1. Let \( S \) be a simplicial set. Show that \( S \) satisfies the unique lifting property for inner horns if and only if \( S \) is the nerve of a usual category \( \mathcal{C} \). Show that the assignment \( \mathcal{C} \mapsto N(\mathcal{C}) \) is a fully faithful embedding from the category of small categories (considered as a usual category, i.e., we discard natural transformations) to the category of simplicial sets.

2. Show that the assertion of Problem 1 remains valid when you replace the word "category" by "groupoid" and "inner horns" by "all horns".

3. Let \( \mathcal{C}_1, \mathcal{C}_2 \) be two categories, and \( F, G : \mathcal{C}_1 \to \mathcal{C}_2 \) be two functors. Show that the set of natural transformations \( F \Rightarrow G \) can be identified with the set of maps of simplicial sets \( \Delta^1 \times N(\mathcal{C}_1) \to N(\mathcal{C}_2) \) that restrict to \( F \) and \( G \) under \( \Delta^0 \Rightarrow \Delta^1 \).

4. Recall the simplicial category \( \mathcal{C}(\mathcal{[n]}) \).
   (a) Convince yourself that for a simplicial category \( \mathcal{C} \), maps of simplicial categories \( \mathcal{C}(\mathcal{[2]}) \to \mathcal{C} \) are the same as a triple of objects \( c_1, c_2, c_3 \in \mathcal{C} \), maps \( f_{i,j} \in \text{Hom}(c_i, c_j)_{0, i < j} \) and a homotopy \( g : f_{2,3} \circ f_{1,2} \Rightarrow f_{1,3} \in \text{Hom}(c_1, c_3)_{1} \).
   (b) Convince yourself that defining a map \( \mathcal{C}(\mathcal{[3]}) \to \mathcal{C} \) is the same as specifying objects \( c_i \in \mathcal{C}, i = 1, 2, 3, 4 \), maps \( f_{i,j} \in \text{Hom}(c_i, c_j)_{0, i < j} \), and a data of "coherent homotopy" between them (or, rather, take it as a definition of coherent homotopy and unravel what it means).

5. Recall the simplicial nerve functor \( N : \text{Cat}_\Delta \to \text{Set}_\Delta \),
   \[
   N_n(\mathcal{C}) := \text{Hom}_{\text{Cat}_\Delta}(\mathcal{C}(\mathcal{[n]}), \mathcal{C}).
   \]
   (a) What’s the precise connection between the simplicial and usual nerve constructions?
   (b) Show that the functor \( N \) above admits a left adjoint, denoted \( \mathcal{C} \). What’s the value of \( \mathcal{C} \) on \( \Delta^n \in \text{Set}_\Delta \)?
   (c) Put (b) in the following framework. Let \( \mathcal{C}_1 \) be a small category, and \( \mathcal{C}_2 \) category closed under colimits. Then colimit preserving functors \( \text{Funct}(\mathcal{C}_1^{op}, \text{Set}) \to \mathcal{C}_2 \) are in bijection with functors \( \mathcal{C}_1 \to \mathcal{C}_2 \).
   (d) Let \( A \) be a poset, considered as an ordinary category. Consider the simplicial set \( N(A) \). Describe \( \mathcal{C}(N(A)) \).

6. Let \( \mathcal{C} \) be a simplicial category such that for any \( c_1, c_2 \in \mathcal{C} \), the simplicial set \( \text{Hom}(c_1, c_2) \) is Kan. Show that \( N(\mathcal{C}) \in \text{Set}_\Delta \) is a quasi-category.

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