Double Integrals

1. Write a double integral \( \int_{\mathcal{R}} f(x, y) \, dx \, dy \) which gives the volume of the top half of a solid ball of radius 5. (You need to specify a function \( f(x, y) \) as well as a region \( \mathcal{R} \).)

Solution. We know that \( \int_{\mathcal{R}} f(x, y) \, dx \, dy \) can be interpreted as the volume of the solid under \( z = f(x, y) \) over the region \( \mathcal{R} \). So, we’d like to think of a function \( f(x, y) \) and a region \( \mathcal{R} \) so that the solid under \( z = f(x, y) \) over \( \mathcal{R} \) is the top half of a solid ball of radius 5.

Let’s look at the sphere of radius 5 centered at the origin. We know that it has equation \( x^2 + y^2 + z^2 = 25 \), so the top half of this sphere can be described by \( z = \sqrt{25 - x^2 - y^2} \). The solid under \( z = \sqrt{25 - x^2 - y^2} \) and over the region \( x^2 + y^2 \leq 25 \) is therefore half of a solid ball of radius 5. So, the double integral

\[
\int_{\mathcal{R}} \sqrt{25 - x^2 - y^2} \, dx \, dy \quad \text{where} \quad \mathcal{R} \text{ is the disk } x^2 + y^2 \leq 25
\]

gives the volume of the top half of a solid ball of radius 5.

2. (a) If \( \mathcal{R} \) is any region in the plane \( (\mathbb{R}^2) \), what does the double integral \( \int_{\mathcal{R}} 1 \, dx \, dy \) represent? Why?

Solution. Remember that we are thinking of the double integral \( \int_{\mathcal{R}} f(x, y) \, dx \, dy \) as a limit of Riemann sums, obtained from the following process:

1. Slice the region \( \mathcal{R} \) into small pieces.
2. In each piece, the value of \( f \) will be approximately constant, so multiply the value of \( f \) at any point by the area \( \Delta A \) of the piece.\(^{(1)}\)
3. Add up all of these products. (This is a Riemann sum.)
4. Take the limit of the Riemann sums as the area of the pieces tends to 0.

Now, if \( f \) is just the function \( f(x, y) = 1 \), then in Step 2, we end up simply multiplying 1 by the area of the piece, which gives us the area of the piece. So, in Step 3, when we add all of these products up, we are just adding up the area of all the small pieces, which gives the area of the whole region.

So, \( \int_{\mathcal{R}} 1 \, dx \, dy \) represents the area of the region \( \mathcal{R} \).

(b) Suppose the shape of a flat plate is described as a region \( \mathcal{R} \) in the plane, and \( f(x, y) \) gives the density of the plate at the point \( (x, y) \) in kilograms per square meter. What does the double integral \( \int_{\mathcal{R}} f(x, y) \, dx \, dy \) represent? Why?

Solution. Following the process described in (a), in Step 2, we multiply the approximate density of each piece by the area of that piece, which gives the approximate mass of that piece. Adding those up gives the approximate mass of the entire plate, and taking the limit gives us the exact mass of the plate.

\(^{(1)}\) Actually, it’s also fine to just approximate the area of the piece.
3. If $R$ is the rectangle $[1, 2] \times [3, 4]$, compute the double integral $\int_R 6x^2 y \, dx \, dy$.

**Solution.** We can rewrite this as an integrated integral in two ways: $\int_1^2 \left( \int_3^4 6x^2 y \, dy \right) \, dx$ or $\int_3^4 \left( \int_1^2 6x^2 y \, dx \right) \, dy$. These will give the same answer (that’s what Fubini’s Theorem says), so let’s just use the first. We need to first do the inner integral, which is $\int_3^4 6x^2 y \, dy$. When we do this integral, we treat $x$ as a constant. So, this integral is equal to $3x^2 y^2 \bigg|_{y=3}^{y=4} = 3x^2 (16 - 9) = 21x^2$. So, our iterated integral becomes $\int_1^2 21x^2 \, dx = 7x^3 \bigg|_{x=1}^{x=2} = 49$.

4. If $R$ is the rectangle $[0, 1] \times [-1, 2]$, compute the double integral $\int_R 2ye^x \, dx \, dy$.

**Solution.** We can rewrite the double integral as an iterated integral in two ways: $\int_0^1 \int_{-1}^1 2ye^x \, dy \, dx$ or $\int_{-1}^1 \int_0^1 2ye^x \, dx \, dy$. Let’s use the first to compute.

$$
\int_0^1 \int_{-1}^1 2ye^x \, dy \, dx = \int_0^1 \left( y^2 e^x \bigg|_{y=-1}^{y=2} \right) \, dx \\
= \int_0^1 3e^x \, dx \\
= 3e^x \bigg|_{x=0}^{x=1} \\
= 3e - 3
$$

5. Find the volume of the solid that lies under $z = x^2 + y^2$ and above the square $0 \leq x \leq 2$, $-1 \leq y \leq 1$.

**Solution.** We know that the volume of the solid lying under a surface $z = f(x, y)$ and above a region $\mathcal{R}$ in the plane is given by the double integral $\iint_{\mathcal{R}} f(x, y) \, dx \, dy$, so the volume we want in this problem is given by the double integral $\iint_{\mathcal{R}} (x^2 + y^2) \, dx \, dy$ where $\mathcal{R}$ is the square $[0, 2] \times [-1, 1]$. We know that this double integral is equal to the iterated integrals $\int_0^2 \int_{-1}^1 (x^2 + y^2) \, dy \, dx$ and $\int_{-1}^1 \int_0^2 (x^2 + y^2) \, dx \, dy$. 

2
Let’s use the first iterated integral:

\[
\int_{0}^{2} \int_{-1}^{1} (x^2 + y^2) \, dy \, dx = \int_{0}^{2} \left[ \left( x^2y + \frac{y^3}{3} \right) \bigg|_{y=-1}^{y=1} \right] \, dx
\]

\[
= \int_{0}^{2} \left( 2x^2 + \frac{2}{3} \right) \, dx
\]

\[
= \frac{2x^3}{3} + \frac{2}{3} \bigg|_{x=0}^{x=2}
\]

\[
= \frac{20}{3}
\]

6. **Find the volume of the solid enclosed by the surfaces** \( z = 4 - x^2 - y^2 \), \( z = x^2 + 2y^2 - 2 \), \( x = -1 \), \( x = 1 \), \( y = -1 \), and \( y = 1 \).

**Solution.** When we study triple integrals, we’ll see another way to do this problem.

First, let’s figure out what this solid looks like. The surface \( z = 4 - x^2 - y^2 \) is a paraboloid which opens downward, with its highest point at \((0, 0, 4)\). The surface \( z = x^2 + 2y^2 - 2 \) is a paraboloid which opens upward, with its lowest point at \((0, 0, -2)\). So, here is a picture of the solid:

![Solid](image)

Here, the top surface is \( z = 4 - x^2 - y^2 \), and the bottom is \( z = x^2 + 2y^2 - 2 \). To find the volume of the solid, let’s imagine approximating it using boxes:

![Box Approximation](image)

Basically, what we are doing is chopping the rectangle \( \mathcal{R} = [-1, 1] \times [-1, 1] \) into lots of small rectangles, each of area \( \Delta A \). Then we look at a particular box:
Its volume is the area $\Delta A$ multiplied by the height of the box. The height of the box is the difference between the $z$-value at the top (on the surface $z = 4 - x^2 - y^2$) and the bottom (on the surface $z = x^2 + 2y^2 - 2$). So, its volume is approximately $[(4 - x^2 - y^2) - (x^2 + 2y^2 - 2)]\Delta A = (6 - 2x^2 - 3y^2)\Delta A$. If we add all of these up and take the limit as $\Delta A \to 0$, we get the double integral $\iint_R (6 - 2x^2 - 3y^2) \, dx \, dy$, which we compute by converting to an iterated integral:

$$
\iint_R (6 - 2x^2 - 3y^2) \, dx \, dy = \int_{-1}^{1} \int_{-1}^{1} (6 - 2x^2 - 3y^2) \, dy \, dx
$$

$$
= \int_{-1}^{1} \left( 6y - 2x^2 y - y^3 \bigg|_{y=-1}^{y=1} \right) \, dx
$$

$$
= \int_{-1}^{1} (10 - 4x^2) \, dx
$$

$$
= 10x - \frac{4}{3}x^3 \bigg|_{x=-1}^{x=1}
$$

$$
= \frac{52}{3}
$$