Flat Modules over $\mathcal{M}_{\rm FG}$ (Lecture 15)

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We have seen that if E is a complex oriented cohomology theory, then the coefficient ring $\pi_* E$ has the structure of an algebra over the Lazard ring $L \simeq \pi_* \, \mathrm{MU}$. Our next goal is to address the converse: suppose we are given a graded ring R equipped with a homomorphism $L \to R$ (corresponding to a graded formal group over R). When can we find a complex oriented cohomology theory E such that $R = \pi_* E$?

There is an obvious way to try to write down such a cohomology theory. Namely, for any space X, we can attempt to define the E-homology of X by the formula $E_*(X) = \mathrm{MU}_*(X) \otimes_{\pi_* \mathrm{MU}} R = \mathrm{MU}_*(X) \otimes_L R$. However, this prescription does not always work: in order to get a homology theory, we to know that certain sequences of abelian groups are exact. The functor $\bullet \mapsto \bullet \otimes_L R$ generally does not preserve exact sequences. Of course, if R is flat over L, there is no problem. However, the condition of flatness over the Lazard ring L is very restrictive, because $L \simeq \mathbf{Z}[t_1, t_2, \ldots]$ is very large. Fortunately, we can get by with much less: we do not need to assume that R is flat over L, only that R is flat over the moduli stack $\mathcal{M}_{\rm FG}$.

We begin by reviewing the notion of quasi-coherent sheaves on a stack.

Definition 1. A *quasi-coherent sheaf* on the moduli stack $\mathcal{M}_{\rm FG}$ is a rule which specifies, for every R-point $\eta \in \mathcal{M}_{\rm FG}(R)$ (corresponding to a formal group over R), an R-module $M(\eta)$. This rule is required to be functorial in the following sense: given a homomorphism $R \to R'$ carrying η to $\eta' \in \mathcal{M}_{\rm FG}(R')$, we have a canonical isomorphism $M(\eta') \simeq M(\eta) \otimes_R R'$.

Remark 2. There is an obvious analogue of Definition 1 if the moduli stack $\mathcal{M}_{\rm FG}$ is replaced by any other stack.

The collection of quasi-coherent sheaves on $\mathcal{M}_{\rm FG}$ forms an abelian category, which we will denote by $\mathrm{QCoh}(\mathcal{M}_{\rm FG})$.

Definition 3. Let M be a quasi-coherent sheaf on $\mathcal{M}_{\rm FG}$. We will say that M is *flat* if, for every R-point $\eta \in \mathcal{M}_{\rm FG}$, the R-module $M(\eta)$ is flat over R. Similarly, we will say that M is *faithfully flat* if each $M(\eta)$ is faithfully flat over R.

Remark 4. The condition that an R-module be flat (or faithfully flat) is local with respect to the Zariski topology on $\mathrm{Spec}\, R$. Consequently, if M is a quasi-coherent sheaf on $\mathcal{M}_{\rm FG}$, then to verify the flatness (or faithful flatness) of M it suffices to test the condition of Definition 3 in the case where $\eta \in \mathcal{M}_{\rm FG}(R)$ classifies a coordinatizable formal group over R. In this case, η is the image of the point $\eta_0 \in \mathcal{M}_{\rm FG}(L)$ classifying the universal formal group law. In other words, M is flat (or faithfully flat) if and only if $M(\eta_0)$ is flat (faithfully flat) as a module over the Lazard ring L.

Let R be any ring, and suppose we are given a map $q : \mathrm{Spec}\, R \to \mathcal{M}_{\rm FG}$ corresponding to a point $\eta \in \mathcal{M}_{\rm FG}(R)$. Then the forgetful functor $M \mapsto M(\eta)$ can be identified with the pullback functor $q^* : \mathrm{QCoh}(\mathcal{M}_{\rm FG}) \to \mathrm{QCoh}(\mathrm{Spec}\, R) \simeq \mathrm{Mod}_R$. This functor has a right adjoint $q_* : \mathrm{Mod}_R \to \mathrm{QCoh}(\mathcal{M}_{\rm FG})$. In concrete terms, if N is an R-module, then $q_*(N)$ is a quasi-coherent sheaf on $\mathcal{M}_{\rm FG}$ given by the formula $q_*(N)(\eta') = q'_*(p')^* M = M \otimes_R B$, where $\eta' \in \mathcal{M}_{\rm FG}(R')$ classifies a map $p : \mathrm{Spec}\, R' \to \mathcal{M}_{\rm FG}$ and B is the $(R \otimes R')$-algebra which classifies the universal isomorphism between the formal groups over R and R'

determined by η and η', so that we have a pullback square

$$\begin{array}{ccc} \operatorname{Spec} B & \xrightarrow{q'} & \operatorname{Spec} R' \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \operatorname{Spec} R & \xrightarrow{q} & \mathcal{M}_{\mathrm{FG}}\,. \end{array}$$

Given $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$, we say that an R-module N is *flat* (or *faithfully flat*) over $\mathcal{M}_{\mathrm{FG}}$ if $q_* N$ is flat (faithfully flat) over $\mathcal{M}_{\mathrm{FG}}$. We will say that q is *flat* (*faithfully flat*) over $\mathcal{M}_{\mathrm{FG}}$ if the R-module R is flat (faithfully flat) over $\mathcal{M}_{\mathrm{FG}}$.

The usefulness of this notion to us is expressed by the following:

Proposition 5. *Let $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ be a map (classifying a formal group $\eta \in \mathcal{M}_{\mathrm{FG}}(R)$) and let N be an R-module which is flat over $\mathcal{M}_{\mathrm{FG}}$. Then the functor $M \mapsto M(\eta) \otimes_R N = q^* M \otimes_R N$ is an exact functor from $\mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}})$ to Mod_R.*

Proof. The question is local on $\operatorname{Spec} R$; we may therefore assume that η classifies a coordinatizable formal group. Let $p : \operatorname{Spec} L \to \mathcal{M}_{\mathrm{FG}}$ classify the universal formal group law, so we have a pullback diagram

$$\begin{array}{ccc} \operatorname{Spec} R[b_0^{\pm 1}, b_1, b_2, \ldots] & \xrightarrow{q'} & \operatorname{Spec} L \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \operatorname{Spec} R & \xrightarrow{q} & \mathcal{M}_{\mathrm{FG}} \end{array}$$

Since p' is a faithfully flat map, it will suffice to show that the functor

$$M \mapsto p'^*(q^* M \otimes_R N) \simeq (qp')^* M \otimes_{R[b_0^{\pm 1}, b_1, \ldots]} p'^* N \simeq (p^* M) \otimes_L p'^* N$$

is exact. The flatness of N is precisely the condition that $p'^* N = N[b_0^{\pm 1}, b_1, \ldots]$ is a flat L-module. □

Corollary 6. *Let M be a graded module over the Lazard ring L. If M is flat over $\mathcal{M}_{\mathrm{FG}}$, then the functor $X \mapsto \mathrm{MU}_*(X) \otimes_L M$ is a homology theory, representable by some spectrum E.*

Example 7. Fix a prime number p, and let $R \simeq \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ be the L-module obtained by taking the quotient of $L_{(p)} \simeq \mathbf{Z}_{(p)}[t_1, t_2, \ldots]$ by the ideal generated by $\{t_i\}_{i+1 \neq p^k}$. We claim that the map $\operatorname{Spec} R \to \operatorname{Spec} L \to \mathcal{M}_{\mathrm{FG}}$ is flat. To prove this, we must show that if we form the fiber product

$$\begin{array}{ccc} \operatorname{Spec} B & \xrightarrow{q} & \operatorname{Spec} L \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \mathcal{M}_{\mathrm{FG}}, \end{array}$$

the map q comes from a flat ring homomorphism $L \to B$. Note that we have two ring homomorphisms $\phi_0, \phi_1 : L \to L[b_0^{\pm 1}, b_1, \ldots]$; ϕ_0 is the obvious map, and ϕ_1 classifies the formal group over $L[b_0^{\pm 1}, b_1, \ldots]$ obtained from the universal formal group by the change of variables $g(t) = b_0 t + b_1 t^2 + \cdots$. Unwinding the definitions, we see that B can be identified with the quotient of $L_{(p)}[b_0^{\pm 1}, b_1, b_2, \ldots]$ by the ideal generated by $\{\phi_1(t_i)\}_{i+1 \neq p^k}$. The proof of Lazard's theorem shows that if $i+1$ is not a power of p, then the image of t_i under the composite map

$$L \xrightarrow{\phi_1} L_{(p)}[b_0^{\pm 1}, b_1, b_2,] \to \mathbf{Z}_{(p)}[b_1, b_2, \ldots]$$

is given by db_i + *decomposables*, where d is invertible in $\mathbf{Z}_{(p)}$. It follows that we can replace b_i by $\phi_1(t_i)$ in our set of polynomial generators for $L_{(p)}[b_0^{\pm 1}, b_1, b_2, \ldots]$, so that B is a polynomial ring over $L_{(p)}[b_0^{\pm 1}]$ and in particular flat over $L_{(p)}$.

It follows from Corollary 6 that the construction $X \mapsto \mathrm{MU}_*(X) \otimes_L R \simeq \mathrm{MU}_*(X)_{(p)}/(t_i)_{i+1 \neq p^k}$ is a homology theory. This homology theory is called *Brown-Peterson homology* and is denoted by $\mathrm{BP}_*(X)$.

Remark 8. The above proof gives more: not only is $\operatorname{Spec} \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ flat over $\mathcal{M}_{\mathrm{FG}}$, it is faithfully flat over the localized moduli stack $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$.

Corollary 6 highlights the importance of the condition of flatness over the moduli stack $\mathcal{M}_{\mathrm{FG}}$. In the next lecture, we will establish the following criterion for flatness:

Theorem 9 (Landweber)**.** *Let M be a module over the Lazard ring L. Then M is flat over $\mathcal{M}_{\mathrm{FG}}$ if and only if, for every prime number p, the sequence $v_0 = p, v_1, v_2, \ldots \in L$ is a regular sequence for M.*

An L-module M satisfying the hypothesis of Theorem 9 is said to be *Landweber-exact*. Every Landweber-exact graded L-module M determines a homology theory E, given by $E_*(X) = \mathrm{MU}_*(X) \otimes_L M$. In particular we have $\pi_* E = E_*(*) \simeq M$.

Remark 10. Recall that if R is a commutative ring and M is an R-module, then a sequence of elements $x_0, x_1, x_2, \ldots \in R$ is said to be *regular for M* if x_0 is not a zero divisor on M, x_1 is not a zero divisor on M/x_0M, x_2 is not a zero divisor on $M/(x_0M + x_1M)$, and so forth. Note that if a module M is trivial, then every element of R is a non zero-divisor on M.

Example 11. Let M be an L-module which is a rational vector space. Then M is Landweber-exact: for every prime p, $v_0 = p$ acts invertibly on M, so $M/v_0M \simeq 0$.

Example 12. Let $R = \mathbf{Z}[\beta, \beta^{-1}]$, where β has degree 2. We have a graded formal group law $f(x, y) = x + y + \beta xy$ over R (a graded version of the multiplicative formal group over $\mathbf{Z}$), which determines a map of graded rings $L \to R$. We claim that R is Landweber exact. Fix a prime p. Then $v_0 = p$ is a non zero-divisor on R. Modulo p, the p-series $[p](t)$ for f is given by the formula $[p](t) = \beta^{p-1} t^p$, so that $v_1 \equiv \beta^{p-1} \mod p$ and therefore v_1 acts invertibly on R/pR (and so every element of R is a nonzero divisor on $R/(v_0R + v_1R)$).

Using Landweber's theorem, we deduce the existence of a homology theory $E_*(X) = \mathrm{MU}_*(X) \otimes_L \mathbf{Z}[\beta, \beta^{-1}]$ with $\pi_* E \simeq R = \mathbf{Z}[\beta, \beta^{-1}]$. We will later see that E_* is given by complex K-theory.

Example 13. Let R be a commutative ring and let E be an elliptic curve defined over R. We can associate to E a formal group $\widehat{E}$, where $\widehat{E}(A) \subseteq \operatorname{Hom}(\operatorname{Spec} A, E)$ is the collection of all A-points of E for which the diagram

$$\begin{array}{ccc} \operatorname{Spec} A/\mathfrak{m} & \longrightarrow & \operatorname{Spec} A \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \xrightarrow{0} & E \end{array}$$

commutes; here $\mathfrak{m}$ denotes the nilradical of A. This construction determines a map from the moduli stack $\mathcal{M}_{\mathrm{Ell}}$ of elliptic curves to the moduli stack $\mathcal{M}_{\mathrm{FG}}$ of formal groups.

If we fix a prime number p and a trivialization of the Lie algebra of E, then $v_1 \in R/pR$ can be identified with the classical *Hasse invariant*: it vanishes precisely on the closed subscheme of $\operatorname{Spec} R/pR$ over which E is supersingular. Moreover, v_2 is invertible in $R/(pR + v_1R)$: that is, the formal group of an elliptic curve is everywhere of height ≤ 2.

To satisfy Landweber's criterion, the pair (E, R) must satisfy the following:

(1) Every prime number p is a non zero-divisor in R: that is, R is flat over $\mathbf{Z}$.

(2) For every prime number p, the Hasse invariant of E is a non zero-divisor in R/pR.

If these conditions are satisfied, then we can define a homology theory $\mathrm{Ell}_*(X)$, where $\mathrm{Ell}_*(X) = \mathrm{MU}_*(X) \otimes_L R[\beta, \beta^{-1}]$. The representing spectrum Ell is sometimes called *elliptic cohomology.*

Remark 14. Conditions (1) and (2) satisfied in "universal" cases; that is, for elliptic curves over $\operatorname{Spec} R$ which define an étale map from $\operatorname{Spec} R$ to the moduli stack $\mathcal{M}_{\mathrm{Ell}}$. In other words, the map of stacks $\mathcal{M}_{\mathrm{Ell}} \to \mathcal{M}_{\mathrm{FG}}$ is flat.