1. Why we care. Start with a ring $A$ and look at the category $P_A$ of finitely generated projective modules. Then $|P_A|$, a coherently commutative monoid. Then $K(A)$ is the group completion of this. There are many explicit constructions. Here’s one: $K(A) = \text{BGL}(A)^+ \times \mathbb{Z}$.

This can be extended to Waldhausen categories (categories with a notion of weak equivalence and cofibrations). Because of the group completion above, $\pi_0 K(A)$ is the (algebraic) group completion of the monoid of finitely-generated projective $A$-modules (up to isomorphism), under direct sums. The idea is that it’s sometimes easier to understand the group completion of something than the starting monoid.

Examples 1.1.

- If you take $K$-theory of the category of pointed sets, you get the sphere spectrum. (This is Barratt-Priddy-Quillen.)
- If you take $K$-theory of the category of vector bundles over $X$, you get topological $K$-theory $K(X)$.
- “$K(\mathbb{Z})$ knows about arithmetic” – it’s related to conjectures in number theory.
- $K(\Sigma^\infty \Omega M_+) \simeq \Sigma^\infty M_+ \vee Wh(M)$ (where the latter is the Whitehead spectrum, a geometric gadget related to the stable $h$-cobordisms of $M$). This equivalence is a big theorem of Waldhausen-Jahren-Rognes. Weiss-Williams showed that you can use this to get information about diffeomorphism groups.

One of the first computations was the algebraic $K$-theory of finite fields.

**Theorem 1.2** (Quillen).

$$\pi_i K(\mathbb{F}_q) = \begin{cases} 
\mathbb{Z} & i = 0 \\
\mathbb{Z}/(q^i - 1)\mathbb{Z} & i = 2j - 1 \\
0 & \text{else.}
\end{cases}$$

**Proof idea.** The main point is to define a homology isomorphism $\theta : BGL(\mathbb{F}_q) \to \text{hofib}(\psi^q - 1)$ such that $\ker \pi_\ast \theta = E(\mathbb{F}_q)$, elementary matrices. (Here $\psi$ is a map $BU \to BU$.)

Another important $K$-theory is $K(\mathbb{Z})$; it’s not complete, because of number theory. This calculation spans over 50 years; information is very scattered, but Weibel has a book about it, and another good reference is lecture notes by Soulé.

$K_0(\mathbb{Z})$ is easy: a finitely-generated projective $\mathbb{Z}$-module is just a free $\mathbb{Z}$-module, and those are classified by dimension.

- $K_0(\mathbb{Z}) = \mathbb{Z}$
- $K_1(\mathbb{Z}) = \mathbb{Z}/2$ (?)
- $K_2(\mathbb{Z}) = \mathbb{Z}/2$
- $K_3(\mathbb{Z}) = \mathbb{Z}/48$ (much harder, due to Lee-Szczarba)
- $K_4(\mathbb{Z}) = 0$ (Rognes, about 2000, very hard)
- $K_5(\mathbb{Z}) = \mathbb{Z}$ (Soulé and others)

**Conjecture 1.3.** $K_{4m}(\mathbb{Z}) = 0$ for $m \geq 1$ (but $K_8$ is not known).

**Theorem 1.4** (Kurihara, ~2000). The above conjecture is equivalent to the Vandiver conjecture.

Everything else is known.

**Theorem 1.5** (Quillen). $K_n(\mathbb{Z})$ is finitely generated.

**Theorem 1.6** (Borel).

$$K_n(\mathbb{Z}) = \begin{cases} 
\mathbb{Z} \oplus \text{finite} & n \equiv 1 \pmod{4} \\
\text{finite} & \text{otherwise}.
\end{cases}$$

**Proof.** For $N \gg q$:
- $H^q(SL_N(\mathbb{Z}); \mathbb{R}) \cong H^q_{cst}(SL_N(\mathbb{R}); \mathbb{R})$.
- $H^*_{cst}(SL_N(\mathbb{R})) = \Lambda(e_5, e_9, \ldots, e_4\left\lfloor \frac{n-1}{2} \right\rfloor)$
- $K_q(\mathbb{Z}) \otimes \mathbb{R} \cong \pi_q BSL(\mathbb{Z})^+ \otimes \mathbb{R}$

The rest of the calculation of $K(\mathbb{Z})$ follows from:

**Theorem 1.7** (Soulé/Dwyer-Friedlander, Voevodsky-Rost).

$$K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Z}_p \cong H^1_{et}(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(n))$$
$$K_{2n-2}(\mathbb{Z}) \otimes \mathbb{Z}_p \cong H^2_{et}(\text{Spec } \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(n))$$

for $n \geq 2$ and $p$ an odd prime.

A slightly more general series of techniques is given by trace methods. The idea: map $K$-theory to a more treatable object: $THH$ or $TC$. This construction is due to Bökstedt-Hsiang-Madsen. These methods were introduced for figuring out the $K$-theoretic Novikov conjecture, which says that the assembly map

$$BG_+ \wedge K(\mathbb{Z}) \to K(\mathbb{Z}[G])$$

is rationally split injective assuming some finiteness conditions on $G$. (This is due to BHM.)

Waldhausen showed that there is a map $\Sigma^\infty_+ X \to K(\Sigma^\infty \Omega X_+) =: A(X)$ that has a splitting.

\[2\]
If $I \subset A$ is an ideal, let $\tilde{K}(A) := \text{hofib}(K(A) \to K(A/I))$. For example, if $I = (x)$ and $A = k[x]/x^n$ for a perfect field, then
\[
\tilde{K}(k[x]/x^n) = \begin{cases} 
W_{nk}(k)/V_n(W_q(h)) & i = 2q - 1 \\
0 & \text{otherwise}.
\end{cases}
\]
(Here $V_n$ is the Verschiebung and $W$ is Witt vectors.) This is due to Hesselholt-Madsen.

You can get information about $K(S)$ in terms of $K(\mathbb{Z})$.

1.2. **The trace.** Topological Hochschild homology is:
\[
\text{THH}(A) = A\hat{\otimes}S^1
\]
Think of this as configurations of points in $S^1$ with labels in $A$. If $A$ is not commutative you can still do this. There is an $S^1$ action on $\text{THH}(A)$. You can talk about the fixed points of this action. Then topological cyclic homology $TC(A)$ is defined using those fixed points.

The trace is a map $K(A) \to TC(A)$. You can extend the $\text{THH}$ construction to categories. The map $K(A) \to TC(A)$ roughly takes $c \mapsto$ the configuration of points labelled by $1_c$.

**Theorem 1.8** (Dundas-Goodwillie-McCarthy). If $B \to A$ is surjective and nilpotent fiber in $\pi_0$ then
\[
\begin{array}{ccc}
K(B) & \longrightarrow & TC(B) \\
\downarrow & & \downarrow \\
K(A) & \longrightarrow & TC(A)
\end{array}
\]
is homotopy cartesian (after $p$-completion).

One example of such a map is the Hurewicz map, which gives rise to a cartesian square
\[
\begin{array}{ccc}
K(S) & \longrightarrow & TC(S) \\
\downarrow & & \downarrow \\
K(\mathbb{Z}) & \longrightarrow & TC(\mathbb{Z})
\end{array}
\]
$K(\mathbb{Z})$ is more-or-less understood. $TC(S)$ is known, by work of BHM, and $TC(\mathbb{Z})$ is known, due to Böstedt-Madsen and Rognes. Blumberg-Mandell used this to show that $K(S)_p^\wedge \to TC(S)_p^\wedge \times K(\mathbb{Z})_p^\wedge$ is split injective (in $\pi_*$) for all odd $p$.

The proof was scattered over 25 years. It’s the main application of Goodwillie calculus.

**Proof idea.**
- Dundas: reduce to simplicial rings.
- Goodwillie: reduce to rings. If you have a simplicial ring, you can either take the $K$-theory in the category of simplicial rings, or you can take $K$-theory levelwise (much easier). In general, these are not at all the same thing, but it works out for the homotopy fiber $K(B) \to K(A)$.
- Reduce inductively to split square-zero extensions $A \ltimes M \to A$. 

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• Extend $K(A \times M)$ to a functor: make the Dold-Thom construction $M(-)$ (bimodule of configurations of points in your space with labels in $M$). So you get a functor $K(A \times M(-)) : \text{Top}_* \to \text{Sp}$. Do the same for $TC$.

• Use calculus: show that $D_1 K \simeq D_1 TC$, by showing $D_1 K \simeq THH(A, M(S^1 \wedge -)) \simeq D_1 TC$ (actually for the last equivalence you have to $p$-complete. This is due to Dundas-McCarthy and Hesselholt). By calculus, $\tilde{K} \simeq \tilde{TC}$.