uses the main results of the preceding two lectures, which we now recall:

**Proposition 12.** *Let $A$ be as in Notation 10, and let $d$ be the smallest positive integer such that $\mathrm{H}_d(A; \mathbf{Z}/p\mathbf{Z})$ is nonzero. Then the Eilenberg-MacLane space $K(\mathbf{Z}/p\mathbf{Z}, d)$ is $P_A$-acyclic.*

**Proposition 13.** *Let $A$ be as in Notation 10, and let $X$ be an arbitrary space. Then the homotopy fibers of the canonical map $f : P_{\Sigma A}(X) \to P_A(X)$ have the form $K(G, d)$, where $G$ is a (p-power) torsion abelian group.*

Proposition 13 was stated in a slightly weaker form in the previous lecture (but not in the notes). For the reader's convenience, we sketch part of the proof. First, we note some elementary observations.

**Lemma 14** ("Fibration Lemma")**.** *Suppose we are given a map of spaces $f : X \to Y$, where each homotopy fiber of $f$ is $P_A$-acyclic. Then the map $P_A(X) \to P_A(Y)$ is an equivalence. In particular, $X$ is $P_A$-acyclic if and only if $Y$ is $P_A$-acyclic.*

**Corollary 15.** *Let $A$ be as in Notation 10 and let $G$ be a $p$-power torsion abelian group. Then $K(G, m)$ is $P_A$-acyclic for $m \geq d$.*

*Proof.* Writing $G$ as a filtered colimit of finite abelian groups, we can reduce to the case where $G$ is finite. Applying Lemma 14 repeatedly, we can further reduce to the case $G = \mathbf{Z}/p\mathbf{Z}$. Using induction on $m$ and Lemma 14, we can further reduce to the case $m = d$, which follows from Proposition 12. $\square$

*Proof of Proposition 13.* Let $F$ be a homotopy fiber of $f$. It was shown in the previous lecture that $F$ is a generalized Eilenberg-MacLane space: that is, it is homotopy equivalent to a product $\prod K(G_m, m)$ for $m \geq 0$. Note that $F$ is $P_A$-acyclic (Remark 4) and therefore $(d-1)$-connected, so the groups $G_m$ vanish for $m < d$. Moreover, since $A$ is rationally trivial, $F$ is also rationally trivial, so that $G_d$ is a torsion group. It now suffices to show that $G_m$ vanishes for $m > d$. Note that $K(G_m, m)$ is a retract of $F$ and is therefore $P_{\Sigma A}$-local (since $P_{\Sigma A}(X)$ and $P_A(X)$ are both $P_{\Sigma A}$-local). Since it is also $P_{\Sigma A}$-acyclic (Corollary 15), it must be contractible: that is, we have $G_m \simeq *$ as desired. $\square$

*Proof of Theorem 11.* We first establish the easy direction. Suppose that $(a)$ is satisfied. Then every $P_{A'}$-acyclic space is also $P_A$-acyclic. Since $A$ is $(d-1)$-connected, it follows that $A$ is $P_{S^d}$-acyclic, so that $A'$ is also $P_{S^d}$-acyclic and therefore $(d-1)$-connected. This proves that $d \leq d'$. Sine $A$ has type $n$, we have $K(n-1)^*_{\mathrm{red}} A \simeq 0$, so that the spaces $\Omega^{\infty+m} K(n-1)$ are $P_A$-local for each $m$. Condition $(a)$ then implies that the spaces $\Omega^{\infty+m} K(n-1)$ are also $P_{A'}$-local, so that $K(n-1)^*_{\mathrm{red}}(A') \simeq 0$ and therefore $n' \geq n$.

We now show that $(b)$ implies $(a)$. Assume that $n \leq n'$ and that $d \leq d'$; we wish to show that $A'$ is $P_A$-acyclic. Using the inequality $n \leq n'$ and the thick subcategory theorem, we deduce that $\Sigma^m A'$ is $P_A$-acyclic for some $m \gg 0$ (as explained in the previous lecture). We will show that $\Sigma^m A'$ is $P_A$-acyclic for *all* $m \geq 0$, using descending induction on $m$. To carry out the inductive step, we must show that if $\Sigma^m A'$ is $P_A$-acyclic then $\Sigma^{m-1} A'$ is also $P_A$-acyclic. Replacing $A'$ by $\Sigma^{m-1} A'$, we can reduce to the case $m = 1$.

Our assumption that $\Sigma A'$ is $P_A$-acyclic guarantees that the canonical map $A' \to P_{\Sigma A'}(A')$ becomes an equivalence after applying $P_A$. Consequently, to show that $A'$ is $P_A$-acyclic, it will suffice to show that $P_{\Sigma A'}(A')$ is $P_A$-acyclic. Since $P_{A'}(A')$ is contractible we can identify $P_{\Sigma A'}(A')$ with the homotopy fiber of the canonical map $P_{\Sigma A'}(A') \to P_{A'}(A')$. It follows from Proposition 13 that $P_{\Sigma A'}(A')$ has the form $K(G, d')$ for some $p$-power torsion abelian group $G$. The desired result now follows from Corollary 15. $\square$

We can summarize Theorem 11 as saying that the homotopy theory of $P_A$-local spaces, where $A$ satisfies the hypotheses of Notation 10, does not depend on the fine details of $A$: it depends only on the type $n = \mathrm{tp}(A)$ and the connectivity $d = \mathrm{cn}(A) + 1$. Our next goal is to show that it does not even very strongly on $d$: that is, the homotopy theory of $P_A$-local spaces is pretty close to homotopy theory of $\Sigma A$-local spaces. However, to articulate this, we need to stay away from "low-degree" behavior: note that the Eilenberg-MacLane space $K(\mathbf{Z}/p\mathbf{Z}, d)$ is $P_A$-acyclic (Lemma **??**) and therefore cannot be $P_A$-local, but it is $(\Sigma A)$-local (since every $d$-truncated space is $\Sigma A$-local).

**Definition 16.** Let $\mathcal{S}_*$ denote the $\infty$-category of pointed spaces. Choose a space $A$ satisfying the requirements of Notation 10, having type $n+1$ and connectivity $d = \mathrm{cn}(A)+1$. We let $L_n^f \mathcal{S}_*^{\langle d \rangle}$ denote the full subcategory of the homotopy category $\mathcal{S}_*$ of pointed spaces spanned by those spaces $X$ which are $d$-connected, $p$-local, and $P_A$-local.

**Warning 17.** It follows from Theorem 11 that the $\infty$-category $L_n^f \mathcal{S}_*^{\langle d \rangle}$ depends only on $n$ and $d$, and not only the space $A$. Beware, however, that $L_n^f \mathcal{S}_*^{\langle d \rangle}$ is defined only when there exists a space $A$ having type $n+1$ and connectivity $d = \mathrm{cn}(A) + 1$. For fixed $n$, such spaces generally exist only when $d$ is sufficiently large.

**Remark 18.** Note that if $X$ is a $d$-connected, $p$-local space, then $P_A X$ is also $d$-connected and $p$-local (since the homotopy fibers of the map $X \to P_A(X)$ are $P_A$-acyclic, and therefore $(d-1)$-connected). It follows that the inclusion functor

$$L_n^f \mathcal{S}_*^{\langle d \rangle} \hookrightarrow \mathcal{S}_*^{\langle d \rangle}$$

admits a left adjoint, which is given by first localizing at the prime $p$ and then applying the localization functor $P_A$.

**Variant 19.** Let $X$ be a $P_A$-local pointed space. Then the $d$-connected cover $X\langle d \rangle$ fits into a fiber sequence

$$\Omega(\tau_{\leq d} X) \to X\langle d \rangle \to X$$

where the base and fiber are $P_A$-local (note that homotopy fibers over points of $X$ which do not belong to the identity component are empty, and therefore also $P_A$-local), so that $X\langle d \rangle$ is also $P_A$-local. Moreover, if $X$ is (simply connected) and $p$-local, then $X\langle d \rangle$ is also simply connected and $p$-local.

**Example 20.** If $n = 0$, we can take $d$ to be any integer $\geq 2$ (by choosing $A$ to be the suspension of the mod $p$ Moore space). In this case, $L_n^f \mathcal{S}_*^{\langle d \rangle}$ is the $\infty$-category of pointed, $d$-connected rational spaces (see Variant 8).

It turns out that for fixed $n$, the differences between the categories $L_n^f \mathcal{S}_*^{\langle d \rangle}$ are entirely due to rational phenomena:

**Theorem 21.** *Let $A$ be a space satisfying the requirements of Notation 10, having type $n+1$ and connectivity $d = \mathrm{cn}(A)+1$. Then the localization functor $P_A$ induces a fully faithful embedding $L_n^f \mathcal{S}_*^{\langle d+1 \rangle} \to L_n^f \mathcal{S}_*^{\langle d \rangle}$, whose essential image is spanned by those objects $X$ for which the rational homotopy group $(\pi_{d+1} X)_{\mathbf{Q}}$ vanishes.*

*Proof.* We first note that if $Y \in L_n^f \mathcal{S}_*^{\langle d+1 \rangle}$, then $P_A(Y)$ belongs to $L_n^f \mathcal{S}_*^{\langle d \rangle}$ (Remark 18). Moreover, the canonical map $Y \to P_A(Y)$ has $P_A$-acyclic homotopy fibers (Remark 4) and therefore induces an isomorphism on rational homotopy groups, which proves the vanishing of $(\pi_{d+1} P_A(Y))_{\mathbf{Q}}$.

Let $X$ be an arbitrary object of $L_n^f \mathcal{S}_*^{\langle d \rangle}$, and let $\widetilde{X}$ be the $(d+1)$-connected covering of $X$, so that we have a fiber sequence

$$\widetilde{X} \to X \to K(G, d+1)$$

for $G = \pi_{d+1}(X)$. Since $X$ and $K(G, d+1)$ are $\Sigma A$-local, it follows that $\widetilde{X}$ is $\Sigma A$-local. We may therefore view the construction $X \mapsto \widetilde{X}$ as a functor

$$L_n^f \mathcal{S}_*^{\langle d \rangle} \to L_n^f \mathcal{S}_*^{\langle d+1 \rangle}$$

We claim that, when restricted to spaces $X$ for which $(\pi_{d+1} X)_{\mathbf{Q}}$ vanishes, this functor is homotopy inverse to $P_A$. For this, we need to prove two things:

$(i)$ For every space $Y \in L_n^f \mathcal{S}_*^{\langle d+1 \rangle}$, the canonical map $f : Y \to P_A(Y)$ exhibits $Y$ as a $(d+1)$-connected covering of $P_A(Y)$: that is, it induces an isomorphism on homotopy groups in degrees $> d+1$. Let $F$ be a homotopy fiber of $f$, so that Proposition 13 guarantees that $F$ has the form $K(G, d)$. We can therefore deloop to a fibration sequence

$$Y \to P_A(Y) \to K(G, d+1)$$

which shows that $Y$ must be the $(d+1)$-connected covering of $P_A(Y)$.

$(ii)$ For every space $X \in L_n^f \mathcal{S}_*^{\langle d \rangle}$ with $(\pi_{d+1} X)_{\mathbf{Q}} \simeq 0$, the canonical map $g : \widetilde{X} \to X$ exhibits $X$ as an $P_A$-localization of $\widetilde{X}$. To prove this, it suffices to show that the homotopy fibers of $g$ are $P_A$-acyclic. These homotopy fibers are Eilenberg-MacLane spaces $K(G, d)$ for $G = \pi_{d+1} X$. Our assumption that $(\pi_{d+1} X)_{\mathbf{Q}}$ guarantees that $G$ is a ($p$-power) torsion group, so the desired result follows from Corollary 15.

□

**Remark 22.** Theorem 21 implies that if we restrict our attention to the subcategories spanned by spaces which are rationally trivial, then the $\infty$-categories $L_n^f \mathcal{S}_*^{\langle d \rangle}$ are independent of $d$. We will elaborate on this point in the next lecture.

**Remark 23.** The $\infty$-category $L_n^f \mathcal{S}_*^{\langle d \rangle}$ is a localization of $\mathcal{S}_*^{\langle d \rangle}$, and therefore has all (homotopy) limits and colimits. They can be described as follows:

- To form the homotopy colimit of a diagram $\{X_\alpha\}$ in $L_n^f \mathcal{S}_*^{\langle d \rangle}$, one first forms the homotopy colimit $\varinjlim X_\alpha$ in the $\infty$-category of pointed spaces (which is automatically $d$-connected and $p$-local), and then applies the localization functor $P_A$ (which preserves the property of being $d$-connected and $p$-local).

- To form the homotopy limit of a diagram $\{X_\alpha\}$ in $L_n^f \mathcal{S}_*^{\langle d \rangle}$, one first forms the homotopy limit $\varprojlim X_\alpha$ in the $\infty$-category of pointed spaces (which is automatically $P_A$-local), and then passes to the $d$-connected cover (which preserves the property of being $P_A$-local; see Variant 19).

The following result will be needed in the next lecture:

**Proposition 24.** *The functor*

$$\mathcal{S}_* \to L_n^f \mathcal{S}_*^{\langle d \rangle} \qquad X \mapsto P_A(X\langle d \rangle_{(p)})$$

*preserves finite homotopy limits.*

The proof of Proposition 24 will require some preliminaries.

**Lemma 25.** *Let $X$ be a $p$-local space. Then $X$ is $P_A$-acyclic if and only if it satisfies the following three conditions:*

($a$) *The space $X$ is $(d-1)$-connected.*
($b$) *The homotopy group $\pi_d(X)$ is $p$-power torsion.*
($c$) *The $d$-connected cover $X\langle d \rangle$ is $P_{\Sigma A}$-acyclic.*

*Proof.* If ($a$), ($b$), and ($c$) are satisfied, then we have a fiber sequence $X\langle d \rangle \to X \to K(G,d)$ where the base and fiber are $P_A$-acyclic (by virtue of Corollary 15 and ($c$), respectively), so that $X$ is $P_A$-acyclic by Lemma 14. Conversely, suppose that $X$ is $P_A$-acyclic. Applying Proposition 14, we deduce that $P_{\Sigma A} X \simeq K(G,d)$ for some $p$-power torsion abelian group $G$. We therefore have a fiber sequence $F \to X \to K(G,d)$, where $F$ is $P_{\Sigma A}$-acyclic. Note that $F$ must be $d$-connected, so this fiber sequence exhibits $F$ as the $d$-connected cover of $X$ and $K(G,d)$ as the $d$-truncation of $X$, which immediately verifies ($a$), ($b$), and ($c$). □

**Remark 26.** It follows from repeated application of Lemma 25 that if $X$ is a $(d-1)$-connected space whose homotopy groups are $p$-power torsion, then the condition that $X$ is $P_A$-acyclic depends only on the connected cover $X\langle n \rangle$ for $n \gg 0$.

**Lemma 27.** *Let $X$ be a $d$-connected pointed space which is $P_A$-acyclic. Then $\Omega X$ is $P_A$-acyclic.*

*Proof.* Using Lemma 25, we see that $X$ is $P_{\Sigma A}$-acyclic. Proposition 5.2 of the previous lecture supplies an equivalence $P_A \Omega X \simeq \Omega P_{\Sigma A} X$, from which the desired result follows immediately. □

**Lemma 28.** *Suppose we are given a homotopy pullback diagram*

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_{01} \end{array}$$

*where $X_0$, $X_1$, and $X_{01}$ are $P_A$-acyclic. If $X_{01}$ is $d$-connected, then $X$ is $P_A$-acyclic.*

*Proof.* Let $F$ be the homotopy fiber of the map $X_1 \to X_{01}$. We then have a homotopy fiber sequence

$$\Omega X_{01} \to F \to X_1.$$

It follows from Lemma 27 that $\Omega X_{01}$ is $P_A$-acyclic. Applying Lemma 14, we conclude that $F$ is $P_A$-acyclic. Using the fiber sequence $F \to X \to X_0$ and applying Lemma 14 again, we conclude that $X$ is $P_A$-acyclic. □

*Proof of Proposition 24.* Note that the functor $X \mapsto X\langle d\rangle$ is right adjoint to the inclusion functor $\mathcal{S}_*^{\langle d\rangle} \hookrightarrow \mathcal{S}_*$, and therefore preserves all homotopy limits. We also note that localization at $p$ preserves finite homotopy limits (on the $\infty$-category $\mathcal{S}_*^{\langle d\rangle}$: this can be checked at the level of homotopy groups. It will therefore suffice to show that the functor $P_A : \mathcal{S}_*^{\langle d\rangle} \to \mathcal{S}_*^{\langle d\rangle}$ preserves finite homotopy limits. Since it clearly preserves final objects, it will suffice to show that it preserves homotopy pullback squares. Fix maps of $d$-connected pointed spaces $X_0 \to X_{01} \leftarrow X_1$. We wish to show that the canonical map

$$\rho : (X_0 \times_{X_{01}} X_1)\langle d\rangle \to (P_A X_0 \times_{P_A X_{01}} P_A X_1)\langle d\rangle$$

is a $P_A$-equivalence (that is, that it has $P_A$-acyclic homotopy fiber). Let $F$ denote the fiber of $\rho$. Then $F$ is $(d-1)$-connected and its homotopy groups are $p$-power torsion. By virtue of Remark 26, it will suffice to show that there exists another $P_A$-acyclic space $F'$ having the same $n$-connected cover as $F$, for some $n \gg 0$. Let $Y_0$ denote the homotopy fiber of the map $X_0 \to P_A X_0$, and define $Y_1$ and $Y_{01}$ similarly. Then $Y_0$, $Y_1$, and $Y_{01}$ are $P_A$-acyclic, so their $d$-connected covers $Y_0\langle d\rangle$, $Y_1\langle d\rangle$, and $Y_{01}\langle d\rangle$ are also $P_A$-acyclic (even $P_{\Sigma A}$-acyclic, by virtue of Lemma 25). We complete the proof by setting $F' = Y_0\langle d\rangle \times_{Y_{01}\langle d\rangle} Y_1\langle d\rangle$ (which is $P_A$-acyclic by virtue of Lemma 28). □