Lecture 25: Differential equations

Differential equations

25.1. A differential equation is an equation for an unknown function $y$ involving derivatives of the function. For example, $y'(t) = y(t)$ is a differential equation for an unknown function $y(t)$. We often think of $t$ as "time".

25.2. Unlike for usual equations like $3x = 4$, where we look for a number as a solution, we now look for a function. A solution to the differential equation is a function $y(t)$ which satisfies the equation. You might notice that there is more than one solution to the equation $y'(t) = y(t)$. Can you see the general solution?

25.3. We are already dealing with differential equations when integrating: the equation $y'(t) = t^2$ has the solution $y(t) = \int_0^t x^2 \, dx + C$, where $C$ is a constant. A specific solution to the above equation $y'(t) = t^2$ is $t^3/3$. The general solution is $t^3/3 + C$.

Growth models

25.4. Here is a typical population growth problem:

Example: If $M(t)$ is the number of Canadian geese on the Charles river. Each geese has 2 off-springs a year in average, while 1 of the geese dies. How many geese are there in 5 years, if there are 5000 geese initially?

Figure 1. A proud Harvard Canadian goose. Photo shot by Oliver on November 7, 2023.
Calculus and Differential equations

We can model this as \( M'(t) = 2M(t) - M(t) = M(t) \). The solution of \( M(t) = 5000e^t \) gives for \( t = 5 \) the number \( 5000e^5 \).

**Decay models**

25.5. An other situation, where differential equations appear are **decay models**. Here is a typical example.

**Example:** The amount \( N(t) \) of Carbon 14 in a sample satisfies
\[
N'(t) = -0.0001216N(t)
\]

The negative sign means that the number of Carbon 14 isotopes **decreases** in time. If we have initially \( N_0 \) atoms, then after time \( t \), we have \( M(t) = e^{-0.0001216t}M(0) \). Since the decay number is \( \log(2)/5700 = -0.0001216 \) we know that in 5700 years, the amount of Carbon 14 is half. We could also write \( M(t) = e^{-t/5700}M(0) \).

**Banking**

25.6.

**Example:** If \( M(t) \) is the bank account, then under a continuous compounding assumption, the balance \( M'(t) \) is \( rM(t) \), where \( r = 0.06 \) is the **interest rate**. If there are \( M(0) = 100'000 \) dollars initially, the equation how much do we have in \( t = 10 \) years?

The equation \( M' = rM \) is solved by \( M(t) = 100'000e^{0.06t} \).

**Example:** If additionally, 50'000 dollars is continuously transferred to the account we can model this with \( M'(t) = M(t) + 50'000 \). Can you see why \( M(t) = 100'000e^{0.06t} + 50000(e^{0.06t} - 1) \) satisfies the differential equation?

Solution: subtract \( M(t) \) from
\[
M'(t) = 0.06 * 100'000e^{0.06t} + 0.06 * 50'000e^{0.06t}
\]
This gives 50'000.

25.7. Compare that if \( r \) is the interest rate and we would compound annually in \( n \) steps, the bank account grows like
\[
(1 + \frac{r}{n})^n.
\]
This is a compound interest formula and for \( n \rightarrow \infty \) it approaches the function \( e^r \).

25.8. Differential equations can be much more general and model much more interesting situations. Coming up with a differential equation which captures the situation means **making a model**. It is not easy to come up with good models. The British statistician **George Box** once expressed this in 1978 with the following aphorism: "**All models are wrong but some are useful**".

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