

The Bousfield Classes of $E(n)$ and $K(n)$ (Lecture 23)

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Let E and E' be homology theories. We say that E and E' are *Bousfield equivalent* if, for every spectrum X , the homology groups $E_*(X)$ vanish if and only if the homology groups $E'_*(X)$ vanish. Bousfield equivalence is an equivalence relation on spectra, and the equivalence classes are called *Bousfield classes*.

Example 1. Let E be a p -local complex oriented cohomology theory and suppose that the associated map $\text{Spec } \pi_* E \rightarrow \mathcal{M}_{\text{FG}} \times \text{Spec } \mathbf{Z}_{(p)}$ is a flat covering of the open substack $\mathcal{M}_{\text{FG}}^{\leq n}$. Then E is Bousfield equivalent to Morava E -theory $E(n)$. Indeed, for every spectrum X , the vanishing of $E_*(X)$ is equivalent to the requirement that, for each k , the localization $(\mathcal{F}_{\Sigma^k})_{(p)}$ of the quasi-coherent sheaf $\mathcal{F}_{\Sigma^k X}$ on \mathcal{M}_{FG} is supported on the closed substack $\mathcal{M}_{\text{FG}}^{\geq n+1}$, which (by the same argument) is equivalent to the vanishing of $E(n)_*(X)$.

Let p be a prime number and an integer $n > 0$. Our main goal is to prove the following:

Proposition 2. *The spectrum $E(n)$ is Bousfield equivalent to $E(n-1) \times K(n)$. Here we agree by convention that $E(0) \simeq H\mathbf{Q}[\beta^{\pm 1}]$, which is Bousfield equivalent to $H\mathbf{Q}$.*

In other words, we claim that a spectrum X is $E(n)$ -acyclic if and only if it is both $E(n-1)$ -acyclic and $K(n)$ -acyclic. To prove this, it will be convenient to replace introduce a different representative for the Bousfield class of $E(n)$.

Construction 3. Recall that there exists an isomorphism $\pi_* \text{MU}_{(p)} \simeq L_{(p)} \simeq \mathbf{Z}_{(p)}[t_1, t_2, \dots]$ with $t_{p^n-1} = v_n$ for $n > 0$, and by convention we have $t_0 = v_0 = p$. For each $k \geq 0$, we let $M(k)$ denote the cofiber of the map $t_k : \Sigma^{2k} \text{MU}_{(p)} \rightarrow \text{MU}_{(p)}$. In the last lecture, we saw that $M(k)$ admits the structure of a homotopy associative algebra in the category of $\text{MU}_{(p)}$ -modules.

For $m \leq n$, we let $Z(m)$ denote the smash product (over $\text{MU}_{(p)}$) of $\text{MU}_p[v_n^{-1}]$ with $M(k)$, where k ranges over all nonnegative integers not of the form $p^{m'} - 1$ for $m \leq m' \leq n$.

By construction, $Z(m)$ is a complex-oriented ring spectrum with $\pi_* Z(m) = \mathbf{Z}_p[v_1, \dots, v_{n-1}, v_n^{\pm 1}]/(v_0, v_1, \dots, v_{m-1})$. We have $Z(n) \simeq K(n)$, and Example 1 shows that $Z(0)$ is Bousfield equivalent to $E(n)$.

Let us now prove Proposition 2. Suppose first that X is an $E(n)$ -acyclic spectrum. Then each $(\mathcal{F}_{\Sigma^k X})_{(p)}$ is supported on the closed substack $\mathcal{M}_{\text{FG}}^{\geq n+1} \subseteq \mathcal{M}_{\text{FG}}$. Since $\mathcal{M}_{\text{FG}}^{\geq n+1} \subseteq \mathcal{M}_{\text{FG}}^{\geq n}$, we deduce immediately that X is $E(n-1)$ -local. Since $Z(0)$ is Bousfield equivalent to $E(n)$, we have $X \otimes Z(0) \simeq 0$. Since $X \otimes K(n) \simeq X \otimes Z(n)$ is obtained from $X \otimes Z(0)$ by smashing (over $\text{MU}_{(p)}$) with $M(p^k - 1)$ for $0 \leq k < n$, we conclude that $X \otimes K(n) \simeq 0$: that is, X is $K(n)$ -acyclic.

Now suppose that X is $K(n)$ -acyclic and $E(n-1)$ -acyclic; we wish to prove that X is $E(n)$ -acyclic. It will suffice to show that X is $Z(0)$ -acyclic. We prove by descending induction on i that X is $Z(i)$ -acyclic for each $i \leq n$. The case $i = n$ follows from our assumption that X is $K(n)$ -acyclic (since $K(n) \simeq Z(n)$). Suppose therefore that $i < n$ and X is $Z(i+1)$ -acyclic. We have a cofiber sequence

$$\Sigma^{2(p^i-1)} Z(i) \xrightarrow{v_i} Z(i) \rightarrow Z(i+1).$$

It follows that multiplication by v_i acts invertibly on $Z(i) \otimes X$, so that $Z(i) \otimes X \simeq Z(i)[v_i^{-1}] \otimes X$. It will therefore suffice to show that X is $Z(i)[v_i^{-1}]$ -acyclic. Since $Z(i)$ is the smash product (over $\text{MU}_{(p)}$) of $Z(0)$

with $M(p^j - 1)$ for $0 \leq j < i$, it will suffice to show that X is $Z(0)[v_i^{-1}]$ -acyclic. Using Example 1, we see that $Z(0)[v_i^{-1}]$ is Bousfield equivalent to $E(i)$: since X is $E(n-1)$ -acyclic and $i < n$, the first part of the proof shows that X is $E(i)$ -acyclic and therefore $Z(0)[v_i^{-1}]$ -acyclic as desired.

We now discuss the relationship between $E(n)$ -localization and $K(n)$ -localization. As a prototype, suppose that M is a finitely generated abelian group and we wish to describe its localization $M_{(p)}$ at a prime p . We can recover this localization as a fiber product

$$\begin{array}{ccc} M_{(p)} & \longrightarrow & \widehat{M} \\ \downarrow & & \downarrow \\ M_{\mathbf{Q}} & \longrightarrow & \widehat{M}_{\mathbf{Q}} \end{array}$$

where \widehat{M} denotes the p -adic completion of M . To obtain a similar picture in our setting, we will need the following nontrivial fact:

Theorem 4 (Smash Product Theorem). *The localization functor $L_{E(n)}$ is smashing.*

Fix a spectrum X , and form a pullback diagram

$$\begin{array}{ccc} X' & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{E(n-1)}X & \longrightarrow & L_{E(n-1)}L_{K(n)}X. \end{array}$$

There is an evident map $\alpha : X \rightarrow X'$.

Proposition 5. *The map α exhibits X' as an E_n -localization of X .*

Since every $E(n)$ -acyclic spectrum is $E(n-1)$ -acyclic, every $E(n-1)$ -local spectrum is $E(n)$ -local; similarly, every $K(n)$ -local spectrum is $E(n)$ -local. Since the collection of $E(n)$ -local spectra is stable under fiber products, we conclude immediately that X' is $E(n)$ -local. To complete the proof, it will suffice to show that the map α is an $E(n)$ -equivalence. By Proposition 2, it suffices to show that α induces both a $K(n)$ -equivalence and an $E(n-1)$ -equivalence. In other words, we must show that the diagrams

$$\begin{array}{ccc} X \otimes K(n) & \longrightarrow & (L_{K(n)}X) \otimes K(n) \\ \downarrow & & \downarrow \\ (L_{E(n-1)}X) \otimes K(n) & \longrightarrow & (L_{E(n-1)}L_{K(n)}X) \otimes K(n) \end{array} \quad \begin{array}{ccc} X \otimes E(n-1) & \longrightarrow & (L_{K(n)}X) \otimes E(n-1) \\ \downarrow & & \downarrow \\ (L_{E(n-1)}X) \otimes E(n-1) & \longrightarrow & (L_{E(n-1)}L_{K(n)}X) \otimes E(n-1) \end{array}$$

are homotopy pullback squares. For the square on the right, this is obvious, since the vertical maps are both homotopy equivalences. For the square on the left, the upper horizontal map is a homotopy equivalence; we are therefore reduced to proving that the map $L_{E(n-1)}X \otimes K(n) \rightarrow (L_{E(n-1)}L_{K(n)}X) \otimes K(n)$ is an equivalence. This is a consequence of the following more general statement:

Lemma 6. *Let X be any spectrum. Then $L_{E(n-1)}X$ is $K(n)$ -acyclic.*

Proof. Since $L_{E(n-1)}$ is smashing, we have $(L_{E(n-1)}X) \otimes K(n) \simeq X \otimes L_{E(n-1)}K(n)$. It therefore suffices to show that $L_{E(n-1)}K(n) \simeq 0$: in other words, that $E(n-1) \otimes K(n) \simeq 0$. This follows from the observation that $E(n-1) \otimes K(n)$ is complex orientable, and the associated formal group must have height $\leq n-1$ and exactly n . \square

Remark 7. According to Proposition 5, if X is an $E(n)$ -local spectrum, then X can be recovered as the homotopy fiber product $L_{E(n-1)}X \times_{L_{E(n-1)}L_{K(n)}X} L_{K(n)}X$. Conversely, suppose that we are given an arbitrary $K(n)$ -local spectrum Y and $E(n-1)$ -local spectrum Z , together with a map $\alpha : Z \rightarrow L_{E(n-1)}Y$. Form a pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & L_{E(n-1)}Y. \end{array}$$

Then X is $E(n)$ -local. Moreover, since the lower horizontal map is an $K(n)$ -equivalence (since the $K(n)$ -homology of both sides vanishes), we deduce that $X \rightarrow Y$ is a $K(n)$ -equivalence: that is, Y can be identified with $L_{K(n)}X$. Similarly, since the right vertical map is an $E(n-1)$ -equivalence, we conclude that $X \rightarrow Z$ is an $E(n-1)$ -equivalence so that Z can be identified with $L_{E(n-1)}X$. It follows that the $E(n)$ -local stable homotopy category can be recovered as the homotopy category of triples $(Y, Z, \alpha : Z \rightarrow L_{E(n-1)}Y)$, where Y is $K(n)$ -local, Z is $E(n-1)$ -local, and α is a map of $E(n-1)$ -local spectra.

In other words, the $E(n)$ -local stable homotopy category admits a “semi-orthogonal” decomposition into the $E(n-1)$ -local and $K(n)$ -local stable homotopy categories.