HILBERT SCHEMES: GEOMETRY, COMBINATORICS, AND REPRESENTATION THEORY

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Abstract. This survey will give an introduction to the theory of Hilbert schemes of points on a surface. These are compactifications of configuration spaces of n distinct particles moving around on a surface. After giving the necessary background and construction, we will explain some of the amazing recent connections found between these spaces, combinatorics and representation theory.

1. Introduction

This survey is an overview to the theory of Hilbert schemes of points on a smooth surface. Hilbert schemes in general were constructed by Grothendieck as solutions to a certain moduli problem. This construction paved the way for the construction of most moduli spaces since then.

The theory of Hilbert schemes of points on a smooth surface exploded once Fogarty proved that $\mathrm{Hilb}^n(X)$ is smooth and irreducible for X a smooth irreducible surface. Then the Hilbert-Chow morphism gives a canonical resolution of the symmetric product and many techniques reserved for smooth varieties and complex manifolds become available.

This note will go over the construction and some interesting properties of $\mathrm{Hilb}^n(X)$. We will mainly restrict to $X = \mathbb{A}^2$ until the last section. There are many more things I wanted to mention but couldn't due to time constraints. These include the connections to representations of finite groups and resolutions of singularity through the *McKay correspondance*, and Haiman's proofs of the $n!$ and $(n+1)^{(n-1)}$ conjectures which were huge milestones in the theory of symmetric functions whose proof relied heavily on the geometry of $\mathrm{Hilb}^n(\mathbb{A}^2)$.

I have attempted to keep the discussion accessible to first year grad students and have tried to include enough of the relevant background. However, I've oversimplified some things and I've also left out references which I will add in the future. As a result there may be some errors so use at your own risk but please let me know if you find any.

1.1. Background.

1.1.1. *Algebraic Geometry.* For this talk we will be working over the complex numbers $\mathbb{C}$. $\mathbb{A}^n$ will denote the complex vector space $\mathbb{C}^n$ viewed as a topological space endowed with the ring of complex polynomial functions from $\mathbb{A}^n \to \mathbb{A}^1$. This is just the ring $\mathbb{C}[x_1, \ldots, x_n]$. We call this space *affine space of dimension* n. An *affine scheme* $V \subset \mathbb{A}^n$ will be the simultaneous zeroes of some ideal of polynomial functions $I \subset \mathbb{C}[x_1, \ldots, x_n]$ endowed with the ring of polynomial functions

$$A(V) := \mathbb{C}[x_1, \ldots, x_n]/I$$

called the *coordinate ring*. The *support* of V is the subset of V without a choice of coordinate ring. We should think of the coordinate ring as the ring of complex valued algebraic functions on the scheme V.

Example 1.1. *(a) $\mathbb{A}^n$ itself is an affine scheme with coordinate ring $\mathbb{C}[x_1, \ldots, x_n]$.*

(b) The quadratic $uw - v^2$ cuts out a quadratic cone $X \subset \mathbb{A}^3$ with coordinate ring $\mathbb{C}[u, v, w]/(uw - v^2)$.

(c) *The nonzero complex numbers* $\mathbb{C}^* \subset \mathbb{A}^1$ *can be given the structure of an affine scheme by identifying* $x \in \mathbb{C}^*$ *with the zero set of* $xy - 1$ *in* $\mathbb{A}^2$*. This gives* $\mathbb{C}^*$ *the coordinate ring* $\mathbb{C}[x, y]/(xy - 1) = \mathbb{C}[x^{\pm 1}]$.

We call the third example $\mathbb{C}^*$ the 1-dimensional *algebraic torus*. More generally, an n-dimensional algebraic torus is the variety $(\mathbb{C}^*)^n$ with coordinate ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Definition 1.2. *A zero-dimensional affine scheme is an affine scheme* Z *such that* $\dim_{\mathbb{C}} A(Z) < \infty$. $\dim_{\mathbb{C}} A(Z)$ *is called the length of* Z.

By abuse of notation, we will sometimes call the length of Z the length of I where I is the defining ideal of Z and denote it $\mathrm{len}(I)$.

Example 1.3. *The defining ideal* I *and the the coordinate ring are crucial to the definition of an affine scheme. For example the rings* $\mathbb{C}[x]/(x)$ *and* $\mathbb{C}[x]/(x^2)$ *both correspond to the point* $0 \in \mathbb{C}$ *as a set. However, the ring of functions on them are very different and we can think of the latter as an infinitesimal first order neighborhood of the origin while the former is just the origin itself.*

A *complex variety* (or more generally *scheme*) structure on a space X is an open cover $\{V_\alpha\}$ where the V_α are affine schemes endowed with their coordinate rings $A(V_\alpha)$. A *morphism* of varieties is a function $f : X \to Y$ so that there are open covers by affines U_α of X and V_α of Y such that f restricts to $f : U_\alpha \to V_\alpha$ and f induces a ring homomorphism $f^\# : A(V_\alpha) \to A(U_\alpha)$.

Thus general varieties or schemes are just topological spaces that you can glue together from affine varieties, and the morphisms are just those that come from ring homomorphisms of affines. This is analogous to a manifold being a space that locally looks like euclidian space and smooth functions between manifolds being functions that locally look like smooth functions between euclidian spaces.

Definition 1.4. *A vector bundle of rank* n *is a morphism* $p : E \to X$ *of varieties so that there exists an open affine cover* $\{U_\alpha\}$ *of* X *such that* $p : p^{-1}(U_\alpha) \to U_\alpha$ *is just the projection* $U_\alpha \times \mathbb{A}^n \to U_\alpha$.

So a vector bundle of rank n is a space that locally looks like a product with an affine space. The preimage $p^{-1}(x) =: E_x$ is a vector space called the *fiber at* p and we can think of a vector bundle as an assignment of vector spaces at each point of X that is glued together in a compatible algebraically varying way.

We can define any operation that we have on vector spaces for vector bundles by doing that operation fiberwise and gluing together. For example, direct sums, tensor products, and duals of vector bundles can be defined as well exact sequences of vector bundles.

1.1.2. *Cohomology.* We will go back and forth between viewing our varieties as schemes and as complex manifolds. For a complex manifold X of dimension n, we will use *compactly supported cohomology* with coefficients in $\mathbb{Q}$ which we will denote by $H^k(X)$. These are $\mathbb{Q}$ vector spaces for $k \geq 0$ such that $H^k(X) = 0$ if $k > 2n$ and $H^k(X) = H_{2n-k}(X)$ where $H_l(X)$ is singular homology. For a compact manifold, these agree with the usual singular cohomology, and in view of this duality, one can think of the cohomology as the same as the homology except where we index by codimension instead of dimension.

We can think of cohomology classes as representing actual subvarieties or submanifolds of X so that $H^k(X)$ consists of classes corresponding to the codimension k submanifolds of X. Then

$$H^*(X) = \bigoplus_{i=0}^{2 \dim X} H^i(X)$$

inherits a graded ring structure that corresponds to intersecting submanifolds. This is the advantage of using cohomology instead of homology.

The ranks of the vector spaces will be denoted

$$b_i(X) := \dim_{\mathbb{Q}} H^i(X)$$

and are the *betti numbers*. The *Euler characteristic* is the alternating sum

$$\chi(X) = \sum_{i=0}^{2\dim X} (-1)^i b_i(X).$$

There is also an analagous ring known as the *Grothendieck ring* or *K theory* of vector bundles on X and denoted $K(X)$. This is an analogue of cohomology where instead of clases representing subvarieties of X, classes now represent vector bundles on X. Addition comes from direct sum of vector bundles and multiplication from the tensor product.

2. The Hilbert Scheme

We will focus on the Hilbert scheme of points on the surface $X = \mathbb{A}^2$ with coordinate ring $\mathbb{C}[x, y]$, though most things we will talk about apply to more general surfaces.

Definition 2.1. *The symmetric product*

$$\mathrm{Sym}^n(X) := X^n/S_n$$

where the symmetric group acts by permuting the coordinates.

This is a compactification of the configuration space of n distinct points on X. However, $\mathrm{Sym}^n(X)$ is not a complex manifold. It has singularities along the diagonal where some of the coordinates coincide. We want to find a compactification of this space that is a smooth complex manifold.

The basic idea is simple. If we have n distinct points in X, that is, a point in the smooth locus of $Z \in \mathrm{Sym}^n(X)$, then we can find a unique ideal I consisting of all functions that vanish on Z. Then the corresponding coordinate ring $A(Z) = \mathbb{C}[x, y]/I$ is an n-dimensional vector space over $\mathbb{C}$. Z is a length n subscheme of the plane. Thus, we may attempt to compactify the smooth locus by adding all ideals of length n.

Theorem-Definition 2.2. *(Grothendieck, Fogarty, Haiman, . . .) There exists an irreducible smooth variety* $\mathrm{Hilb}^n(X)$ *which is a parameter space for zero-dimensional subschemes of* X *compactifying the smooth locus of the symmetric product. As a set, this variety is given by*

$$\mathrm{Hilb}^n(X) = \{I \subset \mathbb{C}[x, y] | \dim_{\mathbb{C}} \mathbb{C}[x, y]/I = n\}.$$

This variety comes equipped with a Hilbert-Chow morphism

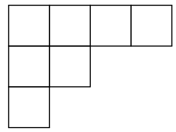
$$\mathrm{Hilb}^n(X) \to \mathrm{Sym}^n(X)$$

giving it as a resolution of singularities of $\mathrm{Sym}^n(X)$.

Remark 2.3. *The existence of Hilbert schemes was originally proved in a much more general context by Grothendieck. Fogarty proved that for any irreducible smooth surface* X, $\mathrm{Hilb}^n(X)$ *is a smooth irreducible variety so that the Hilbert-Chow morphism is a resolution of singularities. The construction we give of the Hilbert scheme for* $\mathbb{A}^2$ *is by Haiman.*

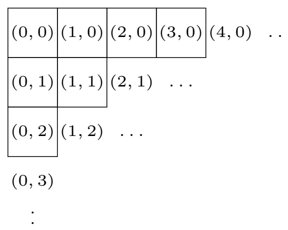
Before we give the proof, we need some combinatorics background.

2.1. **Partitions and Young Diagrams.** A *partition* of n is a nonincreasing list of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ so that $\sum \lambda_i = n$ and $\lambda_k \neq 0$. k is called the *number of parts* of the partition. We can represent a partition pictorally by a diagram of boxes, so that the i^{th} row constists of λ_i boxes and the rows are left aligned. For example, the diagram



corresponds to the partition $4+2+1$ of 7.

Any such box diagram corresponds to a partition of $n = \#$ of boxes and with $k = \#$ of rows parts. This picture is called a *Young diagram.* We will freely switch between talking about the partition and the corresponding Young diagram and will denote both by λ and write $\lambda \vdash n$ for λ is a partition of n. We will give coordinates (i,j) to the boxes in the Young diagram where i denotes the column and j denotes the row and will imagine λ as embedded in the $\mathbb{N}^2$ lattice so we can talk about $(r,s) \in \mathbb{N}^2$ lying outside of λ.



Partitions appear in math as indexing sets for many things as we will see some examples of.

Proposition 2.4. *There is a one to one correspondance between partitions λ of n and monomial ideals $I \subset \mathbb{C}[x,y]$ of length n given by*

$$\lambda \mapsto I_\lambda := (\{x^r y^s | (r,s) \notin \lambda\})$$

with inverse

$$I \mapsto \lambda(I) := \{(i,j) | x^i y^j \notin I\}$$

Proof. To any subset $\{(i,j)\} \subset \mathbb{N}^2$ of size n, we can associate a box diagram similar to the Young diagram λ of a partition. The box diagrams which are Young diagrams are characterized by the fact that if $(i,j) \in \lambda$, then $(i-1,j) \in \lambda$ and $(i,j-1) \in \lambda$.

Now, I_λ is an ideal and $x^i y^j \notin I_\lambda$ if and only if $(i,j) \in \lambda$ so $\mathbb{C}[x,y]/I_\lambda$ is generated by $x^i y^j$ for $(i,j) \in \lambda$ as a vector space. In particular, it has dimension equal to the number of boxes of λ which is n so $\text{len}(I_\lambda) = n$.

Conversely, given an ideal I, we know that if $x^r y^s \in I$ then $x^{r+1} y^s$ and $x^r y^{s+1} \in I$. Therefore, of $x^i y^j \notin I$, then $x^{i-1} y^j$ and $x^i y^{j-1}$ are not in I and $\lambda(I)$ satisfies the characterizing property of Young diagrams. Finally, $\lambda(I)$ is a partition of n because the boxes in $\lambda(I)$ are in one to one correspondance with monomials $x^i y^j$ not in I. These give a basis for $\mathbb{C}[x,y]/I$ which is n-dimensional by assumption so there are n boxes. $\square$

In light of this, we will often picture the Young diagram as representing such a subset of monomials.

1	x	x^2
y	xy	
y^2		
y^3		

We will denote $B_\lambda = \{x^i y^j | (i,j) \in \lambda\}$ and by the proof B_λ is a basis for $\mathbb{C}[x,y]/I_\lambda$.

2.2. **The Existence of** $\mathrm{Hilb}^n(X)$**.** Now we are ready to construct $\mathrm{Hilb}^n(X)$ as a variety. We will follow the construction of Haiman that uses the combinatorics of Young diagrams and monomial ideals established above to give explicit open affine covers of the Hilbert scheme.

For each λ partitioning n, $I_\lambda \in \mathrm{Hilb}^n(X)$. We construct an open affine variety U_λ around each of these special points so that the coordinate rings on these affines are algebraic functions of the ideals $I \in U_\lambda$.

Define $U_\lambda \subset \mathrm{Hilb}^n(X)$ by

$$U_\lambda := \{I \in \mathrm{Hilb}^n(X) | B_\lambda \text{ is a basis for } \mathbb{C}[x,y]/I\}$$

Remark 2.5. *Note that $I_\lambda \in U_\lambda$ and that this is the only monomial ideal contained in U_λ. Thus, all the U_λ are needed to cover* $\mathrm{Hilb}^n(X)$*. The proof that the U_λ do in fact cover* $\mathrm{Hilb}^n(X)$ *will be ommitted but follows from explicit commutative algebra computations on* $\mathbb{C}[x,y]$.

For each $I \in U_\lambda$, we can expand out

$$x^r y^s = \sum_{(i,j)\in\lambda} c_{ij}^{rs}(I) x^i y^j \mod I$$

uniquely since B_λ gives a basis $\mod I$,. This gives algebraic functions c_{ij}^{rs} defined on U_λ for any (r,s) and any $(i,j) \in \lambda$.

Claim 2.6. *U_λ endowed with the ring of functions $\mathbb{C}[c_{ij}^{rs}]$ is an affine scheme. This is a cover of* $\mathrm{Hilb}^n(X)$ *by affines which gives it the structure of an algebraic variety.*

This shows that the Hilbert scheme exists as a variety and that it parametrizes ideals in an algebraically varying way. We won't discuss precisely what this means. Furthermore, we can explicitly construct the Hilbert-Chow morphism. This is just the map

$$\mathrm{Hilb}^n(X) \to \mathrm{Sym}^n(X)$$

sending an ideal I to its support $\mathrm{Supp}(I)$. If I corresponds to n distinct points, then it maps to the sum of those n points in the symmetric product so this is an isomorphism away from the singular locus. Above the singular points, we have families of ideals that the Hilbert scheme parametrizes whose support is all the same.

2.2.1. $\mathrm{Hilb}^2(X)$. Let's look at $\mathrm{Hilb}^2(X)$ explicitly. If we have two distinct points p_1 and p_2 in $\mathbb{A}^2$, then the ideal I consisting of all polynomials that vanish at both points is already a length two ideal and so is a distinct point in $\mathrm{Hilb}^2(X)$.

We can take an automorphism of $\mathbb{A}^2$ sending p_1 to the origin and so our two points are $(0,0)$ and $p \neq 0$. Lets say $p = (a,b)$. The ideal of functions vanishing at $(0,0)$ is (x,y) and the ideal vanishing at (a,b) is $(x-a, y-b)$. Therefore, the ideal vanishing at both consists of $(x,y) \cap (x-a, y-b) = (x(x-a), x(y-b), y(x-a), y(y-b))$ since these points are distinct and so their ideals are coprime.

What happens if we take the limit as p approaches $(0,0)$? Well we need to define how we are taking this limit. Suppose we take the limit so that p lies on a line $y = \alpha x$. Then p is of the form $(t, \alpha t)$ and it approaches 0 as $t \to 0$. Looking at the ideal,

$$I_t = (x(x-t), x(y-\alpha t), y(x-t), y(y-\alpha t))$$

I_t is an algebraic family of ideals and it defines a curve in the Hilbert scheme. The limit as $t \to 0$ should be a length 2 ideal that is supported at the origin. I_t contains the linear form $\alpha x - y$ for all $t \neq 0$. Therefore the limit I_0 must also contain $\alpha x - y$ and so by length considerations, the limit is exactly

$$I_0 = (x^2, xy, y^2, \alpha x - y).$$

We can rewrite I_0 more suggestively as

$$I_0 = \{f | f(0,0) = 0 \text{ and } df_{(0,0)} = rv \text{ for some } r\}$$

where v is a vector in the direction of the line $y = \alpha x$. We see more generally that this is the structure of every point $I \in \operatorname{Hilb}^n(X)$ supported at the origin:

$$\operatorname{Hilb}^n_0(X) = \{I(v) = \{f | f(0) = 0 \text{ and } df_0 = rv\} | v \in T_{(0,0)}\mathbb{C}^2\}$$

where we denote those ideals supported at the origin by $\operatorname{Hilb}^n_0(X)$. This is known as the *punctual* Hilbert scheme. So we see that when points collide on the Hilbert scheme, it keeps track of both where the points collide and the infitessimal information about how they collide.

2.3. **Local Structure of the Hilbert Scheme.** We will compute the tangent space $T_\lambda \operatorname{Hilb}^n(X)$ at a monomial ideal I_λ where we denote I_λ by λ by abuse of notation. The *Zariski cotangent space* of a point p in an affine variety V is the vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$ where $\mathfrak{m}_p$ is the ideal in the coordinate ring consisting of all functions that vanish at p. Then the tangent space $T_pV = (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$ is the dual vector space.

For any monomial ideal $I_\lambda \in \operatorname{Hilb}^n(X)$, either $x^r y^s = 0 \mod I$ if $x^r y^s \in I$ or $x^r y^s = x^i y^j$ if $(r,s) = (i,j) \in \lambda$. Therefore,

$$c^{rs}_{ij}(I_\lambda) = \begin{cases} 1 & \text{if } (r,s) = (i,j) \in \lambda \\ 0 & \text{if } (r,s) \notin \lambda \end{cases}$$

so the maximal ideal of I_λ is

$$\mathfrak{m}_\lambda = \{c^{rs}_{ij} | (r,s) \notin \lambda\}.$$

We can represent each of these functions c^{rs}_{ij} as arrows on the Young diagram λ that start at box (r,s) outside the diagram and end at box (i,j) inside the diagram. We need to understand $\mathfrak{m}_\lambda/\mathfrak{m}_\lambda^2$. By multiplying the defining equation

$$x^r y^s = \sum c^{rs}_{ij}(I) x^i y^j \mod I$$

by x and expanding out both sides in the basis B_λ, we get the relation

$$c^{r+1,s}_{ij} = \sum_{(h,k)\in\lambda} c^{rs}_{hk} c^{h+1,k}_{ij}.$$

Similarly, multiplying by y gives the relation

$$c_{ij}^{r,s+1} = \sum_{(h,k)\in\lambda} c_{hk}^{rs} c_{ij}^{h,k+1}.$$

Taking these equations modulo $\mathfrak{m}_\lambda^2$ kills off every term except when $(h+1,k) = (i,j)$ or $(h,k+1) = (i,j)$ respectively. This reduces to

$$c_{ij}^{r+1,s} = c_{i-1,j}^{rs} \mod \mathfrak{m}_\lambda^2 \qquad c_{ij}^{r,s+1} = c_{i,j-1}^{rs} \mod \mathfrak{m}_\lambda^2$$

So as cotangent vectors, we can move the arrows corresponding to c_{ij}^{rs} either horizontally or vertically and still get the same cotangent vector as long as the tail (r,s) stays outside the diagram and the head (i,j) stays inside the diagram. By abuse of notation, we will use these arrows to denote both the cotangent vectors as above, and the dual tangent vectors in $T_\lambda \operatorname{Hilb}^n(X)$.

3. Torus Actions on the Hilbert Scheme

3.1. **Algebraic Torus Actions.** Let $T = (\mathbb{C}^*)^2$ be the two dimensional algebraic torus. Then T acts on $X = \mathbb{A}^2$ diagonlly by $t \cdot (x,y) = (t_1x, t_2y)$ where $t = (t_1,t_2) \in T$. This naturally lifts to an action on ideals by

$$t \cdot I := \{f(t_1^{-1}x, t_2^{-1}y) | f(x,y) \in I\}.$$

This action preserves degrees of the polynomials and so sends a length n ideal to a length n ideal. Therefore it induces an action of T on $\operatorname{Hilb}^n(X)$. It turns out that we can understand a whole lot of the geometry of $\operatorname{Hilb}^n(X)$ by exploiting the fixed points of this action. Specifically we have the following useful theorems about torus actions on a variety.

Theorem 3.1. *Let Y be a compact normal variety with a torus action by T and let Y^T be the locus of points fixed by T. If all the points in Y^T are smooth then Y is smooth.*

Proof. The singular locus of Y is torus invariant and closed. Therefore it consists of closures of torus orbits. The closure of every torus orbit contains a torus fixed point and so the singular locus of Y must contain a torus fixed point if it is nonempty. If the torus fixed points are all smooth then the singular locus must be empty. □

Theorem 3.2. *If Y has finitely many fixed points so that $Y^T = \{p_1, \ldots, p_n\}$, then there exists a 1-dimensional subtorus $T' \subset T$ so that $Y^{T'} = Y^T$.*

Theorem 3.3. *(Bialynicki-Birula pt.1) Let Y be a smooth variety with a torus action with finitely many fixed points and let T' be a 1-dimensional subtorus as above that also satisfies the following condition:*

$$\lim_{t\to 0} t \cdot y \text{ exists}$$

for all $y \in Y$ where $t \in T'$. Then there exists a decomposition of Y into locally closed subvarieties Y_i given by the following:

$$Y_i := \{y \in Y | \lim_{t\to 0} t \cdot y = p_i\}$$

where again the limit is taken over $t \in T'$. Each Y_i is isomorphic to an affine space $\mathbb{A}^{n_i}$ and we can read the dimension of n_i from the action of T' on $T_{p_i}Y$. Explicitly, this dimension is the dimension of the subspace $T_{p_i}^+Y$ consisting of tangent vectors on which T' acts by positive weight.

Theorem 3.4. *(Bialynicki-Birula pt.2) The cohomology $H^{2k}(Y)$ is generated by the classes of the closures of the cells Y_i such that $\dim Y = k$ and the odd cohomology $H^{2k+1}(Y)$ vanishes. Therefore, the Betti numbers can be read off from the torus action as*

$$b_{2k}(Y) = \#\{p_i | \dim T^+_{p_i} Y = k\}$$

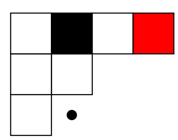
and $\chi(Y) = \#Y^T$.

3.2. **Torus Fixed Points of the Hilbert Scheme.** To apply those above theorems, we need to understand the torus fixed points of $\mathrm{Hilb}^n(X)$ and the limits $\lim_{t\to 0} t \cdot I$ for one dimension subtori. The torus T acts by different weight on x and y. The only way I will be a fixed ideal under this action is if it is generated by f that are homogeneous in both x and y so I must be a monomial ideal. This explains the significance of the I_λ that we discussed above. They are exactly the torus fixed points of a canonical torus action and in fact the open sets U_λ are maximal torus fixed open affine subsets containing I_λ.

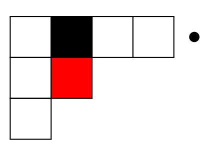
Proposition 3.5. $\mathrm{Hilb}^n(X)$ *is smooth.*

Proof. We need to show that $\mathrm{Hilb}^n(X)$ is smooth at the torus fixed points I_λ. This amounts to computing the dimension of $\mathfrak{m}_\lambda/\mathfrak{m}_\lambda^2$ and showing it is equal to the dimension of $\mathrm{Hilb}^n(X)$ which is $2n$. We can represent cotangent vectors by arrows as above and we can slide the arrows around on the diagram. If the head of an arrow ever leaves the diagram, then that function c^{rs}_{ij} vanishes modulo $\mathfrak{m}_\lambda^2$.

If an arrow is pointing northwest, then we can always move it so it vanishes. If it is pointing weakly northeast, then we can move it to a unique position as far north and to the east as possible. Similarly, if it is pointing weakly southwest, then we can move it as far south and west as possible to a unique position.



The northeast pointing vectors are in one to one correspondance with boxes in the diagram by associating to each box (i, j) (in black above) to the arrow from the $\bullet$ right outside the column i to the red box right inside the row j. Southwest pointing arrows are in one to one correspondance with boxes by associating to the black box the arrow pointing from $\bullet$ to the red box as below:



Thus, to each box in λ, there are exactly 2 tangent vectors, one northeast and one southwest, so the dimension is $2n$ and the Hilbert scheme is smooth.

□

To apply Bialynicki-Birula's theorem, we need to understand the 1-parameter torus limits of ideals in $\mathrm{Hilb}^n(X)$. Each one dimension subtorus $T' \subset T$ is of the form (t^p, t^q) for some weights (p, q). We will denote the weight vector (p, q) by w. Then this weight vector gives an ordering on the monomials $x^r y^s$ by $x^i y^j < x^r y^s$ if and only if $pi + qj < pr + qs$.

Example 3.6. *If we take only monomials up to degree n for some fixed n and pick w so that $p >> q > 0$, then the monomial ordering on these monomials is the one where y is always less than x so that $1 < y < y^2 < \ldots < x < xy < \ldots < x^n$.*

Definition 3.7. *Let $f(x,y) \in \mathbb{C}[x,y]$ and w some weight vector. Then we denote by $\mathrm{in}_w(f)$ the leading monomial term of f with respect to the weight w. For I an ideal, then*

$$\mathrm{in}_w(I) = \{\mathrm{in}_w(f) | f \in I\}.$$

Note that this is a monomial ideal I_λ for some λ by construction.

Then by computing the explicit way (t^p, t^q) acts on a polynomial f, we can see that in the limit as $t \to 0$, only the initial term of f survives. Thus, we have the following.

Proposition 3.8. *Let T' be a one dimensional subtorus from Byalinicki-Birula's theorem and suppose that it corresponds to weight vector w. Then*

$$\lim_{t \to 0} t \cdot I = \mathrm{in}_w(I)$$

Corollary 3.9. *For each suitable weight vector w, $\mathrm{Hilb}^n(X)$ has a stratification into locally closed affine spaces H_λ indexed by partitions λ so that*

$$H_\lambda = \{I \in \mathrm{Hilb}^n(X) | \, \mathrm{in}_w(I) = I_\lambda\}.$$

4. Cohomology of $\mathrm{Hilb}^n(X)$

Now we are in a position to compute the cohomology of $\mathrm{Hilb}^n(X)$ using the cell decomposition above and the Bialynicki-Birula theorem. The defining equations for c_{ij}^{rs} as coefficients in the basis expansion must be torus invariant from the definition of the torus action since the basis B_λ is a torus invariant basis for all λ. From this, we get that the torus acts on c_{ij}^{rs} by

$$(t_1, t_2) \cdot c_{ij}^{rs} = t_1^{r-i} t_2^{s-j} c_{ij}^{rs}.$$

In particular, a 1-dimensional subtorus with weight vector $w = (p,q)$ acts by

$$t \cdot c_{ij}^{rs} = t^{p(r-1)+q(s-j)} c_{ij}^{rs}.$$

The weight of the 1-d torus on each arrow c_{ij}^{rs} is exactly $p(r-i) + q(s-j)$ and so by the Bialynicki-Birula theorem, for any fixed weight vector w, the dimension of the cell H_λ is exactly the number of arrows c_{ij}^{rs} such that $p(r-1) + q(s-j) > 0$.

Definition 4.1. *For each w, there exists a statistic d_w on the set of partitions of n so that*

$$d_w(\lambda) = \#\{c_{ij}^{rs} \in T_\lambda \mathrm{Hilb}^n(X) | p(r-i) + q(s-j) > 0\}$$

Then we have essentially proved the following.

Theorem 4.2. *Fixing an appropriate weight vector $w = (p,q)$, we can compute the betti numbers numbers of $\mathrm{Hilb}^n(X)$ as*

$$b_{2k}(\mathrm{Hilb}^n(X)) = \#\{\lambda | d_w(\lambda) = k\}$$

Corollary 4.3.

$$\chi(\mathrm{Hilb}^n(X)) = \#\{\lambda \vdash n\}$$

Corollary 4.4. *Each choice of w gives you an equidstributed statistic d_w on the set of partitions of n. That is, the number of λ with a specific value of $d_w(\lambda)$ is independent of w. Thus there is a 2-dimensional family of equidistributed statistics on the set of partitions.*

Proof. By the theorem, the number of λ with some fixed value of $d_w(\lambda)$ counts the betti numbers of $\mathrm{Hilb}^n(X)$ which are an intrinsic topological invariant not depending on w. $\square$

4.1. **Specific Statistics and the Formula of Ellingsrud and Stromme.** Ellingsrud and Stromme found an explicit weight w such that $d_w(\lambda) = n + l(\lambda)$ where $l(\lambda)$ is the number of parts of the partition λ. This lets us compute the generating function for the Betti numbers.

Definition 4.5. *Let Y be a smooth complex manifold. The Poincare polynomial $P_t(Y)$ is the polynomial*

$$P_t(Y) = \sum b_k(Y)t^k.$$

Then with Ellingsrud and Stromme's statistic, we have the equality

$$P_t(\mathrm{Hilb}^n(X)) = \sum_{\lambda \vdash n} t^{2n+2l(\lambda)}.$$

Even better, we can package these all together so that

$$\sum_n P_t(\mathrm{Hilb}^n(X))q^n = \sum_n \sum_{\lambda \vdash n} t^{2n+2l(\lambda)} = \prod_{m=1}^{\infty} \frac{1}{1 - t^{2m+2}q^m}.$$

By the discussion above, this generating function up to a change of variables must be the generating function of partitions under any statistic d_w defined by a weight w. A question one could ask is if there are combinatorial interpretations for different statistics. Loehr and Warrington solved this question with a strong affirmative. They contructed for each of these weights, explicit discriptions of the statistics in terms of the combinatorics of λ and then, inspired by corllary 4.4, they proved the equidistributivity of these statistics by constructing explicit combinatorial bijections that related Young diagrams with associated statistics to Eulerian tours, cylindrical lattice paths, directed multigraphs, and oriented trees.

4.2. **Gottsche's Formula and the Nakajima Construction.** Until now we have been working very explicitly with $X = \mathbb{A}^2$. However, the construction for Hilbert schemes of points on $\mathbb{A}^2$ can be generalized to any smooth surface by using the fact that smooth surfaces locally look like $\mathbb{A}^2$. Here we have to take care in what we mean by locally since it does not necessarily mean there exists an open affine subvariety $\mathbb{A}^2$. Rather, locally here means locally analytically in a formal power series neighborhood.

An immediate question is what is the generalization of Ellingsrud and Strommes generating function for the betti numbers of $\mathrm{Hilb}^n(X)$ for general X? Gottsche settled this question by giving a very beautiful formula.

Theorem 4.6. *(Gottsche) Let X be a smooth projective surface. Then*

$$\sum_n P_t(\mathrm{Hilb}^n(X))q^n = \prod_m \frac{(1+t^{2m-1}q^m)^{b_1(X)}(1+t^{2m+1}q^m)^{b_3(X)}}{(1-t^{2m-2}q^m)^{b_0(X)}(1-t^{2m}q^m)^{b_2(X)}(1-t^{2m+2}q^m)^{b_4(X)}}$$

This showed that the best way to study the cohomology of the Hilbert scheme is by viewing all the Hilbert schemes at once. Witten and Vafa made an observation that this generating function is the character formula for an irreducible representation of a Heisenberg algebra on a bosonic Fock space. Nakajima and Grojnowski showed this observation was not a coincidence.

Definition 4.7. *The Heisenberg algebra $\mathbb{H}$ is the algebra generated by creation and annihilator operators $P[k]$ for $k \in \mathbb{Z} \setminus \{0\}$ under commutation relations*

$$[P[m], P[n]] = m\delta_{m+n,0}K$$

where K is some constant.

Theorem 4.8. *(Nakajima, Grojonwski) The cohomology of all the Hilbert schemes* $\mathrm{Hilb}^n(X)$ *for all* n

$$\bigoplus_n H^*(\mathrm{Hilb}^n(X))$$

is an irreducible representation of $\mathbb{H}$ *with character given by the Gottsche formula.*

The idea is that the Nakajima operators $P[i]$ actually act like creation and annihilation operators on the Hilbert scheme by adding i points (or subtracting if i is negative). For example, naively, we can think of the action of the operators $P[1]$ as sending a diagram $\lambda \vdash n$ to the sum of all diagrams $\lambda^+ \vdash n+1$ obtained by adding exactly one box to λ.

$$\ydiagram{3,1} \xrightarrow{P[1]} \ydiagram{4,1} + \ydiagram{3,2} + \ydiagram{3,1,1}$$

Similarly, $P[-1]$ can be thought of as acting by sending λ to the sum of all partitions obtained by removing a box.

$$\ydiagram{3,1} \xrightarrow{P[-1]} \ydiagram{2,1} + \ydiagram{3}$$

This is not quite precise but it is a good heuristic as to why we may expect a creation/annihilation action on the cohomologies of the Hilbert scheme.

4.3. **K-Theory and the Representations of S_n.** One other place that Young diagrams appear in mathematics is as the indexing set for irreducible representations of S_n. To each $\lambda \vdash n$, we can associate an irreducible representation V_λ of S_n constructed explicitly from the combinatorics of λ. Since every representation can be decomposed into a sum of irreducibles, we can make the following definition.

Definition 4.9. *The representation ring* Rep_{S_n} *is the ring generated by the isomorphism classes of irreducible representations* V_λ *with addition given by* $\oplus$ *and multiplication given by* $\otimes$.

If V is a representation of S_n and W is a representation of S_m, then $V \otimes W$ can be induced to representation of S_{n+m} by the natural inclusion of $S_n \times S_m \to S_{n+m}$. We will denote this representation by $(V \otimes W)_{n+m}$.

Definition 4.10. *The Littlewood-Richardson coefficients are the coefficients* $c^\nu_{\lambda,\mu}$ *in the expansion*

$$(V_\lambda \otimes V_\mu)_{n+m} = \oplus_\nu c^\nu_{\lambda,\mu} V_\nu$$

of the induced representation in terms of irreducibles of S_{n+m}*, where* $\lambda \vdash n$*,* $\mu \vdash m$ *and* $\nu \vdash n+m$.

Rep_{S_n} and $H^*(\mathrm{Hilb}^n(\mathbb{A}^2))$ are both rings generated by classes indexed by partitions λ. One could ask is there any relation between these two rings? That is, is there a natural algebraic correspondance between representations V_λ of S_n and cells H_λ of $\mathrm{Hilb}^n(\mathbb{A}^2)$ generating the cohomology? It turns out the answer is yes if we switch from the cohomology ring of $\mathrm{Hilb}^n(\mathbb{A}^2)$ to the K-theory of vector bundles $K_0(\mathrm{Hilb}^n(\mathbb{A}^2))$.

Theorem 4.11. *(Haiman) There exists a one to one correspondance between irreducible representations* V_λ *of* S_n *and certain vector bundles* P_λ *on* $\mathrm{Hilb}^n(\mathbb{A}^2)$ *such that the bundles* P_λ *generate* $K_0(\mathrm{Hilb}^n(\mathbb{A}^2))$ *and that the correspondance respects the ring structures between* Rep_{S_n} *and* $K_0(\mathrm{Hilb}^n(\mathbb{A}^2))$.

Remark 4.12. *I'm being purposely vague here. The correct statement of the correspondance involves an equivalence between derived categories of coherent sheaves which lifts to an equivalence between certain equivariant* K_0 *groups but I didn't want to get into that here.*

Haiman proved this as part of his proofs of the celebrated $n!$ and $(n+1)^{(n-1)}$ conjectures. These are important results in the theory of symmetric functions which Haiman proved by carefully studying the geometry of $\mathrm{Hilb}^n(\mathbb{A}^2)$.

Question 4.13. *Can we extend the Nakajima operators to the K-theory of* $\mathrm{Hilb}^n(\mathbb{A}^2)$ *so that they give a geometric interpretation of the Littlewood-Richardson coefficients in terms of the natural vector bundles on* $\mathrm{Hilb}^n(\mathbb{A}^2)$*?*