Localizations and the Adams-Novikov Spectral Sequence (Lecture 30)

April 17, 2010

Throughout this lecture, we fix a ring spectrum E. We will assume for simplicity that E is a structured ring spectrum. To any spectrum X, we can associate the cosimplicial ring spectrum $[n] \mapsto X \otimes E^{\otimes n+1}$, which we will denote by $X^{\bullet}$. The homotopy inverse limit of $X^{\bullet}$ is called its totalization and denoted $\mathrm{Tot}(X^{\bullet})$. It is given as an inverse limit of partial totalizations

$$\cdots \to \mathrm{Tot}^2(X^{\bullet}) \to \mathrm{Tot}^1(X^{\bullet}) \to \mathrm{Tot}^0(X^{\bullet}) \simeq X \otimes E,$$

called the *Adams tower* for X with respect to E. There is a canonical map $X \to \mathrm{Tot}(X^{\bullet})$. We ask how closely this map approximates a homotopy equivalence.

The first observation is that $X^{\bullet}$ depends only on the localization $L_E X$: any E-homology equivalence $X \to Y$ induces a homotopy equivalence of cosimplicial spectra $X^{\bullet} \to Y^{\bullet}$. On the other hand, $\mathrm{Tot}\, X^{\bullet}$ is a homotopy inverse limit of E-modules, and is therefore automatically E-local. The best possible situation, then, is that $\mathrm{Tot}\, X^{\bullet}$ is an E-localization of X: equivalently, the map $X \to \mathrm{Tot}\, X^{\bullet}$ induces an isomorphism in E-homology. This is equivalent to the assertion that $E \otimes X \to E \otimes (\mathrm{Tot}\, X^{\bullet})$ is a homotopy equivalence. The right hand side also admits a map to $\mathrm{Tot}(E \otimes X^{\bullet})$. The augmented cosimplicial object $[n] \mapsto E \otimes X \otimes (E^{\otimes (n+1)})$ is *split*: that is, it admits an extra codegeneracy map. It follows formally that the composite map

$$E \otimes X \to E \otimes \mathrm{Tot}\, X^{\bullet} \to \mathrm{Tot}(E \otimes X^{\bullet})$$

is a homotopy equivalence. Consequently, we obtain the following:

Proposition 1. *Let E be a structured ring spectrum and X a spectrum. Then the canonical map $X \to \mathrm{Tot}\, X^{\bullet}$ exhibits $\mathrm{Tot}\, X^{\bullet}$ as an E-localization of X if and only if $E \otimes \mathrm{Tot}\, X^{\bullet} \simeq \mathrm{Tot}(E \otimes X^{\bullet})$.*

Note that $\mathrm{Tot}(E \otimes X^{\bullet}) \simeq \varprojlim \mathrm{Tot}^n(E \otimes X^{\bullet})$. Each partial totalization Tot^n is given by a finite homotopy inverse limit, and therefore commutes with smash products. It follows that $\mathrm{Tot}(E \otimes X^{\bullet})$ can be identified with $\varprojlim E \otimes \mathrm{Tot}^n(X^{\bullet})$. Consequently, the condition of Proposition 1 can be restated as follows: the canonical map

$$E \otimes \varprojlim \mathrm{Tot}^n(X^{\bullet}) \to \varprojlim E \otimes \mathrm{Tot}^n(X^{\bullet})$$

is a homotopy equivalence.

To understand this condition better, it is convenient to work in the setting of *pro-spectra*. A pro-spectrum is a formal inverse limit "$\varprojlim X_\alpha$" of a filtered diagram of spectra (for our needs, it will be sufficient to consider inverse limits of towers). Morphism spaces are computed by the formula

$$\mathrm{Map}(\text{“}\varprojlim X_\alpha\text{”}, \text{“}\varprojlim Y_\beta\text{”}) = \varprojlim_{\beta} \varinjlim_{\alpha} \mathrm{Map}(X_\alpha, Y_\beta).$$

The collection of all pro-spectra form a homotopy theory, which we will denote by $\mathrm{Pro}(\mathrm{Sp})$. There is a forgetful functor $U : \mathrm{Pro}(\mathrm{Sp}) \to \mathrm{Sp}$, which carries a diagram "$\varprojlim X_\alpha$" to its homotopy inverse limit $\varprojlim X_\alpha$. We say that a pro-spectrum "$\varprojlim X_\alpha$" is *constant* if, in $\mathrm{Pro}(\mathrm{Sp})$, it is homotopy equivalent to a constant tower

$$\cdots X \to X \to X.$$

In this case, we have a canonical equivalence $\varprojlim X_\alpha \simeq X$.

If "$\varprojlim X''_\alpha$ is a pro-spectrum and E is any spectrum, then we can define a new prospectrum $E \otimes$"$\varprojlim X''_\alpha =$ "$\varprojlim E \otimes X''_\alpha$. We then have a natural map $E \otimes U(''\varprojlim X''_\alpha) \to U(E \otimes$ "$\varprojlim X''_\alpha)$. This map is not always an equivalence, but it is obviously an equivalence when "$\varprojlim X''_\alpha$ is constant. Applying this to our situation, we obtain the following:

Proposition 2. *The equivalent conditions of Proposition 1 are satisfied whenever the tower*

$$\cdots \to \mathrm{Tot}^2 X^\bullet \to \mathrm{Tot}^1 X^\bullet \to \mathrm{Tot}^0 X^\bullet$$

is constant as a pro-spectrum.

Consequently, it is of interest for us to have a criterion for determining when a tower of spectra

$$\cdots \to Y(2) \to Y(1) \to Y(0)$$

is constant as a pro-spectrum. Recall that any such tower determines a spectral sequence $\{E_r^{p,q}, d_r\}$, which (in good cases) converges to $\pi_q \varprojlim Y(n)$. Our goal is to establish the following criterion (a very imprecise version of a criterion of Bousfield):

Proposition 3 (Bousfield)**.** *Let $\cdots \to Y(2) \to Y(1) \to Y(0)$ be a tower of spectra. Suppose that there exists an integer $s \geq 1$ with the following property: for every finite spectrum F, if $\{E_r^{p,q}, d_r\}$ is the spectral sequence associated to the tower*

$$\cdots \to F \otimes Y(2) \to F \otimes Y(1) \to F \otimes Y(0),$$

then the groups $E_s^{p,q}$ vanish for $p \geq s$. Then the tower $\cdots \to Y(2) \to Y(1) \to Y(0)$ is constant as a pro-object.

To prove Proposition 3, we begin by fixing a tower of spectra

$$\cdots Y(2) \to Y(1) \to Y(0)$$

and assume that the associated spectral sequence $\{E_r^{p,q}\}$ satisfies $E_s^{p,q} \simeq 0$ for $p \geq s$. To exploit this hypothesis, we need to recall the details of the definition of the spectral sequence $\{E_r^{p,q}, d_r\}$. For $m \leq n$ let $F(m,n)$ denote the homotopy fiber of the map $Y(n) \to Y(m)$ (here we adopt the convention that $Y(m) \simeq 0$ for $m < 0$). Then $E_r^{p,q}$ is defined as the image of the map $\pi_q F(p+r-1, p-1) \to \pi_q F(p, p-r)$, and the differential d_r carries $E_r^{p,q}$ into $E_r^{p+r,q-1}$. If $p < 0$, then $F(p, p-r)$ is contractible so that $E_r^{p,q}$ automatically vanishes. If $p \geq s$, then $E_r^{p,q}$ vanishes for $r \geq s$ by assumption. It follows that if $r \geq s$, then at least one of the groups $E_r^{p,q}$ and $E_r^{p+r,q-1}$ vanishes, so that the differential d_r is identically zero. This proves:

($*$) The groups $E_r^{p,q}$ are independent of r for $r \geq s$. That is, the spectral sequence $\{E_r^{p,q}, d_r\}$ collapses at the s-page.

Now suppose $r > p$. Since $F(p, p-r) \simeq Y(p)$, we have $\pi_q F(p, p-r) \simeq \pi_q Y(p)$. In this case, $E_r^{p,q}$ is the image of the composite map

$$\pi_q F(p+r-1, p-1) \to \pi_q Y(p+r-1) \to \pi_q Y(p).$$

The image of the first map is the kernel of the map $\pi_q Y(p+r-1) \to \pi_q Y(p-1)$. We therefore have:

($*'$) For $r > p$, the group $E_r^{p,q}$ is the intersection $\mathrm{Im}(\pi_q Y(p+r-1) \to \pi_q Y(p)) \cap \ker(\pi_q Y(p) \to \pi_q Y(p-1))$.

Combining ($*$) and ($*'$), we deduce:

($*''$) The intersection $\mathrm{Im}(\pi_q Y(p+r) \to \pi_q Y(p)) \cap \ker(\pi_q Y(p) \to \pi_q Y(p-1)$ is independent of r, provided that $r \geq p, s$.

Lemma 4. *For every integer $k \geq 0$, the intersection $\mathrm{Im}(\pi_q Y(p+r) \to \pi_q Y(p)) \cap \ker(\pi_q Y(p) \to \pi_q Y(p-k))$ is independent of r, provided that $r \geq p, s$.*

Proof. We use induction on k. The case $k = 0$ is trivial, so assume that $k > 0$. Suppose that $r \geq p, s$, and that $x \in \pi_q Y(p+r)$ has trivial image in $\pi_q Y(p-k)$. Let $y \in \pi_q Y(p)$ be the image of x; we wish to show that y lifts to $\pi_q Y(p+r+1)$. Let y' denote the image of y in $\pi_q Y(p-1)$. Then y' belongs the kernel of the map $\pi_q Y(p-1) \to \pi_q Y(p-k)$. Since y' lifts to $\pi_q Y(p+r)$, the inductive hypothesis implies that y' can be lifted to an element $x' \in \pi_q Y(p+r+1)$. Subtracting the image of x' from x, we can reduce to the case $y' = 0$. Then $y \in \ker(\pi_q Y(p) \to \pi_q Y(p-1))$, and the desired result follows from $(*'')$. □

Taking $k = p+1$ in Lemma 4, we deduce that the image of the map $\pi_* Y(p+r) \to \pi_* Y(p)$ is independent of r, so long as $r \geq p, s$. Let us denote this image by $A(p)_*$. By construction, we have a sequence of surjections

$$\cdots A(3)_* \to A(2)_* \to A(1)_* \to A(0)_*.$$

By construction, each of these surjections fits into a short exact sequence

$$0 \to E_\infty^{p,*} \to A(p)_* \to A(p-1)_* \to 0$$

By assumption, the groups $E_\infty^{p,*}$ vanish for $p \geq s$. We deduce:

$(*''')$ The maps $A(p)_* \to A(p')_*$ are isomorphisms for $p \geq p' \geq s$.

Let us now consider the tower of graded abelian groups

$$\cdots \to \pi_* Y(4s) \xrightarrow{\theta_2} \pi_* Y(2s) \xrightarrow{\theta_1} \pi_* Y(s).$$

For $m \geq 0$, let $K(m)_* \subseteq \pi_* Y(2^m s)$ be the kernel of the map θ_m. Note that $K(m)_* \cap A(2^m s)_* = 0$, since each θ_m induces an isomorphism $A(2^m s)_* \to A(2^{m-1} s)_*$. For any class $x \in \pi_* Y(2^m s)$, the image $\theta_m(x) \in A(2^{m-1} s)_*$, so that $\theta_m(x) = \theta_m(x')$ for some $x' \in A(2^m s)_*$. It follows that $x = x' + x''$, where $x' \in A(2^m s)_*$ and $x'' \in K(m)_*$. In other words, for $m \geq 1$ we have a direct sum decomposition

$$\pi_* Y(2^m s) \simeq A(2^m s)_* \oplus K(m)_*.$$

It follows that, as a pro-object in graded abelian groups, the tower $\{\pi_* Y(2^m s)\}$ is equivalent to the constant group $A(s)_*$.

Let $Y = \varprojlim Y(p) \simeq \varprojlim_m Y(2^m s)$. The Milnor exact sequence

$$0 \to \varprojlim{}^1 \pi_{*+1} Y(p) \to \pi_* Y \to \varprojlim \pi_* Y(p) \to 0$$

gives $\pi_* Y \simeq A(s)_*$. For each integer $p \geq 0$, let $Y(p)/Y$ denote the cofiber of the canonical map $Y \to Y(p)$. It follows that the maps $\pi_* Y(2^m s) \to \pi_* Y(2^m s)/Y$ induce a composite isomorphism

$$K(m)_* \subseteq \pi_* Y(2^m s) \to \pi_* Y(2^m s)/Y.$$

We conclude that the tower of spectra

$$\cdots \to Y(4s)/Y \to Y(2s)/Y \to Y(s)/Y$$

has the following property: each map in the tower is trivial on all homotopy groups.

Let us now return to the setting of Proposition 3: that is, we assume that the spectral sequence $\{E_r^{p,q}, d_r\}$ has vanishing $E_s^{p,q}$ for $p \geq s$ not only for the tower $\{Y(p)\}$, but also for $\{Y(p) \otimes F\}$ for every finite spectrum F. The same reasoning shows that the maps

$$\cdots \to (Y(4s)/Y)_* \to (Y(2s)/Y)_* F \to (Y(s)/Y)_* F$$

are zero. In other words, each of the maps $Y(2^m s)/Y \to Y(2^{m-1} s)/Y$ is a phantom.

Lemma 5. *A composition of two phantom maps is zero.*

Proof. Fix a spectrum X, and consider a map $u : \bigoplus F_\alpha \to X$, where the sum ranges over all homotopy equivalence classes of maps from finite spectra into X. Using the argument given in Lecture 17, we see that the homotopy fiber X' of u is equivalent to a retract of a sum of finite spectra. Now suppose we are given phantom maps $f : X \to Y$ and $g : Y \to Z$. Since f is a phantom, $f \circ u \simeq 0$ and therefore f is equivalent to a composition $X \to \Sigma X' \to Y$. Consequently, $g \circ f$ factors through the composition $\Sigma X' \xrightarrow{v} Y \xrightarrow{g} Z$. Since g is a phantom and $\Sigma X'$ is a retract of a sum of finite spectra, the composition $g \circ v$ is nullhomotopic and therefore $g \circ f \simeq 0$. □

Applying this to our situation, we deduce that the maps

$$\cdots \to Y(16s)/Y \to Y(4s)/Y \to Y(s)/Y$$

are nullhomotopic, so that the pro-spectrum $\{Y(p)/Y\}$ is trivial. This proves that the tower $\{Y(p)\}$ is equivalent (as a pro-spectrum) to the constant spectrum Y.