n. Hence there exists a sequence $n_1, n_2, \ldots$ and numbers $a_1, a_2, \ldots$ such that for each m

$$\phi_r(z_m) = f_{n_r}(z_m) \to a_m$$

as $r \to \infty$. [Note that the sequence (n_r) and the functions ϕ_r do not depend on D_1'.] Now let D_2' be any closed set interior to D, a the distance from D_2' to the frontier of D, and let D_1' (so far arbitrary) be the set of points of D whose distance from the frontier is not less than $\frac{1}{2}a$. Let δ be the number associated with D_1' and the inequality (1). We may allow δ to be diminished, and may therefore suppose that $\delta < \frac{1}{2}a$. If now z is any point of D_2' there exists a z_m within a distance δ of z, where $m \leqslant M(\delta)$, and z and z_m belong to D_1'. Now, given *any* m, there exists a $\nu(\epsilon, m)$ such that

$$|\phi_r(z_m) - \phi_s(z_m)| < \epsilon \quad (r, s \geqslant \nu);$$

hence, if

$$\operatorname*{Max}_{m \leqslant M} \nu(\epsilon, m) = N = N(\epsilon, \delta) = N(\epsilon, D_2'),$$

we have

(2) $$|\phi_r(z_m) - \phi_s(z_m)| < \epsilon \quad (r, s \geqslant N, \ m \leqslant M).$$

In the inequality

$$|\phi_r(z) - \phi_s(z)| \leqslant |\phi_r(z) - \phi_r(z_m)| + |\phi_s(z) - \phi_s(z_m)| + |\phi_r(z_m) - \phi_s(z_m)|$$

each term on the right is less than ϵ (if $r, s \geqslant N$), the first two in virtue of $|z - z_m| < \delta$ and (1), the last in virtue of (2). Thus

$$|\phi_r(z) - \phi_s(z)| \leqslant 3\epsilon$$

for $r, s \geqslant N(\epsilon)$, and all z of D_2'. It follows that $\phi_r(z)$ tends to a limit function $f(z)$ uniformly in D_2'. Since D_2' is arbitrary, $f(z)$ cannot depend on it; finally f is continuous at any point of D, as the uniform limit of a continuous function ϕ_r. This completes the proof.

4. *Theory of functions of a real variable.*

4.1. In this section we set out those parts of the theory of real functions which we require later. We sometimes carry our developments beyond what is strictly necessary, but have not tried to be exhaustive. The extent of knowledge required is nothing like so great as is sometimes supposed. There are three principles, roughly expressible in the following terms: Every (measurable) set is nearly a finite sum of intervals; every function (of class L^λ) is nearly continuous; every convergent sequence of functions is nearly uniformly convergent. Most of the results of the present section are fairly intuitive applications

of these ideas, and the student armed with them should be equal to most occasions when real variable theory is called for. If one of the principles would be the obvious means to settle a problem if it were "quite" true, it is natural to ask if the "nearly" is near enough, and for a problem that is actually soluble it generally is.

When our results are capable of extension to functions that may take complex values—and they generally are—the extended form is either deducible trivially from the real one, or else the *proof* for the real case applies with trivial modifications. We shall therefore take such extensions for granted, giving only the proof in the real case, and sometimes only the *enunciation* in the real case, when either procedure suits our convenience.

4.21. THEOREM 6†. *Suppose that $f(\theta)$ is of class L^λ in E_0, and let δ, ϵ be given. Then there exist (a) a continuous ϕ, (b) a stretchwise constant‡ ("step-function") ϕ, with the further properties:*

(1) $|f - \phi| < \epsilon$, *except in a set e of measure less than δ,*

(2) $$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f - \phi|^\lambda \, d\theta \leqslant \epsilon^\lambda,$$

(3) $$\phi(\pi) = \phi(-\pi).$$

If further f is bounded on one or both sides, such a ϕ can be found with the same bound or bounds.

Theorem 2 shows that if, given f, we can find, first an $f^{(1)}$ of a certain type such that $M_\lambda(f - f^{(1)})$ is arbitrarily small, then, for fixed $f^{(1)}$, an $f^{(2)}$ of another type such that $M_\lambda(f^{(1)} - f^{(2)})$ is arbitrarily small, and so on to $f^{(r)}$; then, given f, we can find a function of the last type, $f^{(r)}$, such that $M_\lambda(f - f^{(r)})$ is arbitrarily small. We shall use this argument frequently, and we have here its first occasion. A precisely similar principle is available for a result of the type (1).

Let $f_n = [f]_n$ and let e_n be the set in which $f_n \neq f$. Then $n = |f_n| \leqslant |f|$ in e_n,

$$n^\lambda e_n \leqslant \int_{e_n} |f|^\lambda \, d\theta \leqslant \int_{E_0},$$

and so $e_n \to 0$ as $n \to \infty$, and

$$\int_{E_0} |f - f_n|^\lambda \, d\theta \leqslant \int_{e_n} |2f|^\lambda \, d\theta \to 0.$$

† This is the second principle of § 4.1.

‡ That is to say: E_0 can be divided into a finite number of intervals, in each of which ϕ is a constant.