Intrinsic wildness of homeomorphisms. (M. Freedman.) Let $S^1 = \mathbb{R}/\mathbb{Z}$ and consider the annulus $A = [0,1] \times S^1$.

Theorem. Given a modulus of continuity $m(s) \to 0$ as $s \to 0$, there exists a homeomorphism $F : A \to A$ such that for every homeomorphism $h$ of $A$, the modulus of continuity of $hFh^{-1}$ is worse than $m(s)$.

Let us start out by constructing a wild map $f : [0,1] \to \mathbb{R}$. First choose a sequence of integers $a_n \to \infty$ rapidly. Then set

$$f(x) = \sum_{n=0}^{\infty} 2^{-n} \sin(2\pi a_n x).$$

Let

$$\text{var}(f, \epsilon) = \sup \{|f(x) - f(y)| : |x - y| < \epsilon\}.$$

Note that along $[0,1]$, $f(x)$ oscillates about $a_n$ times with laps of size $2^{-n}$ or larger. The same is true for $f \circ h$, for any homeomorphism $h : [0,1] \to [0,1]$. Thus there are two points separated by at most $1/a_n$ that form the endpoints of one lap. Therefore

$$\text{var}(f \circ h, 1/a_n) \geq 2^{-n}.$$

On the other hand, if $f \circ h$ has modulus of continuity $m(s)$, then

$$\text{var}(f \circ h, 1/a_n) \leq m(1/a_n).$$

So by choosing $a_n$ large enough one can defeat any given modulus of continuity $m(s)$. (By ‘defeat’ we mean there exist $r_n \to 0$ such that

$$\text{var}(f \circ h, r_n) > m(r_n)$$

for all $n$.)

Now define $F : A \to A$ by

$$F(x,t) = (x, t + f(x)).$$

Note that under iteration, we have

$$F^q(x,t) = (x, t + qf(x)).$$

Let $q = 2^n$. Then whenever $f(x)$ varies by size larger than $2^{-n}$ over an interval $J$, there is a subinterval $J'$ such that $F$ fixes the boundary of $J' \times S^1$ and performs a power of a Dehn twist on the interior.

Since $f(x)$ has at least $a_n$ laps of size $2^{-n}$, the map $F^q$ fixes an ordered sequence of $a_n$ circles $C_i = \{x_i\} \times S^1$, $x_1 < x_2 < \ldots$, and performs a nonzero
power of a Dehn twist on the annulus between $C_i$ and $C_{i+1}$. Two of these circles must be within distance $1/a_n$ of each other, by area considerations. Let $L$ be the shortest arc joining them; we have $|L| < 1/a_n$. Since $F^q(L)$ must wrap at least once around $A$, we find that:

$$\text{var}(F^q, 1/a_n) \geq 1/3.$$ 

The same reasoning applies to any conjugate of $F$ by a homeomorphism.

On the other hand, if $F$ has modulus of continuity $m(s)$, then

$$\text{var}(F^q, 1/a_n) \leq m^q(1/a_n),$$

where $m^q$ means we iterate $s \mapsto m(s)$ $q$ times. So again, by choosing $a_n$ large enough, we can defeat any modulus of continuity $m(s)$, and the argument applies not just to $F$ but uniformly to all conjugates $hFh^{-1}$. 

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