1. (a) **Parameterize the elliptic paraboloid** \(z = x^2 + y^2 + 1\). **Sketch the grid curves defined by your parameterization.**

**Solution.** There are several ways to parameterize this. Here are a few.

i. One way to think of parameterizing is simply that we want to describe the surface using 2 variables. This amounts to describing \(x, y,\) and \(z\) using just 2 variables. In this case, \(z\) is already written in terms of \(x, y,\) and \(z,\) so we can describe the surface just using \(x\) and \(y.\) That is, we can use the parameterization \(\vec{r}(x, y) = (x, y, x^2 + y^2 + 1).\) We often use the variables \(u\) and \(v\) as the parameters (just as we usually used \(t\) for the parameter when parameterizing curves), so we could also write this as \(\vec{r}(u, v) = (u, v, u^2 + v^2 + 1).\) (It is certainly not necessary to use \(u\) and \(v\) though.)

Here is a picture of this parameterization:

![Parameterization Picture 1](image1)

The gray curves are where \(u\) is constant, and the black curves are where \(v\) is constant. The right picture shows a way of coloring the \(uv\)-plane; the paraboloid is colored according to the corresponding \(u\) and \(v\) value at each point. For instance, from the right picture, we see that \(u = 4, v = -4\) is colored pink. Therefore, the point \(\vec{r}(4, -4) = (4, -4, 17)\) on the paraboloid is colored pink.

ii. Another possibility is to use cylindrical coordinates to rewrite the surface as \(z = r^2 + 1.\) Then, every point can be described in terms of \(r\) and \(\theta\) since \(z = r^2 + 1.\) Converting back to Cartesian coordinates, \(x = r \cos \theta, y = r \sin \theta,\) and \(z = r^2 + 1,\) so we have the parameterization \(\vec{r}(r, \theta) = (r \cos \theta, r \sin \theta, r^2 + 1).\)

Here is a picture of this parameterization:

![Parameterization Picture 2](image2)
The gray curves are where $r$ is constant, and the black curves are where $\theta$ is constant.

iii. Yet another possibility is to think of slicing (or taking cross-sections or traces). This approach is slightly more difficult than the previous one, but it’s also more flexible. Looking at the surface, we know that taking traces in $z = k$ gives us circles, and circles are curves that we know how to parameterize. If we imagine slicing at a particular $z$-value, then the slice is the circle $x^2 + y^2 = z - 1$, which is a circle centered at $(x, y) = (0, 0)$ with radius $\sqrt{z - 1}$. Therefore, we know that $x$ and $y$ can be described by $x = \sqrt{z - 1} \cos t, y = \sqrt{z - 1} \sin t$. This gives the parameterization $\vec{r}(z, t) = \langle \sqrt{z - 1} \cos t, \sqrt{z - 1} \sin t, z \rangle$.

Here is a picture of this parameterization:

The gray curves are where $z$ is constant, and the black curves are where $t$ is constant.
(b) If we only want to parameterize the part of the elliptic paraboloid under the plane $z = 10$, what restrictions would you place on the parameters you used in (a)?

Solution.

i. For the parameterization $\vec{r}(u, v) = (u, v, u^2 + v^2 + 1)$, we need to restrict $u$ and $v$. Since we want $z$ (the last component) to be less than 10, we need $u^2 + v^2 < 9$.

ii. For the parameterization $\vec{r}(r, \theta) = (r \cos \theta, r \sin \theta, r^2 + 1)$, we need to restrict $r$ and $\theta$. Since the paraboloid was written as $z = r^2 + 1$ in cylindrical coordinates and we want $z < 10$, we need $r < 3$. We know that $\theta$ can be anything, so our restrictions are $0 \leq r < 3, 0 \leq \theta < 2\pi$.

iii. For the parameterization $\vec{r}(z, t) = (\sqrt{z - 1} \cos t, \sqrt{z - 1} \sin t, z)$, we need to restrict $z$ and $t$. We already know that we want $z < 10$. Looking at the paraboloid, we also want $z \geq 1$,\(^{(1)}\)

Looking back, we used $t$ to parameterize a circle, and the parameterization we chose means $0 < t < 2\pi$ is a good restriction. So, for this parameterization, we have $1 \leq z < 10, 0 \leq t < 2\pi$.

2. Parameterize the plane that contains the 3 points $P(1,0,1)$, $Q(2,-2,2)$, and $R(3,2,4)$.

Solution. One way to parameterize the plane is to let the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ define our grid. We can think of $P$ as an “origin” for the plane and the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ as a set of “axes” for the plane. That is, we can reach any point in the plane by starting at $P$, going in the direction of $\overrightarrow{PQ}$ for a while, and then going in the direction of $\overrightarrow{PR}$ for a while.

In this case, $\overrightarrow{PQ} = (1,-2,1)$ and $\overrightarrow{PR} = (2,2,3)$, so our parameterization is

$$ \vec{r}(u, v) = (1,0,1) + u(1,-2,1) + v(2,2,3) = (1 + u + 2v, -2u + 2v, 1 + u + 3v) $$

(You can think of this as saying: start at $P(1,0,1)$, go off in the direction of $\overrightarrow{PQ} = (1,-2,1)$ for a bit — how long is determined by $u$ — and then go off in the direction of $\overrightarrow{PR} = (2,2,3)$ for a bit.)

Alternatively, you could find the equation of the plane (see the worksheet “Lines and Planes”) — it is $8x + y - 6z = 2$. Then, we can write any one of the variables in terms of the other two and use those other two as parameters. For instance, $y = 2 - 8x + 6z$ expresses $y$ in terms of $x$ and $z$. If we want to use $u$ and $v$ as our parameters, then we can just have $x = u, z = v$, and $y = 2 - 8u + 6v$, which gives the parameterization $\vec{r}(u, v) = (u, 2 - 8u + 6v, v)$.

Of course, there are many other parameterizations. One way to check whether your parameterization is reasonable is to remember that you are supposed to be parameterizing $8x + y - 6z = 2$. So, however you parameterize, this relationship should be satisfied. For instance, in our first parameterization $\vec{r}(u, v) = (1 + u + 2v, -2u + 2v, 1 + u + 3v)$, you can easily check that $8(1 + u + 2v) + (-2u + 2v) - 6(1 + u + 3v) = 2$.

3. Parameterize the hyperboloid $x^2 - 4y^2 + z^2 = 1$.

Solution. The traces in $y = k$ of this surface will be circles. In particular, since $x^2 + z^2 = 1 + 4y^2$, the trace in $y = k$ will be a circle centered at $(x, z) = (0, 0)$ with radius $\sqrt{1 + 4y^2}$. We can parameterize this by taking $x = \sqrt{1 + 4y^2} \cos u, z = \sqrt{1 + 4y^2} \cos u$ with $0 \leq u < 2\pi$. Our other parameter is just $y$; if we rename it $v$, then we have the parameterization $\vec{r}(u, v) = (\sqrt{1 + 4v^2} \cos u, v, \sqrt{1 + 4v^2} \sin u)$ with $0 \leq u < 2\pi$ ($v$ can be anything).

\(^{(1)}\) Notice that this is implied in our parameterization since $\sqrt{z - 1}$ is not defined if $z < 1$. 

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Here is a picture of the parameterization.

The gray curves are where \( u \) is constant, and the black curves are where \( v \) is constant.

4. **Parameterize the ellipsoid** \( 9x^2 + 4y^2 + z^2 = 36 \).

**Solution.** There is not much work to do here if we take a clever approach. Let’s rewrite the given equation as \( \left( \frac{x}{\frac{3}{\sqrt{2}}} \right)^2 + \left( \frac{y}{\frac{2}{\sqrt{2}}} \right)^2 + \left( \frac{z}{\frac{6}{\sqrt{2}}} \right)^2 = 1 \). We know that \( X^2 + Y^2 + Z^2 = 1 \) can be parameterized as \( X = \sin \phi \cos \theta, Y = \sin \phi \sin \theta, \) and \( Z = \cos \phi, \) and if we think of \( X \) as being \( \frac{x}{\frac{3}{\sqrt{2}}} \), \( Y \) as being \( \frac{y}{\frac{2}{\sqrt{2}}} \), and \( Z \) as being \( \frac{z}{\frac{6}{\sqrt{2}}} \), then we get \( x = 2 \sin \phi \cos \theta, y = 3 \sin \phi \sin \theta, z = 6 \cos \phi \). That is, our parameterization is \[ \vec{r}(\theta, \phi) = (2 \sin \phi \cos \theta, 3 \sin \phi \sin \theta, 6 \cos \phi) \] with \( 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi \).

Here is a picture of the parameterization.
The gray curves are where $\theta$ is constant, and the black curves are where $\phi$ is constant.

5. Consider the curve $z = 2 + \sin y$, $0 \leq y \leq 4\pi$ in the $yz$-plane. Let $S$ be the surface obtained by rotating this curve about the $y$-axis. Find a parameterization of $S$.

Solution. The traces in $y = k$ of this surface will be circles. In particular, if we look at the trace in $y = k$, we see a circle centered at $(x, z) = (0, 0)$ with radius $2 + \sin y$. We can parameterize this by taking $x = (2 + \sin y) \cos t$, $z = (2 + \sin y) \sin t$ with $0 \leq t < 2\pi$. Thus, a parameterization of the surface is $\mathbf{r}(y, t) = ((2 + \sin y) \cos t, y, (2 + \sin y) \sin t)$ with $0 \leq y \leq 4\pi$, $0 \leq t < 2\pi$.

Here is a picture of the parameterization.

The gray curves are where $y$ is constant, and the black curves are where $t$ is constant.
6. Here are three surfaces.

Match each function with the surface it parameterizes. Which curves are where \( u \) is constant and which curves are where \( v \) is constant?

(a) \( \vec{r}(u, v) = \left( \cos \frac{u}{4} + \cos v, \frac{\sin u}{4} + \sin v, v \right), 0 \leq u \leq 2\pi, 0 \leq v \leq 4\pi. \)

**Solution.** If \( u \) is a constant, then \( \vec{r}(u, v) \) has the form \( \langle C_1 + \cos v, C_2 + \sin v, v \rangle \) where \( C_1 \) and \( C_2 \) are constants. You should recognize this as a helix (remember \( \langle \cos t, \sin t, t \rangle \)), shifted.

On the other hand, if \( v \) is a constant, then \( \vec{r}(u, v) \) has the form \( \langle \frac{1}{4} \cos u + C_1, \frac{1}{4} \sin u + C_2, C_3 \rangle \) where \( C_1 \), \( C_2 \), and \( C_3 \) are constants. You should recognize this as parameterizing a circle (which is parallel to the \( xy \)-plane since the \( z \)-component does not vary with \( u \)).

The surface which has helices as one set of grid curves and circles as the other is (II). The grid curves with \( u \) constant are shown in red; the grid curves with \( v \) constant are shown in blue.

(b) \( \vec{r}(u, v) = \left( \cos u, \sin u, u + \frac{v}{4} \right), 0 \leq u \leq 4\pi, 0 \leq v \leq 2\pi. \)

**Solution.** If \( u \) is a constant, then \( \vec{r}(u, v) \) has the form \( \langle C_1, C_2, C_3 + \frac{v}{4} \rangle \) where \( C_1 \), \( C_2 \), and \( C_3 \) are constants. This simply parameterizes a vertical line segment (of length \( \frac{\pi}{2} \) since \( v \) varies between 0 and \( 2\pi \)).

On the other hand, if \( v \) is a constant, then \( \vec{r}(u, v) \) has the form \( \langle \cos u, \sin u, u + C \rangle \) where \( C \) is a constant. This parameterizes a helix.

The surface which has vertical line segments as one set of grid curves and helices as the other is (I). The grid curves with \( u \) constant are shown in red; the grid curves with \( v \) constant are shown in blue.

(c) \( \vec{r}(u, v) = \langle u \cos v, u \sin v, uv \rangle, 0 \leq u \leq 2\pi, 0 \leq v \leq 4\pi. \)

**Solution.** If \( u \) is a constant, then \( \vec{r}(u, v) \) has the form \( \langle C \cos v, C \sin v, Cv \rangle = C \langle \cos v, \sin v, v \rangle \) where \( C \) is a constant. This parameterizes a helix. Notice that, unlike in the two previous parts, the helices here can be different sizes.

On the other hand, if \( v \) is a constant, then \( \vec{r}(u, v) \) has the form \( \langle C_1 u, C_2 u, C_3 u \rangle \) where \( C_1 \), \( C_2 \), and \( C_3 \) are constants. You should recognize this as parameterizing a line segment; since \( u \) starts at 0, this line segment always starts at the origin.

The matching surface is (III). The grid curves with \( u \) constant are shown in red; the grid curves
with $v$ constant are shown in blue.