The isomorphism $\mathrm{GL}_4(\mathbf{F}_2) \cong A_8$ (outline)

Noam D. Elkies

The simple groups A_8 and $\mathrm{GL}_4(\mathbf{F}_2)$ both have $\frac{1}{2}8! = 15 \cdot 14 \cdot 12 \cdot 8 = 20160$ elements. One might then expect that they are isomorphic. This in fact turns out to be the case. One way to show this is to construct a 4-dimensional vector space V over $\mathbf{F}_2$ on which A_7 acts nontrivially by linear transformations; this will embed A_7 as an index-8 subgroup of $\mathrm{GL}_4(\mathbf{F}_2)$, and thus a homomorphism of $\mathrm{GL}_4(\mathbf{F}_2)$ into S_8. Since $\mathrm{GL}_4(\mathbf{F}_2)$ is simple, this homomorphism will have trivial kernel, and image contained in A_8. It will thus yield an embedding of $\mathrm{GL}_4(\mathbf{F}_2)$ into A_8, which must then be an isomorphism because the group orders coincide.

Let $H \subset S_7$ be the automorphism group of the projective plane Π_2. We know that $\#H = 168$. Moreover, H is simple, so $H \subset A_7$. Thus H has $[A_7 : H] = \frac{1}{2}7!/168 = 15$ cosets in A_7. To these 15 we will add $\mathbf{0}$ to obtain V, and show that V has the structure of an $\mathbf{F}_2$-vector space. That is, we will define for each $P, Q \in V$ a natural element $P + Q \in V$ and show that this addition law satisfies the axioms of an $\mathbf{F}_2$-vector space.

The cosets of H in S_7 are the 30 different ways to identify an unstructured 7-element set Σ with the points of Π_2. We'll show that two of these fall in the same A_7-orbit if and only if they have exactly one line in common. (Otherwise they share either three lines or none, but we will not need this.) We do this by noting that given one Π_2 structure P on Σ and one of its seven lines l, there are two other Π_2 structures in the same A_7-orbit whose only common line with P is l, namely the images of P under g, g^2 where $g \in A_7$ is a 3-cycle permuting l. This is enough because it accounts for all $1 + 2 \cdot 7 = 15$ structures in the orbit of P. We now define $P + Q$ for $P, Q \neq 0$ as follows: if $P = Q$ then of course $P + Q = 0$; else P, Q have a unique line l in common, and $P + Q$ is the third Π_2-structure in the same A_7 containing l. The only obstruction to verifying that this makes V into an $\mathbf{F}_2$-vector space is checking that the identity $(P + Q) + R = P + (Q + R)$ for distinct $P, Q, R \in V - \{\mathbf{0}\}$ without a common line.

Fortunately A_7 acts (simply) transitively on the $15 \cdot 14 \cdot 12 = 2520$ such triples (P, Q, R), so it is enough to check the identity on any one of them. To prove the transitivity it suffices to verify that if $g \in A_7$ fixes each of P, Q, R then g is the identity. But then g fixes each of the lines common to two of P, Q, R, which are three 3-element subsets of Σ any two of which meet at a point. They thus separate Σ into singletons, and so g cannot move any element of Σ. [Note that it follows in particular that the action of A_7 on $V - \{\mathbf{0}\}$ is doubly transitive.] Confirming that $(P + Q) + R = P + (Q + R)$ for some appropriate P, Q, R is straightforward. This completes the identification of $\mathrm{GL}_4(\mathbf{F}_2)$ with A_8.

Alternatively we could have obtained V directly as a representation of A_8, by identifying the nonzero points with orbits of $\mathrm{AGL}_3(\mathbf{F}_2) \subset A_8$, i.e. with half of the 30 ways of arranging an 8-element set into a 3-(8,4,1) Steiner system. That makes it somewhat trickier to define the linear structure on V, though.

We can even recover S_8 from $\mathrm{GL}_4(\mathbf{F}_2)$. Recall that $S_8 = \mathrm{Aut}(A_8)$, so we need only find an outer automorphism of $\mathrm{GL}_4(\mathbf{F}_2)$. But we already know such an automorphism: the "contragredient" involution $A \leftrightarrow {}^t A^{-1}$.