$$\frac{G}{g} \geqslant \frac{\pi R'^2}{\pi r'^2}, \tag{2}$$

with equality holding only when $\phi(\zeta) = c_0 + c_1\zeta$. But $G \leq \pi R^2$ and $g \geq \pi r^2$. Consequently, from (2) we have

$$\frac{R}{r} \geqslant \frac{R'}{r'}, \tag{3}$$

with equality holding, obviously, only for $\phi(\zeta) = c_1\zeta$, where $|c_1| = 1$.

The assertion of the theorem follows because, if we map the extended domain B' onto the annulus $r < |z| < R$, the domain B is mapped onto a subdomain of that annulus, to which (3) is applicable with inequality holding. Then, the ratio on the right is the modulus of the domain B. This completes the proof of the theorem.

In conclusion, we note that, for the first part of the existence theorem given above, we can give a proof based (like the proof of Riemann's theorem) on the solution of an extremal problem: Let B denote a doubly connected domain that is contained in the domain $|z| > 1$ and that has as one of its boundary continua the circle $|z| = 1$. Out of the family $\mathfrak{M}$ of all functions $\zeta = f(z)$ that map the domain B univalently onto domains of the same type in such a way that the circle $|z| = 1$ is mapped onto itself, find the one for which the quantity

$$M(f) = \sup_{z \in B} |f(z)|$$

has the greatest value. The function representing the solution of this extremal problem maps the domain B univalently onto a circular annulus.

§2. Univalent mapping of a multiply connected domain onto a plane with parallel rectilinear cuts

Let us now investigate univalent conformal mapping of arbitrary multiply connected domains onto canonical domains of various kinds. The simplest such domain is a plane with parallel rectilinear cuts. In this case, our investigation will be based on the solution of certain extremal problems.

Lemma 1. *Out of all functions* $F(z) = z + \alpha_1/z + \cdots$ *that are univalent in* $|z| > R$, *the quantity* $\Re(e^{-2i\theta}\alpha_1)$, *where* θ *is a given real number, is maximized by a function mapping the domain* $|z|>R$ *onto the plane with rectilinear cut that makes an angle* θ *with the real axis, and only by that function. For it,*

$\Re(e^{-2i\theta}\alpha_1) = R^2$.

Proof. It follows from the area theorem of §4 of Chapter II as applied to the functions

$$\frac{1}{R}F(R\zeta) = \zeta + \frac{\alpha_1}{R^2\zeta} + \cdots,$$

that $|\alpha_1| \leq R^2$ with equality holding only when $F(z) = z + R^2e^{i\alpha}/z$. Therefore,

$$\Re(e^{-2i\theta}\alpha_1) \leqslant R^2,$$

with equality holding, obviously, only for the function $F(z) = z + R^2e^{2i\theta}/z$, which maps the domain $|z| > R$ onto the plane with cut at an angle θ to the real axis.

Lemma 2. *If the function* $\zeta = F(z) = z + \alpha_0 + \alpha_1/z + \cdots$ *is univalent in* $|z| > R$, *then* $|F(z) - \alpha_0| \leq 2|z|$ *in the domain* $|z| > R$ *and the entire boundary of the image of the domain* $|z| > R$ *under the mapping* $\zeta = F(z)$ *is contained in the disk* $|\zeta - \alpha_0| \leq 2R$.

Proof. If $|z_0| > R$, then the function

$$\zeta = F_1(z) = \frac{1}{z_0}F(z_0z) - \frac{\alpha_0}{z_0} = z + \frac{\alpha_1}{z_0^2z} + \cdots$$

is univalent in $|z| > 1$. Consequently, in accordance with §4 of Chapter II, the entire boundary of the image of the domain $|z| > 1$ under this function is contained in the disk $|\zeta| \leq 2$. In particular, $|F_1(1)| \leq 2$; that is, $|f(z) - \alpha_0| \leq 2|z|$, where $|z| > R$. Then, if we shift from $|z| > R$ to $|z| = R$, we obtain the second part of the lemma.

Theorem 1.[1] *Every domain* B *in the* z*-plane can be mapped univalently onto a domain* B' *in the extended* ζ*-plane such that an arbitrary continuum in the complement of the domain* B' *with respect to the plane is a straight line segment of given inclination* θ *to the real axis. Also, this mapping is such that a given point* a *of the domain* B *is mapped into* $\zeta = \infty$ *and the expansion of the mapping function about* $z = a$ *is of the form*

$$\frac{1}{z-a} + \alpha_1(z-a) + \cdots \quad \textit{or} \quad z + \frac{\alpha_1}{z} + \cdots,$$

according as a *is finite or infinite.*

Proof. It will be sufficient to prove the theorem for the case in which $a = \infty$, because, in case of finite a, we can map the domain B by means of the preliminary transformation $z^* = 1/(z-a)$ onto the domain B^* containing $z^* = \infty$,

1) Beginning with the extremal property of the mapping function, which was indicated in the proof given below, this theorem was proved by de Possel [1931] and Grötzsch [1932].

which reduces the problem to the original case.

In the case of a simply connected domain B, the theorem is easily proved. Specifically, if the boundary of the domain B has more than a single boundary point, then it follows from Riemann's theorem that it can be mapped univalently onto an arbitrary domain constituting the entire ζ'-plane with cut at inclination θ to the real axis in such a way that $z=\infty$ is mapped into $z'=\infty$ and the expansion of the function $\zeta'=\zeta'(z)$ about $z=\infty$ is of the form $\alpha_{-1}z+\alpha_0+\alpha_1/z+\cdots$, where $\alpha_{-1}>0$. Then, the function $\zeta=(\zeta'(z)-\alpha_0)/\alpha_{-1}$ provides the required mapping of the domain B.

Let us turn to the general case. Consider the family $\mathfrak{M}$ of all functions $f(z)$ that map the domain B univalently and have the expansion $f(z)=z+\alpha_1/z+\cdots$ in a neighborhood of $z=\infty$. An example of such a function is $f(z)\equiv z$. Let us pose the extremal problem: Out of all the functions of the family $\mathfrak{M}$, find the one that maximizes the quantity $\Re(e^{-2i\theta}\alpha_1)$.

Let us first prove that this problem has a solution. Suppose that the entire boundary of the domain B is contained in the disk $|z|<R$. Then all functions of the family $\mathfrak{M}$ are univalent in the domain $|z|>R$. Consequently, in accordance with Lemma 1, for these functions we have $\Re(e^{-2i\theta}\alpha_1)\le R^2$; that is, the numbers $\Re(e^{-2i\theta}\alpha_1)$ are uniformly bounded. Let A denote the least upper bound of these numbers. If this upper bound is not attained, there exists a sequence of functions $f_n(z)\in\mathfrak{M}$ for which the sequence $\{\Re(e^{-2i\theta}\alpha_1)\}$ converges to A. But, from Lemma 2, for functions of the family $\mathfrak{M}$ we have $|f(z)/z|\le 2$ in $|z|>R$. This shows that the condensation principle can be applied to the functions $f_n(z)/z$, that is, that the sequence $\{f_n(z)/z\}$ contains a subsequence $\{f_{n_k}(z)/z\}$ that converges uniformly in the interior of the domain $|z|>R$ to a regular function $f_0(z)/z=1+\alpha_1^{(0)}/z^2+\cdots$ such that $\Re(e^{-2i\theta}\alpha_1^{(0)})=A$. But in accordance with Lemma 2, we have $|f_{n_k}(z)|\le 2R'$ for functions $f_{n_k}(z)$ on $|z|=R'$, where $R'>R$. By virtue of the univalence of $f_{n_k}(z)$ in B, the same inequality holds also at points of the domain B that belong to the disk $|z|<R'$. Consequently, the functions $f_{n_k}(z)$ are uniformly bounded in the interior of the domain B_0 obtained from B by removing the point $z=\infty$. In accordance with Vitali's theorem, the sequence of these functions converges inside B_0 to a regular function $f_0(z)$, which, in accordance with Theorem 2 of §1 of Chapter I, is univalent in B_0 and hence in B. Therefore, $f_0(z)\in\mathfrak{M}$, which contradicts our assumption that the least upper bound A is not attained by the numbers $\Re(e^{-2i\theta}\alpha_1)$. Thus, the existence of a solution of the extremal problem posed is proved.

Let us now show that the extremal function $f_0(z)$ provides the mapping referred to in the theorem. Let us suppose that the complement of the domain B with respect to its image B' under the function $\zeta=f_0(z)$ contains a continuum other than a straight line segment of inclination θ with the real axis. Let B_1 denote whichever of the simply connected domains complementary to this continuum in the ζ-plane contains the point $\zeta=\infty$. Suppose that the function $w=w(\zeta)$ maps B_1 univalently onto the w-plane with rectilinear cut of inclination θ with the real axis and that it has the expansion $\zeta+\beta_1/\zeta+\cdots$ in a neighborhood of $\zeta=\infty$. Let us show that $\Re(e^{-2i\theta}\beta_1)>0$.

In accordance with Riemann's theorem, the domain B_1 can be mapped univalently onto the domain $|t|>R'$ in such a way that the function $\zeta=\zeta(t)$ inverse to the mapping function has the form $\zeta(t)=t+\gamma_0+\gamma_1/t+\cdots$. The function $w=w(\zeta(t))-\gamma_0=t+(\beta_1+\gamma_1)/t+\cdots$ maps the domain $|t|>R'$ univalently onto the w-plane with rectilinear cut of inclination θ with the real axis, and the function $\zeta=\zeta(t)-\gamma_0=t+\gamma_1/t+\cdots$ maps $|t|>R'$ onto a different domain.

Therefore, on the basis of Lemma 1 we have

$$\Re(e^{-2i\theta}\gamma_1)<\Re(e^{-2i\theta}(\beta_1+\gamma_1)),$$

that is, $\Re(e^{-2i\theta}\beta_1)>0$.

Now, let us look at the function $w=w(f_0(z))$. This function maps the domain B univalently and has the expansion $z+(\alpha_1^{(0)}+\beta_1)/z+\cdots$ in a neighborhood of $z=\infty$.

Consequently, $w(f_0(z))\in\mathfrak{M}$ and we have

$$\Re(e^{-2i\theta}(\alpha_1^{(0)}+\beta_1))=\Re(e^{-2i\theta}\alpha_1^{(0)})+\Re(e^{-2i\theta}\beta_1)>A,$$

which, however, is impossible. This shows that the function $f_0(z)$ meets all of the requirements of the theorem. At the same time, we have established an important extremal property of the mapping function, namely, the fact that it maximizes the quantity $\Re(e^{-2i\theta}\alpha_1)$.

In the case of finitely connected domains, this theorem can be formulated as follows:

Theorem 1′ (Hilbert). *Every n-connected domain B in the z-plane can be mapped univalently onto the ζ-plane with n parallel finite cuts of inclination θ with the real axis in such a way that a given point $z=a$ is then mapped into*

$\zeta=\infty$, and the expansion of the mapping function about $z=a$ has the form

$$\frac{1}{z-a}+\alpha_1(z-a)+\ldots \quad \text{or} \quad z+\frac{\alpha_1}{z}+\ldots,$$

according as a is finite or not. Some of the cuts referred to may consist of single points.

Let us now look at the question of the correspondence of boundaries under univalent conformal mapping of multiply connected domains. We shall confine ourselves here to finitely connected domains that have only accessible boundary points. We may assume that the point ∞ is in the interior of the domain. Let B denote such a domain and let $K_1, K_2, \cdots, K_n$ denote its boundary continua. Let us map the domain B univalently onto a domain bounded by closed analytic Jordan curves. This can be done as follows: we map the simply connected domain including $z=\infty$ and bounded by the continuum K_1 onto the interior of a circle. Under such a mapping, the domain B is mapped into a domain B' bounded by a circle K_1' and the continua $K_2', \cdots, K_n'$. Then, we map the simply connected domain containing the domain B' and bounded by the continuum K_2' onto the interior of a circle. Under this mapping, the domain B' is mapped into a domain B'' bounded by a circle K_2'', an analytic curve K_1'' and the continua $K_3'', \cdots, K_n''$. Continuing in this way, after n steps we arrive at an n-connected domain $B^{(n)}$ whose boundary consists of n closed analytic Jordan curves. The univalent mapping of the domain B onto the domain $B^{(n)}$ is a composite of successive mappings of univalent domains. Since these mappings define a one-to-one correspondence between the boundaries, the correspondence between the points of these boundaries defined by the mapping of B onto $B^{(n)}$ is also one-to-one. Furthermore, as in §3 of Chapter II, we can show that the function inverse to the mapping function is continuous in $\overline{B^{(n)}}$ except at the point ∞. If the domain B is bounded only by closed Jordan curves, we can show in addition that the mapping of $\overline{B}$ onto $\overline{B^{(n)}}$ is also bicontinuous. If we now map the domain $B^{(n)}$ onto the plane with rectilinear cuts, then, just as in the proof of Theorem 5 of §3 of Chapter II, we can show that the mapping function is regular on the entire boundary of the domain $B^{(n)}$. Combining all that we have said, we conclude that, in the present case, the mapping mentioned in Theorem 1′ is one-to-one except for boundary points of the domain B.

An important supplement to the existence theorem proved above is the following uniqueness theorem:

Theorem 2. *There exists only one function performing the mapping mentioned in Theorem 1′.*

Proof. Let us suppose first that B is a bounded domain with boundary consisting of closed analytic Jordan curves $K_1, \cdots, K_n$. Let us suppose that there are two functions $\zeta'=f_1(z)$ and $\zeta''=f_2(z)$ that map the domain B univalently onto the plane with parallel rectilinear cuts of inclination θ and normalized as indicated in Theorem 1′. Then both these functions are regular in $\overline{B}$ except at the point $z=a$. On K_ν, $\nu=1,\cdots,n$, they assume values that lie respectively on certain straight lines $\Im(e^{-i\theta}\zeta')=c_\nu'$ and $\Im(e^{-i\theta}\zeta'')=c_\nu''$, for $\nu=1,\cdots,n$; that is, on K_ν we have $\Im(e^{-i\theta}f_1(z))=c_\nu'$ and $\Im(e^{-i\theta}f_2(z))=c_\nu''$, where c_ν' and c_ν'', $\nu=1,\cdots,n$, are constants. It follows from this that the difference $\zeta=F(z)=f_1(z)-f_2(z)$ is regular in B and assumes on K_ν (for $\nu=1,\cdots,n$) values lying respectively on certain straight lines d_ν. Consequently, if we take an arbitrary point ζ_0 that does not lie on any of the straight lines d_ν, we conclude that $\arg[F(z)-\zeta_0]$ does not change as we move around any of the curves K_ν. Therefore, on the basis of Cauchy's familiar theorem on the zeros of analytic functions, it follows that the function $F(z)-\zeta_0$ has no zeros in B, that is, that $F(z)$ does not assume the value ζ_0 in B. Thus, all values that $F(z)$ assumes in B lie on the straight lines d_ν. However, this is possible only when $F(z)\equiv\text{const}$ in B. Since $F(a)=0$, it follows that $F(z)\equiv 0$, that is, $f_1(z)=f_2(z)$, so that the theorem is proved in this case.

Now, let B denote an arbitrary finitely connected domain. Here, we may assume that it has no isolated boundary points. If there were two distinct functions performing the mapping described in Theorem 1′, we could, by mapping the domain B univalently onto the bounded domain B_0 with boundary consisting of closed analytic Jordan curves, find two distinct functions mapping the domain B_0 univalently as indicated in Theorem 1′, which, on the basis of what was said above, is impossible. This completes the proof of the theorem.

In conclusion, we present an important relationship for the functions mentioned in Theorem 1′ for various values of θ.

Let B denote an n-connected domain bounded by closed analytic Jordan curves $K_1, \cdots, K_n$ and let $\zeta=j_\theta(z,a)$ denote a function that maps the domain B univalently as indicated in Theorem 1′. Let us show that the following relationship holds for arbitrary θ:

$$j_\theta(z, a) = e^{i\theta}(\cos\theta\, j_0(z, a) - i\sin\theta\, j_{\frac{\pi}{2}}(z, a)). \qquad (1)$$

The difference $d(z)$ between the two sides of this asserted equation is a function that is regular in the domain B and that vanishes at $z = a$. Furthermore, all the values that $d(z)$ assumes on any of the curves K_ν lie on a straight line $\Im(e^{-i\theta}\zeta) = \text{const}$.

Reasoning as in the proof of Theorem 2, let us show that $d(z) \equiv 0$; that is, let us prove (1). This relation, which has been proved for a domain bounded by closed analytic Jordan curves is also valid for an arbitrary finitely connected domain B. For this, it is sufficient to map the domain B onto a domain B^* bounded by closed analytic Jordan curves, apply (1) to B^*, and then return to the domain B. This yields the relation (1) for B.

Equation (1) yields $j_\theta(z, a)$ for arbitrary θ as soon as we know $j_0(z, a)$ and $j_{\pi/2}(z, a)$.

§3. Univalent mapping of a multiply connected domain onto a helical domain

In an analogous manner, we shall find the answer to the question of univalent mapping of multiply connected domains onto a plane with cuts along arcs of logarithmic spirals and, as limiting cases, onto the plane with radial cuts and with cuts along circular arcs of concentric circles.

For constant θ and c, the equation $\Im(e^{-i\theta}\log\zeta) = c$ defines a logarithmic spiral in the ζ-plane with asymptotic point at the origin. This spiral has the property that it is intersected by an arbitrary ray issuing from the origin at an angle θ. This last follows, for example, from the fact that, if we shift to the plane $t = \log\zeta$, this logarithmic spiral is mapped into the straight line $\Im(e^{-i\theta}t) = c$ with inclination θ to the real axis, and the ray referred to is mapped into a straight line parallel to the real axis. For $\theta = 0$, the logarithmic spiral degenerates into a ray issuing from the origin. For $\theta = \pi/2$, it degenerates to a circle with center at the origin. If we hold θ constant and vary c, we obtain various curves constituting the family of logarithmic spirals of inclination θ. In all that follows, when we speak of logarithmic spirals of inclination θ, these are what we mean.

Let us show that, for any simply connected domain B that has boundary points, it is possible to map B onto the ζ-plane with cut along an arc of a logarithmic spiral of inclination θ in such a way that given points a and b of the domain B are mapped into 0 and ∞ and the expansion of the mapping function about $z = b$ has the form

$$\frac{1}{z-b} + \alpha_0 + \alpha_1(z-b) + \dots \quad \text{or} \quad z + \alpha_0 + \frac{\alpha_1}{z} + \dots, \qquad (1)$$

according as b is finite or infinite. In the case in which the domain B has a single boundary point, this is obvious, and then the arc of the logarithmic spiral referred to degenerates to a point. On the other hand, if the boundary of the domain B is a continuum, let us first map B conformally onto the domain $|z'| > 1$ in such a way that the point $z = b$ is mapped into $z' = \infty$. Then, the mapping function $z' = \phi(z)$ has in a neighborhood of $z = b$, the expansion

$$\frac{\alpha_{-1}}{z-b} + \alpha_0 + \alpha_1(z-b) + \dots \quad \text{or} \quad \alpha_{-1}z + \alpha_0 + \frac{\alpha_1}{z} + \dots,$$

according as b is finite or infinite. This is possible by virtue of Riemann's theorem. Suppose that the point $z = a$ is mapped into a point $z' = a'$. It now remains to establish the possibility of mapping the domain $|z'| > 1$ onto the ζ-plane with cut along a logarithmic spiral of inclination θ. But this possibility follows from the solution of the problem on the minimum of the quantity $\Re(e^{-2i\theta}\log F'(a'))$ in the class Σ. This problem was studied in §3 of Chapter IV (the first application of Theorem 1 with $\alpha = \pi/2 - \theta$). It was shown there that this problem has a solution and that the extremal function $\zeta = F(z')$ normalized by the condition $F(a') = 0$ provides the required mapping.

As a result, we see that the function $\zeta = cF(\phi(z)) = f(z)$ with $f(0) = 0$ and $f'(\infty) = \infty$ with suitably chosen c provides a mapping of the domain B onto the plane with cut along a logarithmic spiral and has an expansion of the form (1).

This result for simply connected domains can be generalized to multiply connected domains. For this generalization, we need the analogue of the minimal property proved in §3 of Chapter IV, which we formulate as a

Lemma. *Of all functions $\zeta = F(z) = z + \alpha_0 + \alpha_1/z + \cdots$ that map the domain $|z| > R$ univalently in such a way that a given finite point a and the point ∞ are mapped into the points 0 and ∞ respectively, the quantity $\Re(e^{-2i\theta}\log F'(a))$ is minimized by the function $F_0(z)$ that maps the disk $|z| > R$ onto the ζ-plane with cut along an arc of a logarithmic spiral of inclination θ. Here $\log F'(z)$ means the branch that approaches 0 as $z \to \infty$.*

We now have

Theorem 1.[1)] *Every domain B in the z-plane can be mapped univalently onto*

1) Grötzsch [1931].

a domain B' in the ζ-plane that includes the points 0 and ∞ such that an arbitrary continuum of the complement of the domain B' with respect to the ζ-plane is an arc of a logarithmic spiral of given inclination θ. This mapping maps given points a and b of the domain B into 0 and ∞ and the expansion of the mapping function about $z = b$ has the form $(z-b)^{-1} + \alpha_0 + \alpha_1(z-b) + \cdots$ or $z + \alpha_0 + \alpha_1 z^{-1} + \cdots$ according as b is finite or infinite.

Proof. We may assume that $b = \infty$ because, if this is not the case, we can use the transformation $z^* = 1/(z-b)$ to switch from B to a new domain B_* and we can then prove the theorem for this last domain, replacing a and b with $1/(a, b)$ and ∞.

In the case of simply connected domains, the theorem was proved at the beginning of this section. To prove it in the case of a multiply connected domain B, let us consider the family $\mathfrak{M}$ of all functions $f(z)$ that map B univalently in such a way that a and ∞ are mapped respectively into 0 and ∞ and that have an expansion $f(z) = z + \alpha_0 + \alpha_1/z + \cdots$ in a neighborhood of $z = \infty$. An example of such a function is $f(z) = z - a$. We pose the extremal problem: Out of all functions of the family $\mathfrak{M}$, find the one that minimizes the quantity $\Re(e^{-2i\theta}\log f'(a))$, where $\log f'(z)$ denotes the single-valued branch in the domain B that approaches 0 as $z \longrightarrow \infty$.

Let us show that this problem has a solution. To do this let us choose a simply connected domain B' contained in B that includes a and ∞. In this domain, all functions $f(z) \in \mathfrak{M}$ are univalent. Consequently, for these functions, the quantity $\Re(e^{-2i\theta}\log f'(a))$ is bounded below by the same number as was calculated for a function that maps the domain B' univalently onto the ζ-plane with cut along an arc of a logarithmic spiral of inclination θ in such a way that a and ∞ are mapped into 0 and ∞. Let us denote the greatest lower bound of the quantity $\Re(e^{-2i\theta}\log f'(a))$ for functions in $\mathfrak{M}$ by A.

Let us show that this greatest lower bound is attained. We assume the opposite. Then, there exists a sequence of functions $f_n(z) \in \mathfrak{M}$, for $n = 1, 2, \cdots$, such that, as $n \longrightarrow \infty$,

$$\Re(e^{-2i\theta}\log f_n'(a)) \to A.$$

Suppose that the entire boundary of the domain B lies in the disk $|z| < R$. According to Lemma 2 of §2, it follows that the entire boundary of the image of the domain $|z| > R$ under the function $w = f_n(z) = z + \alpha_0^{(n)} + \alpha_1^{(n)}/z + \cdots$ (where $n = 1, 2, \cdots$) lies in the disk $|\zeta - \alpha_0^{(n)}| \le 2R$ and that $|f_n(z) - \alpha_0^{(n)}| \le 2|z|$ in the domain $|z| > R$. If $|a| > R$, this last inequality with $z = a$ implies that $|a_0^{(n)}| \le 2|a|$. On the other hand, if $|a| < R$, then the point $\zeta_0 = f_n(a) = 0$ does not belong to the image of the domain $|z| > R$ under the mapping $\zeta = f_n(z)$; hence, it belongs to the disk $|\zeta - \alpha_0^{(n)}| \le 2R$. Therefore, $|\alpha_0^{(n)}| \le 2R$. Thus, in all cases, $|\alpha_0^{(n)})| \le 2\max(|a|, R) = M$. But then, $|f_n(x)/z| \le 2 + M/R$ in $|z| > R$. Consequently, the condensation principle can be applied the sequence $\{f_n(z)/z\}$, so that it contains a subsequence $\{f_{n_k}(z)/z\}$ that converges uniformly in $|z| > R$ to a regular function. The corresponding sequence $\{f_{n_k}(z)\}$ converges uniformly in an arbitrary closed bounded subset of the domain $|z| > R$ to a function $f_0(z)$ that is univalent in $|z| > R$. But since the values of $\zeta = f_{n_k}(z)$ corresponding to points of the domain B that lie in $|z| < 2R$ belong to the disk $|\zeta - \alpha_0^{(n)}| \le 4R$ and hence to the disk $|\zeta| \le 4R + M$, the set of functions $f_{n_k}(z)$ is uniformly bounded inside the domain B with the point $z = \infty$ excluded. Therefore the uniform convergence, mentioned above, to $f_0(z)$ holds also in the interior of the domain B, and $f_0(z) \in M$. The function $f_0(z)$ thus proves to be a solution of the extremal problem posed, so that this problem does have a solution.

Let us show that the extremal function $f_0(z)$ provides the mapping indicated in the theorem. Let us suppose that the complement of the image of the domain B under the mapping $\zeta = f_0(z)$ contains a continuum other than an arc of a logarithmic spiral of inclination θ. Let B_1 denote whichever of the simply connected domains in the ζ-plane that are complementary to this continuum contains 0 and ∞. Suppose that the function $w = \phi(\zeta)$ maps B_1 univalently onto the w-plane with cut along an arc of a logarithmic spiral of inclination θ in such a way that $\phi(0) = 0$ and $\phi(\infty) = \infty$ and suppose that $\phi(\zeta)$ has an expansion $\zeta + \beta_0 + \beta_1/\zeta + \cdots$ in a neighborhood of $\zeta = \infty$. Let us show that

$$\Re(e^{-2i\theta}\log\varphi'(0)) < 0.$$

In accordance with Riemann's theorem, the domain B_1 can be mapped univalently onto a domain $|t| > R$ in such a way that the inverse function has an expansion of the form $\zeta = \zeta(t) = t + \gamma_0 + \gamma_1/t + \cdots$. Suppose that the point $t = a'$ corresponds to the point $\zeta = f_0(a)$. The function $w = \phi(\zeta(t))$ maps the domain $|t| > R$ univalently onto the w-plane with cut along an arc of a logarithmic spiral of inclination θ in such a way that the point $t = a'$ is mapped into $w = 0$. From the preceding lemma, we have

$$\Re(e^{-2i\theta}(\log\varphi'(0) + \log\zeta'(a'))) < \Re(e^{-2i\theta}\log\zeta'(a')).$$