

Diophantine and ergodic foliations on surfaces

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1 Introduction

This paper gives a topological characterization of Diophantine and recurrent laminations on surfaces. It also establishes an upper bound for the number of ergodic components of a measured foliation, in terms of limits of the corresponding geodesic ray in $\overline{\mathcal{M}}_{g,n}$. Taken together, these results give a new approach to Masur's theorem on unique ergodicity.

Teichmüller rays. Let $\mathcal{M}_{g,n}$ denote the moduli space of Riemann surfaces of genus g with n punctures. Consider the Teichmüller ray

$$\gamma : [0, \infty) \rightarrow \mathcal{M}_{g,n}$$

generated by a holomorphic quadratic differential $q = q(z) dz^2$ on $X \in \mathcal{M}_{g,n}$. If there exists a compact set $K \subset \mathcal{M}_{g,n}$ such that $\gamma(t) \in K$ for all t , we say γ is *Diophantine*. If $\gamma(t_i) \in K$ for a sequence $t_i \rightarrow \infty$, we say γ is *recurrent*.

In §2 we will show:

Theorem 1.1 *The Teichmüller ray generated by q is Diophantine iff the vertical lamination λ of q satisfies*

$$\inf_S |S| i(S, \lambda) > 0.$$

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It is recurrent iff

$$\limsup_{T \rightarrow \infty} \inf_{|S| \leq T} T \cdot i(S, \lambda) > 0.$$

Here S ranges over all simple closed curves in $\mathcal{ML}_{g,n}$, and $|S|$ is a combinatorial measure of its length (e.g. the number of 1-cells required to describe S in a fixed triangulation of $\Sigma_{g,n}$).

Corollary 1.2 *The Diophantine or recurrent character of q depends only on its vertical lamination.*

This observation also follows from [Iv1, Thm. 3.1].

We say a measured lamination or foliation is Diophantine (resp. recurrent) if it generates only Diophantine (resp. recurrent) Teichmüller geodesic rays. Theorem 1.1 shows that these properties are monotone, in the following sense.

Corollary 1.3 *If $\lambda' \geq \lambda$ and λ is Diophantine (resp. recurrent), then so is λ' .*

Here $\lambda \leq \lambda'$ means $i(S, \lambda) \leq i(S, \lambda')$ for all simple closed curves S .

Ergodicity. In §4 we will use the moduli space $Q\overline{\mathcal{M}}_{g,n}$ of quadratic differentials on stable curves to formulate and prove the following result.

Theorem 1.4 *Let $(X_t, q_t) \subset Q_1\mathcal{M}_{g,n}$ be an orbit of the Teichmüller flow, and suppose*

$$(X_{t_i}, q_{t_i}) \rightarrow (Y, q) \in Q_1\overline{\mathcal{M}}_{g,n}$$

for some sequence $t_i \rightarrow \infty$. Let k be the number of components of Y^ where q is nonzero. Then the vertical foliation $\mathcal{F}(q_0)$ has at most k ergodic components.*

Here Y^* is the smooth part of the stable curve Y . The conclusion means X_0 can be partitioned into at most k sets of positive measure, each saturated by the leaves of $\mathcal{F}(q_0)$. The proof leverages the area-preserving property of Teichmüller mappings and a general lemma in ergodic theory, given in the Appendix.

Combining these results, we obtain:

Corollary 1.5 (Masur) *If X_t is recurrent in $\mathcal{M}_{g,n}$, then the vertical lamination λ of q_0 is uniquely ergodic.*

Proof. Theorem 1.4 implies any recurrent lamination is ergodic with respect to its underlying measure. If λ is not uniquely ergodic, then by adding in a second, independent transverse invariant measure, we obtain a $\lambda' \geq \lambda$ which is not ergodic. But λ' is still recurrent, by Corollary 1.3, so this is a contradiction. ■

This Corollary is a restatement of [Mas, Thm. 1.1]; see also [MT, Thm. 3.8]. The latter survey also describes its striking applications to polygonal billiards. The approach given here avoids technical issues that may arise from singular measures and the zeros and poles of limits (Y, q) .

Veech examples. A well-known construction of Veech provides examples of quadratic differentials (X_0, q_0) of genus two such that $\mathcal{F}(q_0)$ is minimal but not ergodic [V], [MT, §3]. To conclude, we show Theorem 1.4 yield the sharp bound $k = 2$ for the number of ergodic components of these examples.

Theorem 1.6 *The flow line in $Q_1\overline{\mathcal{M}}_2$ generated by a Veech example accumulates on a stable differential (Y, q) such that Y^* has two components, each of genus one.*

Proof. By Veech's construction, we have a differential (E_0, r_0) of genus one and a degree two holomorphic map $\pi_0 : X_0 \rightarrow E_0$ such that $q_0 = \pi_0^*(r_0)$. Applying the Teichmüller flow to both (X_0, q_0) and (E_0, r_0) , we obtain a family of maps $\pi_t : X_t \rightarrow E_t$ with $\pi_t^*(r_t) = q_t$. Since $\mathcal{F}(q_0)$ is minimal, so is $\mathcal{F}(r_0)$, and thus (E_t, r_t) is recurrent in $Q_{1/2}\mathcal{M}_1$; say $(E_{t_n}, r_{t_n}) \rightarrow (E, r)$. The differentials arising from 2-fold branched covers of (E, r) have compact closure $K \subset Q_1\overline{\mathcal{M}}_2$, so (X_t, q_t) has *some* accumulation point $(Y, q) \in K$. Since $\mathcal{F}(q_0)$ is not ergodic, q must be supported on $k > 1$ components of Y^* ; hence Y^* has exactly two components, each of genus one. ■

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Notation. The notation $A \asymp B$ means A/B and B/A are both bounded above by an implicit constant C .

2 Diophantine and recurrent laminations

In this section we discuss quadratic differentials, measured foliations and laminations, and the Teichmüller geodesics they generate. For more on these topics, see e.g. [FLP], [IT], [Iv2], [Bon].

Proofs of the statements in Theorem 1.1 follow. We conclude with a discussion of harmonic forms with 2 periods, and various other sources of Diophantine and recurrent geodesics.

Quadratic differentials. Let $\mathcal{M}_{g,n}$ denote the moduli space of Riemann surfaces of genus g with n punctures. For each $X \in \mathcal{M}_{g,n}$ we let $Q(X)$ denote the space of holomorphic quadratic differentials on X whose total mass

$$m(X, q) = \int_X |q|$$

is finite. This means q has at worst simple poles at the punctures of X . The zeros of q form a finite set $Z(q) \subset X$, provided $q \neq 0$.

The moduli space of quadratic differentials

$$Q\mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,n}$$

consists of pairs (X, q) with $0 \neq q \in Q(X)$. The corresponding bundle over Teichmüller space will be denoted $Q\mathcal{T}_{g,n}$. We let $Q_1\mathcal{M}_{g,n}$ denote the locus where $m(X, q) = 1$. There is a natural action of $\mathrm{PSL}_2(\mathbb{R})$ on $Q\mathcal{M}_{g,n}$, preserving $m(X, q)$, with the diagonal matrices giving the Teichmüller geodesic flow.

Laminations. Let $\mathcal{ML}_{g,n}$ denote the space of (nonzero) measured geodesic laminations on a standard topological surface $\Sigma_{g,n}$. The intersection number of a pair of laminations will be denoted by $i(\lambda, \mu)$.

By straightening the vertical foliation $\mathcal{F}(q)$ of a quadratic differential, we obtain a measured lamination $\Lambda(q)$ (cf. [Le]). The map $q \mapsto (\Lambda(q), \Lambda(-q))$ gives an embedding

$$Q\mathcal{T}_{g,n} \hookrightarrow \mathcal{ML}_{g,n} \times \mathcal{ML}_{g,n},$$

which sends a Teichmüller geodesic (X_t, q_t) to a path of the form $(e^t\lambda, e^{-t}\mu)$. A pair of laminations is in the image iff $i(\lambda, \xi) + i(\mu, \xi) > 0$ for all $\xi \in \mathcal{ML}_{g,n}$.

Remarks.

1. One might expect X_t to converge to $[\lambda] \in \mathbb{P}\mathcal{ML}_{g,n}$ as $t \rightarrow \infty$. This is true often, but not always [Len].
2. The process of straightening a foliation to a lamination is similar to the process of resolving an interval exchange transformation $f : I \rightarrow I$, by replacing all points of discontinuity of all iterates f^n with two points. The result is a measure-preserving homeomorphism $F : K \rightarrow K$, usually on a Cantor set, to which the usual methods of topological dynamics can be applied.

Lengths. It is useful to introduce a measure $|S|$ of the *combinatorial length* of a simple closed curve $S \in \mathcal{ML}_{g,n}$. Many definitions are possible; for concreteness, we fix a triangulation of $\Sigma_{g,n}$, and we let $|S|$ denote the minimum number of 1-cells in a cycle homotopic to S .

For any nonzero $q \in Q(X)$ we let $L(S, |q|)$ denote the length of S in the conformal metric with area form $|q|$. That is,

$$L(S, |q|) = \inf \left\{ \int_{S'} |q|^{1/2} : S' \text{ is isotopic to } S \right\}.$$

If the vertical and horizontal foliations of q are given by the laminations (λ, μ) , then we have

$$1 \leq \frac{i(S, \lambda) + i(S, \mu)}{L(S, |q|)} \leq 2. \quad (2.1)$$

Indeed, if X is compact we can replace S by a geodesic in the $|q|$ -metric; then in local coordinates where $q = dz^2$, $z = x + iy$ we have

$$L(S, |q|) = \int_S |dz|, \quad i(S, \lambda) = \int_S |dx|, \quad \text{and} \quad i(S, \mu) = \int_S |dy|,$$

and (2.1) follows from the inequality $|z| \leq |x| + |y| \leq 2|z|$. In the noncompact case, S need not be represented by a geodesic, but the same argument applies to a length minimizing sequence of representatives.

Since $|S| \asymp L(S, |q|)$ we also have

$$|S| \asymp i(S, \lambda) + i(S, \mu), \quad (2.2)$$

where the implicit constants depend on q .

Compactness. As is well-known, a sequence $X_i \in \mathcal{M}_{g,n}$ tends to infinity iff the length of the shortest closed hyperbolic geodesic on X_i tends to zero [Mum]. The metrics coming from quadratic differentials also detect divergence.

Proposition 2.1 *Given $(X_i, q_i) \in Q_1\mathcal{M}_{g,n}$, the sequence X_i diverges in moduli space iff*

$$\inf_S L(S, |q_i|) \rightarrow 0. \quad (2.3)$$

Here the infimum is over all simple closed curves $S \in \mathcal{ML}_{g,n}$.

Proof. The function $\inf_S L(S, |q|)$ is positive and continuous, so (2.3) implies $(X_i, q_i) \rightarrow \infty$. But the projection $Q_1\mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,n}$ is proper, so $X_i \rightarrow \infty$. Conversely, if $X_i \rightarrow \infty$ then there are simple closed curves S_i whose extremal lengths $\mathcal{E}(S_i, X_i)$ tend to zero. Since $|q_i|$ is a conformal metric of area one, we have $L(S_i, |q_i|)^2 \leq \mathcal{E}(S_i, X_i)$, which implies (2.3). ■

Diophantine geodesics. Let $\gamma : [0, \infty) \rightarrow \mathcal{M}_{g,n}$ be the Teichmüller ray generated by $(X, q) \in Q\mathcal{M}_{g,n}$. The ray γ is *Diophantine* if it is contained in a compact subset of moduli space.

Theorem 2.2 *The ray γ generated by q is Diophantine iff the vertical lamination $\lambda = \Lambda(q)$ satisfies*

$$\inf_S |S| i(S, \lambda) > 0. \quad (2.4)$$

Proof. Let (λ, μ) be the vertical and horizontal laminations of q , and let $\gamma(t) = (X_t, q_t)$ be the corresponding geodesic ray. Then the laminations of q_t are $(e^t \lambda, e^{-t} \mu)$. By the proposition above, γ is Diophantine iff

$$\inf_S \inf_{t \geq 0} e^t i(S, \lambda) + e^{-t} i(S, \mu) > 0. \quad (2.5)$$

Consider any fixed S , and let $(A, B) = (i(S, \lambda), i(S, \mu))$. Then $(\inf_{t \geq 0} e^t A + e^{-t} B)^2 \asymp A(A + B)$. By (2.2), we have $A + B \asymp |S|$, and thus (2.5) is equivalent to (2.4). \blacksquare

The same reasoning shows:

Corollary 2.3 *The complete geodesic $\gamma : \mathbb{R} \rightarrow \mathcal{M}_{g,n}$ generated by q is bounded iff its vertical and horizontal laminations satisfy*

$$\inf_S i(S, \lambda) \cdot i(S, \mu) > 0.$$

Remark: Diophantine slopes on the torus. Up to scale, a lamination λ on the square torus $\Sigma_1 = \mathbb{R}^2/\mathbb{Z}^2$ is given by a slope α , say $0 < \alpha < 1$. A simple closed curve S is determined by a pair of relatively prime integers (p, q) . Their intersection number is given by

$$i(S, \lambda) = \left| \det \begin{pmatrix} \alpha & 1 \\ p & q \end{pmatrix} \right| = |q\alpha - p|,$$

which can only be small if $p < q$. Under this assumption, we can take $|S| = |q|$; then (2.4) becomes the standard *bounded type* condition

$$\inf_S |S| i(S, \lambda) = \inf_{p,q} q|\alpha - p/q| > 0,$$

or equivalently $|\alpha - p/q| > C/q^2$. Thus we recover the well-known fact that α is the endpoint of a bounded geodesic ray in \mathcal{M}_1 iff its continued fraction expansion is bounded.

Recurrence. A ray $\gamma : [0, \infty) \rightarrow \mathcal{M}_{g,n}$ is *divergent* if $\gamma(t) \rightarrow \infty$ in $\mathcal{M}_{g,n}$; otherwise it is *recurrent*.

Theorem 2.4 *The ray γ generated by q is recurrent iff $\lambda = \Lambda(q)$ satisfies*

$$\limsup_{T \rightarrow \infty} \inf_{|S| \leq T} T \cdot i(S, \lambda) > 0. \quad (2.6)$$

Proof. Let $\delta(T) = \inf_{|S| \leq T} T \cdot i(S, \lambda)$. We will show that γ diverges iff $\delta(T) \rightarrow 0$. Reasoning as in the Diophantine case, we find γ is divergent iff

$$\epsilon(T) = \inf_S T \cdot i(S, \lambda) + T^{-1} \cdot |S| \rightarrow 0$$

as $T \rightarrow \infty$. Suppose this is the case. Then $\epsilon(T) \leq 1$ for all T large enough. For each such T , we have a simple curve S with $|S| \leq \epsilon(T)T \leq T$ and $T \cdot i(S, \lambda) \leq \epsilon(T)$. Hence $\delta(T) \leq \epsilon(T)$, so $\delta(T) \rightarrow 0$.

Now suppose $\delta(T) \rightarrow 0$. Choose a positive continuous function with $\alpha(T) \rightarrow 0$ such that

$$\frac{\delta(\alpha(T)T)}{\alpha(T)} \rightarrow 0.$$

(This is always possible.) Then we have $\delta(\alpha(T)T) \leq 1$ for all T sufficiently large. For each such T , there is an S such that $|S| \leq \alpha(T)T$ and $\alpha(T)T \cdot i(S, \lambda) \leq \delta(\alpha(T)T)$. This shows

$$\epsilon(T) \leq T \cdot i(S, \lambda) + T^{-1} \cdot |S| \leq \frac{\delta(\alpha(T)T)}{\alpha(T)} + \alpha(T).$$

Since the right-hand side tends to zero as $T \rightarrow \infty$, γ is divergent. ■

Harmonic forms with two periods. The following criterion is sometimes useful for giving examples. Consider a nonzero holomorphic 1-form ω on a Riemann surface $X \in \mathcal{M}_g$ with vertical foliation $\mathcal{F} = \mathcal{F}(\omega^2)$. This foliation depends only on the harmonic form $\rho = \operatorname{Re}(\omega)$ and the smooth structure on X . The *relative periods* of ρ are given by

$$\operatorname{Per}(\rho) = \left\{ \int_C \rho : C \in H_1(X, Z(\rho); \mathbb{Z}) \right\}.$$

Theorem 2.5 *Suppose the saddle connections of \mathcal{F} form a forest, and $\operatorname{Per}(\rho) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$. Let $\theta = \alpha/\beta$. Then \mathcal{F} is recurrent if θ is irrational, and \mathcal{F} is Diophantine if θ has bounded type.*

Proof. This result is immediate for the foliation \mathcal{E} of the square torus $E = \mathbb{R}^2/\mathbb{Z}^2$ determined by the 1-form $\xi = \alpha dx + \beta dy$. To handle the general case, choose a smooth map $f : X \rightarrow E$ such that $f^*(\xi) = \rho$ and f

sends $Z(\rho)$ to a single point $p \in E$. (Such a map exists by our assumptions on $\text{Per}(\rho)$; indeed, we can write $\rho = \alpha\xi_1 + \beta\xi_2$ as a linear combination of two integral cohomology classes, and construct the two coordinates of f by integrating ξ_1 and ξ_2 .)

To complete the proof, it suffices to show that \mathcal{F} is Diophantine (resp. recurrent) whenever the same is true of \mathcal{E} .

Consider an essential simple closed curve S on X . Up to isotopy, S is represented by a geodesic on $(X, |\omega|)$ consisting of a chain of saddle connections S_1, \dots, S_n . This representative satisfies $i(S, \mathcal{F}) = \sum |\int_{S_i} \rho|$ and $\sum |S_i| \asymp |S|$. Let $T_i = f(S_i)$. Each T_i is a loop based at p , satisfying $|T_i| = O(|S_i|)$ and

$$i(T_i, \mathcal{E}) = \left| \int_{T_i} \xi \right| = \left| \int_{S_i} \rho \right|.$$

Combining these bounds we find

$$|S| i(S, \mathcal{F}) \geq C \left(\sum |T_i| \right) \left(\sum i(T_i, \mathcal{E}) \right),$$

where $C > 0$ is independent of S . Since \mathcal{F} has no loop made up of saddle connections, $i(T_i, \mathcal{E}) \neq 0$, and thus each T_i is isotopic to a multiple of a nontrivial simple loop T on E . Consequently we have

$$\inf_S |S| i(S, \mathcal{F}) \geq C \inf_T |T| i(T, \mathcal{E}).$$

This shows \mathcal{F} is Diophantine whenever \mathcal{E} is, by Theorem 2.2. The same reasoning works for recurrence as well, using Theorem 2.4. \blacksquare

A similar result, expressed in the language of interval exchange transformations, appears in [Bo].

Example: genus two. Let x' denote the Galois conjugate of $x \in \mathbb{Z}(\sqrt{D})$. Choose $\alpha, \beta > 0$ in $\mathbb{Z}(\sqrt{D})$ such that $\alpha > \alpha'$ and $\beta > \beta'$. Consider the L -shaped polygon P shown in Figure 1. By gluing opposite sides with vertical and horizontal translations, we obtain a Riemann surface X of genus 2.

We claim the diagonal foliation \mathcal{F} of X , defined by the harmonic form $\rho = dx - dy$, is Diophantine. Since the periods of ρ lie in $\mathbb{Q}(\sqrt{D})$, we only need to rule out saddle connections. Consider the first return map to the horizontal edges of P/\sim under the diagonal flow. Locally this map has the form $x \mapsto x + n + \beta$ or $x + m + \alpha + \beta$, with $n, m \in \mathbb{Z}$. Both of these forms strictly increase $x - x'$, so the first return map has no periodic points and hence \mathcal{F} has no saddle connections. (For related arguments, see [Mc, §2].)

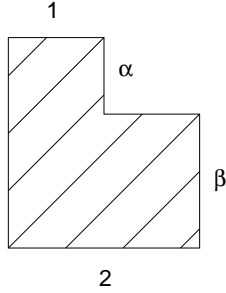


Figure 1. A Diophantine foliation in genus two.

Square tilings. In fact, \mathcal{F} is equivalent to a foliation of a square-tiled surface by straight lines with quadratic irrational slope. Such a foliation is clearly Diophantine.

To see this square tiling, let ρ' be a 1-form representing the Galois conjugate of $[\rho]$ in $H^1(X, \mathbb{Z}[\sqrt{D}])$ such that $\rho \wedge \rho' > 0$ outside $Z(\rho)$. (An explicit form of this type is given by $\rho' = dx - f(y) dy$, where $f(y) = \alpha'/\alpha$ in the top rectangle of P and $f(y) = \beta'/\beta$ in the bottom. Even though ρ' is discontinuous, we have $d\rho' = 0$ as a current.) Let $E = \mathbb{R}^2/\mathbb{Z}[\sqrt{D}]$ using the embedding $x \mapsto (x, x')$. Integrating (ρ, ρ') , we obtain a covering map $\pi : X \rightarrow E$. The map π is branched over one point, and \mathcal{F} is simply the preimage of the horizontal foliation of E . By choosing a linear isomorphism $E \cong \mathbb{R}^2/\mathbb{Z}^2$, we obtain the desired square tiling of X . (This tiling gives X a new complex structure.)

Further examples: Schottky groups and winning sets. Any lamination in the limit set of a convex cocompact subgroup of the mapping-class is necessarily Diophantine. Examples of such subgroups, similar to Schottky groups acting on \mathbb{H}^3 , are discussed in [FM].

In [KW] it is shown that for any quadratic differential $q \in \mathcal{QM}_{g,n}$, the vertical foliation of $e^{i\theta}q$ is Diophantine for all θ in a set of Hausdorff dimension one. In fact the Diophantine directions form an absolute winning set [CCM].

3 Ergodicity of pseudo-Anosov foliations

Let $\gamma \subset \mathcal{M}_{g,n}$ be a closed Teichmüller geodesic generated by $(X, q) \in \mathcal{QM}_{g,n}$. Then we have an associated pseudo-Anosov mapping $f : X \rightarrow X$,

locally of the form

$$f(x + iy) = Kx + iK^{-1}y, \quad K > 1$$

in coordinates where $z = x + iy$ and $q = dz^2$. (Such coordinates exist away from the zeros of q .) The length of γ is $\log K$ in the Teichmüller metric.

In this section we will show:

Theorem 3.1 *The stable foliation $\mathcal{F}(q)$ of a pseudo-Anosov mapping is ergodic.*

This result is well-known, and in fact $\mathcal{F}(q)$ is uniquely ergodic (see e.g. [FLP, Exposé 12]). The proof we give below generalizes readily from closed geodesics to recurrent geodesics, after which it too will yield unique ergodicity (see the proof of Corollary 1.5).

Distribution of leaves. We begin with the setup for a general foliation. Let $(X, q) \in \mathcal{Q}_1\mathcal{M}_{g,n}$ be a unit-area quadratic differential, and let $\mathcal{F} = \mathcal{F}(q)$ be the vertical foliation. The conformal metric $|q|$ determines a probability measure

$$m(A) = \int_A |q|$$

on X . This measure is given locally by the product $|dz|^2 = |dx| \cdot |dy|$ of arclength measure $|dy|$ along the leaves of \mathcal{F} with the transverse invariant measure $|dx|$.

A set $A \subset X$ is *saturated* if any leaf of \mathcal{F} meeting A is contained in A . The foliation is *ergodic* if any measurable saturated set satisfies $m(A) = 0$ or 1. Since \mathcal{F} has only finitely many singular leaves (through the zeros and poles of q), these play no role in the definition of ergodicity.

Let \bar{X} denote the compact surface obtained by filling in the punctures of X . For any continuous function $h \in C(\bar{X})$ and any $x \in X$, we can average the values of h with respect to arclength along the leaves of \mathcal{F} through x . If these averages converge (for leaves in both directions through x), they define a probability measure ν_x on \bar{X} . Note that $\nu_x = \nu_y$ whenever x and y lie on the same leaf of \mathcal{F} .

The measure $m = |q|$ on X is invariant under the (locally defined) unit speed flow along the leaves of \mathcal{F} . By the ergodic theorem, this implies ν_x exists for all x in a set $E \subset X$ with $m(E) = 1$. In fact, for almost every x and $h \in C(\bar{X})$ we have

$$I(x, h) = \int h \nu_x = H(x), \tag{3.1}$$

where H is the projection of h to the closed subspace of $L^2(X, m)$ consisting of functions that are constant on the leaves of \mathcal{F} . In particular,

$$\int_X I(x, h) dm(x) = \int h dm.$$

Thus ν_x is supported in X for almost every x . Formula 3.1 implies:

Proposition 3.2 *The foliation \mathcal{F} is ergodic iff ν_x is constant a.e. on X .*

In the ergodic case, $\nu_x = m$ a.e., and hence almost every leaf of \mathcal{F} is uniformly distributed.

The pseudo-Anosov case. Now let $f : X \rightarrow X$ be a pseudo-Anosov mapping adapted to q . Given an index set $S = \{s_1 < s_2 < \dots\} \subset \mathbb{N}$, let

$$\lim_S f^s(x) = \lim_{i \rightarrow \infty} f^{s_i}(x)$$

if the limit exists. A *rectangle* is a region $R \subset X$ such that $(R, q|R)$ is isomorphic to $([a, b] \times [c, d], dz^2)$ in \mathbb{C} .

A well-known argument (see e.g. [Mas, p. 383]) shows:

Proposition 3.3 *Suppose $x, y \in E$ and there is an index set S such that the limits*

$$(x', y') = (\lim_S f^s(x), \lim_S f^s(y))$$

both exist and lie in the same rectangle R of (X, q) . Then $\nu_x = \nu_y$.

Proof. Consider, for each $s \in S$, the rectangle $R_s = f^{-s}(R)$. Since f^{-s} expands the vertical sides of R by K^s and contracts the horizontal sides by K^{-s} , the region R_s is close to a long leaf of \mathcal{F} through x . Any $h \in C(\overline{X})$ is uniformly continuous in the metric $|q|$, and thus $(1/m(R_s)) \int_{R_s} h dm \rightarrow \int h d\nu_x$. The same reasoning applies to y , and hence $\nu_x = \nu_y$. ■

Since f preserves the measure m , a general result in ergodic theory (see Appendix A) guarantees:

Proposition 3.4 *There is a countable set $A \subset E$ and an index set S such that $F(x) = \lim_S f^s(x)$ exists for all $x \in A$, and $F(A) \subset X$ is dense.*

Proof of Theorem 3.1. Choose S and A as above. We may assume $F(A) \cap Z(q) = \emptyset$. By Proposition 3.3, $\nu_a = \nu_b$ whenever $F(a)$ and $F(b)$ lie in a rectangle. Since any two points of $X - Z(q)$ lie in a finite chain of overlapping rectangles, we conclude that ν_a takes on only one value μ , for all $a \in A$.

Let $B = \{x \in E : \nu_x \neq \mu\}$. Suppose $m(B) > 0$. Let $\omega(S, x)$ denote the set of accumulation points of $\langle f^s(x) : s \in S \rangle$, and let $\omega(S, B) = \bigcup_{x \in B} \omega(S, x)$. By Lemma A.1, $m(\omega(S, B)) > 0$. Thus there exists an $x \in B$ and an index set $T \subset S$ such that $y = \lim_T f^t(x)$ is not a zero or pole of q . Then y lies in a rectangle R for (X, q) . But R meets $F(A)$, so $\nu_y = \nu_a = \mu$ for some $a \in A$, a contradiction.

Thus $m(B) = 0$, and hence \mathcal{F} is ergodic. ■

4 Ergodic components and stable curves

In this section we will discuss quadratic differential on stable curves, and prove the bound on the number of ergodic components of $\mathcal{F}(q_0)$ stated in Theorem 1.4. For more on the stable curves and their quadratic differentials, see e.g. [IT], [HM], [HK].

Stable quadratic differentials. Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli space of stable curves Y . Let $Q(Y)$ denote the holomorphic quadratic differentials on the smooth points Y^* of Y , with at worst simple poles at punctures and at worst double poles, with *equal* residues, at nodes. The moduli space $Q\overline{\mathcal{M}}_{g,n}$ consists of pairs (Y, q) with $0 \neq q \in Q(Y)$. Note that $m(Y, q) = \int |q|$ is finite iff all poles of q are simple.

As in the case of holomorphic 1-forms, the bundle of projective spaces $\mathbb{P}Q\overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ is compact. However mass can be lost in the limit, so the locus $Q_1\overline{\mathcal{M}}_{g,n}$ of differentials with $m(Y, q) = 1$ is *not* compact.

Teichmüller maps and area-preserving maps. To set up the proof of Theorem 1.4, we first relabel (X_{t_i}, q_{t_i}) as (X_i, q_i) . The natural Teichmüller mapping $F_i : X_0 \rightarrow X_i$ sends the measure $|q_0|$ to $|q_i|$, while shrinking the leaves of $\mathcal{F}(q_0)$ by a factor of $K_i = \exp(-t_i)$.

Let $W \subset Y$ denote the support of q , i.e. the union of the irreducible components of Y where q is not identically zero. Since $m(X_i, q_i) = m(Y, q) = 1$, we can find a sequence of piecewise smooth, bijective maps $H_i : X_i \rightarrow W$ sending the measure $|q_i|$ to the measure $|q|$. (We do not require that H_i is continuous). We can also arrange that the differentials $(H_i)_*(q_i)$ converge to q smoothly on compact subsets of $W^* - Z(q)$.

(Here is one way to construct the maps H_i . Choose a compact set $K \subset W^* - Z(q)$ carrying most of the mass of $|q|$. Then for all $i \gg 0$ we have holomorphic maps $h_i : K \rightarrow X_i$ such that $h_i^*(q_i) \rightarrow q$ on K . Thus $\rho_i = |h_i^*(q_i)|/|q| \rightarrow 1$ smoothly on K , and in particular $\int_K \rho_i |q| < \int_W |q|$ for all $i \gg 0$. We can therefore find area-preserving maps $s_i : W \rightarrow W$, converging smoothly to the identity on K , such that $s_i|_K$ sends the measure $\rho_i |q|$ to $|q|$ (cf. [Mos]). The map $H_i = s_i \circ h_i^{-1}$ is then smooth and area-preserving on $h_i(K)$, and $(H_i)_*(q_i) = (s_i)_*(q) \rightarrow q$ on K as required. It is now easy to extend these maps to the rest of X_i and let K grow with i to complete the construction.)

The compositions $f_i = H_i \circ F_i : X_0 \rightarrow W$ then give a sequence of area-preserving maps, with respect to the area measures $|q_0|$ and $|q|$ on their domain and range.

Proof of Theorem 1.4. Let k be the number of components of W^* . We wish to show $\mathcal{F}(q_0)$ has no more than k ergodic components.

As in §3, for a.e. $x \in X_0$ we have a probability measure ν_x describing the distribution of the leaf of $\mathcal{F}(q_0)$ through x . Again we make two observations:

1. If $x' = \lim_S f_s(x)$ and $y' = \lim_S f_s(y)$ both lie in an open rectangle R of (W, q) , then $\nu_{x'} = \nu_{y'}$.
2. There exists a countable set $A \subset X_0$ and an index set S such that $F(x) = \lim_S f_s(x)$ exists for all $x \in A$, and $F(A)$ is dense in W^* .

To see the first observation, use the fact that $H_i^{-1}(R)$ is contained in a nearly isomorphic rectangle $R_i \subset X_i$ for all $i \gg 0$. The second observation follows from Appendix A.

Now let W_1, \dots, W_k denote the connected components of W^* , and let $A_i = \{x \in A : F(x) \in W_i\}$. Since any two points of W_i are joined by a chain of rectangles, ν_a is constant for $a \in A_i$. Call this constant value μ_i .

Let $Z_i = \{x \in X_0 : \nu_x = \mu_i\}$. Each of these sets is saturated, and $\mathcal{F}|_{Z_i}$ is ergodic. It remains only to show that $B = X - \bigcup_1^k Z_i$ has measure zero. But if $m(B) > 0$, then (again by Lemma A.1) there is an $x \in B$ and an index set $T \subset S$ such that $y = \lim_T f_t(x)$ is neither a zero of q nor a node of W . Then y lies in a rectangle R meeting $F(A_i)$ for some i , which implies $\nu_x = \mu_i$, a contradiction.

So in fact $X_0 = \bigcup_1^k Z_i$ gives the ergodic decomposition of \mathcal{F} . ■

Examples. Let $(E_i, q_i) \in Q\mathcal{M}_1$, $i = 1, 2$ be a pair of quadratic differentials of genus one with Diophantine vertical foliations. Slit each surface open

along a unit vertical arc, and then glue them together to obtain a new differential $(X_0, q_0) \in Q\mathcal{M}_2$.

The slits give a pair of vertical saddle connections forming a loop $\alpha_0 \subset X_0$. The corresponding loop $\alpha_t \subset X_t$ shrinks to a node as $t \rightarrow \infty$.

It is easy to see that all accumulation points (Y, η) of (X_t, q_t) in $Q\overline{\mathcal{M}}_g$ consist of two forms (F_1, η_1) , (F_2, η_2) of genus one joined at a node, and that many such accumulation points exist. Indeed, each component (F_i, η_i) is simply an accumulation point of the bounded flow line in $Q\mathcal{M}_1$ generated by (E_i, q_i) . In particular, there is no loss of mass; we have

$$m(Y, \eta) = m(E_1, q_1) + m(E_2, q_2) = m(X_0, q_0).$$

Thus we have an instance of Theorem 1.4 with $k = 2$. And indeed $\mathcal{F}(q_0)$ has two ergodic components, separated by α_0 .

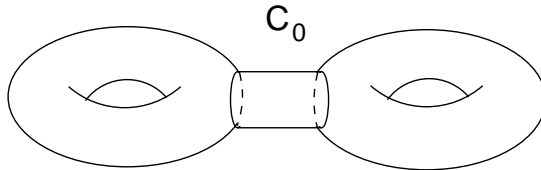


Figure 2. Inserting a cylinder.

Now repeat the construction, but replace α_0 with a cylinder C_0 of positive area (Figure 2) foliated by closed leaves. Then the accumulation points of (X_t, q_t) remain the same, but the vertical foliation $\mathcal{F}(q_0)$ has infinitely many ergodic components. The difference is that the mass of C_0 now disappears in the limit, so we no longer have an instance of Theorem 1.4.

This example shows that recurrence in $Q_1\overline{\mathcal{M}}_{g,n}$, unlike recurrence in $\mathcal{M}_{g,n}$, is not a monotone property of the vertical foliation.

A Appendix: Orbits and measures

This section gives an elementary result in ergodic theory, used in §3 and §4.

Let X be a compact metric space equipped with a complete probability measure m of full support, and let

$$f_n : X \rightarrow X$$

be a sequence of measure-preserving mappings. Assume f_n^{-1} sends open sets to Borel sets; this regularity certainly holds if f_n is piecewise continuous.

Let $\omega(x) \subset X$ denote the set of all limits of accumulation points of the sequence $\langle f_n(x) \rangle$, and let $\omega(E) = \bigcup_{x \in E} \omega(x)$.

Lemma A.1 *For any Borel set E , we have $m(\omega(E)) \geq m(E)$.*

Proof. It suffices to show $\omega(E)$ meets any compact set $B \subset X$ with $m(B) > m(X - E)$. To see this, let $B_N = \bigcup_{n > N} f_n^{-1}(B)$; then we have $B_N \supset B_{N+1}$ and $m(B_N) \geq m(f_N^{-1}(B)) = m(B)$, so $m(\bigcap B_N) \geq m(B) > m(X - E)$. Hence there is an $x \in E \cap \bigcap B_N$; but this means $f_n(x) \in B$ for arbitrarily large n and hence $\omega(x)$ meets B by compactness of B . ■

Note. The set $\omega(E)$ is the projection to X of the Borel set

$$E' = \{(x, y) : \forall n > 0 \exists m > 0 : d(f_m(x), y) < 1/n\} \subset E \times X.$$

The projection of a Borel set is not always Borel, but it is analytic and hence measurable for any complete measure [Ke].

Corollary A.2 *If $E \subset X$ has full measure then $\omega(E)$ is dense.*

As in §3, given $S = \{s_1 < s_2 < \dots\} \subset \mathbb{N}$ we let $\lim_S f_s(x) = \lim_{i \rightarrow \infty} f_{s_i}(x)$.

Corollary A.3 *If E has full measure, then there exists an index set S and a countable set $A \subset E$ such that $F(x) = \lim_S f_s(x)$ exists for all $x \in A$, and $F(A)$ is dense in X .*

Proof. Let (U_i) be a countable base for X . Since $\omega(E)$ is dense, there is an $a_1 \in X$ and an $S_1 \subset \mathbb{N}$ such that $\lim_{S_1} f_s(a_1) \in U_1$. Restricting attention now to $\{f_s : s \in S_1\}$, we find there is an $a_2 \in X$ and an $S_2 \subset S_1$ such that $\lim_{S_2} f_s(a_2) \in U_2$. Continuing inductively and then diagonalizing, we obtain the required set $A = \{a_1, a_2, \dots\}$ and index set S . ■

Remark. These results apply just as well to a sequence of measure-preserving maps $f_n : Y \rightarrow X$.

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