

Cuspidal Representations of p -adic groups

Mikayel Mkrtchyan

Harvard University

1. INTRODUCTION

1.1. These are the notes for the author's talks at the Langlands Support Group seminar at Harvard on February 20 and March 13, 2019. In these talks, the notions of compact and cuspidal representations of p -adic groups were introduced, thus facilitating the first steps in the program of reducing the problem of classifying all smooth irreducible representations of a p -adic group to the problem of classifying cuspidal representations.

1.2. **Notation.** Throughout the note, F will denote a p -adic field with ring of integers \mathcal{O} and fixed uniformizer π . G will always denote the F -points of a reductive group. All representations will be assumed to be smooth, and $\text{Rep}(G)$ will denote the category of smooth representations of G .

We will be following Bernstein's course notes [Ber].

2. BRIEF REFRESHER ON GENERALITIES ON G -REPRESENTATIONS

In this section, G is any p -adic group.

Definition 2.0.1. Let V be a \mathbb{C} -vector space. A continuous representation $\rho : G \rightarrow \text{GL}_n(V)$ is called smooth if for any $v \in V$, the stabilizer $\text{Stab}_G(v) \subset G$ is open.

Smoothness is thought of as a reasonable finiteness condition. This is elucidated by the following

Proposition 2.0.2. A continuous representation V is smooth if and only if $V = \bigcup_{K \subset G} V^K$, where $K \subset G$ varies through the set of all compact subgroups of G .

From now on, we will restrict our attention to smooth representations.

Definition 2.0.3. A smooth representation V is called admissible if for any open compact subgroup $K \subset G$, the space of invariants V^K is finite-dimensional.

Admissibility allows one to study the representation "finite-dimensional piece at a time". Recall further the definition of the Hecke algebras associated to G :

Definition 2.0.4. Let $K \subset G$ be an open compact subgroup. The corresponding Hecke algebra \mathcal{H}_K is defined to be the algebra of locally constant, compactly supported, bi- K -invariant distributions on G , viewed as an algebra under convolution. The Hecke algebra for $K = \{1\}$ is denoted \mathcal{H} and called the Hecke algebra of G .

The key property of Hecke algebras is

Proposition 2.0.5. For any compact open subgroup $K \subset G$, there is an equivalence of categories

$$\text{Rep}(G)_{K\text{-inv}} \simeq \mathcal{H}_K\text{-mod}$$

where $\text{Rep}(G)_{K\text{-inv}}$ denotes the full subcategory of $\text{Rep}(G)$ consisting of representations containing a non-zero vector fixed by all elements of K . The functor in one direction sends $V \in \text{Rep}(G)_{K\text{-inv}}$ to V^K .

We will use juxtaposition or $*$ interchangeably to denote the multiplication on \mathcal{H} and \mathcal{H}_K . For a compact open subgroup $K \subset G$, $e_K \in \mathcal{H}$ will denote the unique bi- K -invariant distribution supported on K with integral 1. For $g \in G$, let $\delta_g \in \mathcal{H}$ denote the delta distribution supported at g . The following proposition follows straightforwardly from the definitions:

Proposition 2.0.6.

- (1) For any compact open subgroups $K_1, K_2 \subset G$, $e_{K_1 K_2} = e_{K_1} * r_{K_2}$.
- (2) For any $g \in G$ and compact open $K \subset G$, $e_{g K g^{-1}} = \delta_g * e_K * \delta_{g^{-1}}$.

3. COMPACT REPRESENTATIONS

In this section, we will develop the theory of *compact* representations - a simpler version of cuspidal representations that we will make use of in the sequel.

Throughout this section, we will be assuming that G is any unimodular, countable at infinity group.

3.1. Recall that the *matrix coefficient* of a G -representation (ρ, V) associated to $v \in V$ and $\tilde{v} \in \tilde{V}$ is the function $D_{v, \tilde{v}} : G \rightarrow \mathbb{C}$ defined by

$$g \mapsto \langle \tilde{v}, \rho(g^{-1})v \rangle.$$

The smoothness assumption on V guarantees that $m_{v, \tilde{v}}$ is locally constant.

Definition 3.1.1. A representation V of G is called *compact* if all its matrix coefficients are compactly supported, i.e. for all $v \in V$ and $\tilde{v} \in \tilde{V}$, the function $m_{v, \tilde{v}}$ has compact support.

Compact representations behave similarly to representations of compact groups. In particular, we will show that they are semisimple, and copies of irreducible compact representations can be split off as direct summands from any representation. Of course, both these properties fail for general non-compact representations.

The following is a convenient technical characterization of compact representations.

Proposition 3.1.2. Let (ρ, V) be a G -representation. Then V is compact if and only if for any open compact subgroup $K \subset G$ and $v \in V$, the function $D_{K, v} : G \rightarrow V$ defined by

$$v \mapsto \rho(e_K)\rho(g)v$$

has compact support.

Proof. Fix $v \in V$ and an open compact subgroup $K \subset G$. It suffices to prove that the image of $D_{v, K}$ is finite-dimensional, as the contrary would directly lead to a matrix coefficient with non-compact support. If this was false, then there would exist an sequence of group elements $\{g_i\}$ such that the $v_i = D_{v, K}(g_i)$ are linear independent. Since the function $D_{v, K}$ is locally constant, this implies that the $\{g_i\}$ are not contained in any compact set. Define $\tilde{v} \in \tilde{V}^K$ by $\langle \tilde{v}, v_i \rangle = 1$ and extend by zero. Then $\{g_i\} \subset \text{supp}(m_{v, \tilde{v}})$, which contradicts the assumption that all matrix coefficients are compactly supported.

Conversely, let (ρ, V) be compact, and choose any $v \in V$ and $\tilde{v} \in \tilde{V}$; let K be a compact open subgroup stabilizing v . Then the support of $m_{v, \tilde{v}}$ is contained in the support of $D_{K, v}$, which is compact by assumption. \square

Proposition 3.1.3. Finitely generated compact representations are admissible.

Proof. Let (ρ, V) be any compact representations, generated by some v_1, \dots, v_n . Then $V^K = e_K V$ is generated by $\{\rho(e_K)\rho(g)v_i \mid g \in G, i \in \{1, \dots, n\}\}$. For each i , the function $g \mapsto \rho(e_K)\rho(g)v_i$ has compact support by assumption, hence the space $\{\rho(e_K)\rho(g)v_i \mid g \in G\}$ is finite-dimensional and hence V^K is finite-dimensional. \square

Corollary 3.1.4. *Irreducible compact representations are admissible.*

We will later prove that all irreducible representations are admissible.

3.2. The formal dimension.

Proposition 3.2.1. *Let (ρ, W) be any compact representation of G . Then the natural map $\alpha : W \otimes \widetilde{W} \rightarrow (\text{End}_{\mathbb{C}}(W))_{sm}$ defined by $\alpha(v \otimes \widetilde{v})(u) = \langle \widetilde{v}, u \rangle v$ is an isomorphism of $G \times G$ -modules.*

Here $\text{End}_{\mathbb{C}}(W)$ is a $G \times G$ -module in the usual way, using the action $(g_1, g_2)(f) = \rho(g_1)f\rho(g_2)^{-1}$.

Proof. Injectivity is clear. For surjectivity, note that both sides are finitely generated compact representations of $G \times G$, hence are admissible by Proposition 3.1.3. This allows us to deduce surjectivity from injectivity by working at a finite level at a time: for any compact open subgroup $K \subset G$, the image of $\alpha((W \otimes \widetilde{W})^{K \times K})$ lies in $(\text{End}_{\mathbb{C}}(W))^{K \times K}$, and since α is injective, this implies that $\dim((W \otimes \widetilde{W})^{K \times K}) \leq \dim(\text{End}_{\mathbb{C}}(W))^{K \times K} = \dim(\text{End}_B C(W)_{sm})^{K \times K}$. On the other hand, $\text{End}(W)^{K \times K}$ is contained in $\text{End}(W^K)$, which implies that

$$\dim(\text{End}_{\mathbb{C}}(W)^{K \times K}) \leq \dim(\text{End}_{\mathbb{C}}(W^K)) = \dim(W^K)^2 \leq \dim(W \otimes \widetilde{W})^{K \times K},$$

which implies that $\alpha : (W \otimes \widetilde{W})^{K \times K} \rightarrow \text{End}_{\mathbb{C}}(W)^{K \times K}$ is an isomorphism. Since both $W \otimes \widetilde{W}$ and $\text{End}_{\mathbb{C}}(W)_{sm}$ are smooth, the claim follows. \square

Fix a two-sided Haar μ_G on G ; this gives an algebra isomorphism between \mathcal{H} and the algebra of locally constant, compactly supported functions $G \rightarrow \mathbb{C}$. Let (ρ, W) be an irreducible compact representation, and consider the composition

$$W \otimes \widetilde{W} \xrightarrow{v \otimes \widetilde{v} \mapsto m_{v, \widetilde{v}}} \mathcal{H} \xrightarrow{\rho} (\text{End}_{\mathbb{C}} W)_{sm} \xrightarrow{\alpha^{-1}} W \otimes \widetilde{W}.$$

Let us denote the map $W \otimes \widetilde{W} \rightarrow \mathcal{H}$ by m and the composition $\mathcal{H} \rightarrow W \otimes \widetilde{W}$ by φ . The composition $W \otimes \widetilde{W} \rightarrow W \otimes \widetilde{W}$ defines a map of $G \times G$ -modules; since $W \otimes \widetilde{W}$ is irreducible, this composition must be a scalar, which we will denote $d(\rho)$ and call the *formal dimension* of (ρ, W) .

Lemma 3.2.2. *The formal dimension $d(\rho)$ of any (non-zero) compact representation (ρ, W) is non-zero.*

Proof. Pick any $w \in W \otimes \widetilde{W}$ be any element such that $h = m(w)$ is non-zero (where $m : W \otimes \widetilde{W} \rightarrow \mathcal{H}$ is the matrix coefficient map); it suffices to prove that $\varphi(h) \neq 0$. We will prove this by proving the following two facts: for any irreducible representation (τ, V) not isomorphic to (ρ, W) , $\tau(h) = 0$, and for any non-zero $x \in \mathcal{H}$, there exists an irreducible representation (π, U) such that $\pi(x) \neq 0$.

To prove the first claim, let (τ, V) be any irreducible representation of G not isomorphic to ρ , and consider any $v \in V$. Consider the morphism of G -modules

$$\begin{aligned} W \otimes \widetilde{W} &\rightarrow V \\ w \otimes \widetilde{w} &\mapsto \tau(m(w \otimes \widetilde{w}))v, \end{aligned}$$

where $W \otimes \widetilde{W}$ is considered a G -module via its usual action on the first component and the trivial action on the second. With this G -module structure, $W \otimes \widetilde{W}$ is isomorphic to a direct sum of copies of W , which implies that the same is true of the image of $W \otimes \widetilde{W}$ in V . Since V is irreducible and not isomorphic to W , this implies that the image is zero, and in particular, $\tau(h)v = 0$. This proves that $\tau(h) = 0$.

The second claim is Lemma 3.2.3, which we will (independently) prove below. \square

Lemma 3.2.3. *For any $0 \neq h \in \mathcal{H}$, there exists an irreducible representation (ρ, V) of \mathcal{H} such that $\rho(h) \neq 0$.*

This lemma may be thought of as establishing that the Hecke algebra of G behaves somewhat like a semisimple algebra. Before proving it, we shall require a purely algebraic lemma.

Lemma 3.2.4. *Let A be an associative \mathbb{C} -algebra with unit, of countable dimension over \mathbb{C} . Then for any non-nilpotent $a \in A$, there exists a simple A -module M such that $a|_M \neq 0$.*

Remark 3.2.5. The non-nilpotency condition cannot be relaxed: the only simple module over $A = \mathbb{C}[x]/(x^2)$ is $M = \mathbb{C}$, with x acting as 0.

Proof. First let us see that there exists some $\lambda \in \mathbb{C}$ such that $a - \lambda$ is not invertible in A . This is evident if $a \in \mathbb{C}$; if not, assume the contrary, and consider the elements $\{(a - \lambda)^{-1}, \mu \in \mathbb{C}\}$. Since A has countable dimension over \mathbb{C} , there exists some linear dependence of the form

$$\sum_{i=1}^k c_i (a - \mu)^{-1} = 0.$$

Multiplying by $\prod_{i=1}^k (a - \lambda_i)$ and factoring the resulting polynomial tells us that

$$a^{m_0} \prod_j (a - \lambda_j) = 0$$

for some $\lambda_i \in \mathbb{C}$. Since a is not nilpotent, this implies that the $a - \lambda_i$ are zerodivisors and hence not invertible, as desired.

To prove the lemma, pick any λ such that $a - \lambda$ is not invertible, and let M be any irreducible quotient of $A/(a - \lambda)A$ (which is non-zero precisely because $a - \lambda$ is not invertible). Then $a|_M = \lambda|_M \neq 0$. \square

Proof of Lemma 3.2.3. Recall that G is assumed to be unimodular; this allows us to write $h = \phi \mu_G$, where μ_G is a two-sided Haar measure on G and ϕ is a compactly supported, locally constant function $G \rightarrow \mathbb{C}$. Define $h^\vee = \phi^\vee \mu_G$, where $\phi^\vee(g) = \overline{\phi(g^{-1})}$. Then $\mu := hh^\vee = \psi \mu_G$, where

$$\psi(g) = \int_{a \in G} \phi(a) \overline{\phi(ga)} da.$$

Setting $g = 1$, we see that $\mu \neq 0$. Moreover, an identical argument proves that $\mu^2 \neq 0$, and in general, $\mu^i \neq 0$ for any $i \in \mathbb{N}$. Thus, μ is a non-nilpotent element of \mathcal{H}_K , and it suffices to find a representation of \mathcal{H}_K that doesn't map μ to 0. Since \mathcal{H}_K is countable over \mathbb{C} and has a unit element e_K , the lemma follows from Lemma 3.2.3. \square

3.3. Semisimplicity of compact representations. In this section, we will prove

Theorem 3.3.1. *Let (ρ, W) be a compact irreducible representation of G . Then for any $V \in \text{Rep}(G)$, there exists a decomposition $V \simeq V_W \oplus V_W^\perp$, where $V_W \simeq W^{\oplus I}$ and no Jordan-Holder factor of V_W^\perp is isomorphic to W .*

The theorem is clearly implied by the following proposition:

Proposition 3.3.2. *Let (ρ, W) be an irreducible compact representation of G , and let m and φ be as in the previous section; let (η, V) be any representation of G . Define*

$$\begin{aligned} \mathcal{E}_{W,K} &= d(\rho)^{-1}m(\varphi(e_K)) \\ V_0 &= \sum_{K \subset G} \text{Im}(\eta(\mathcal{E}_{W,K})) \\ V_1 &= \text{Ker}_{K \subset G}(\eta(\mathcal{E}_{W,K})) \end{aligned}$$

where K runs through all compact open subgroups of G . Then

- (1) V_0 and V_1 are G -submodules of V
- (2) $V = V_0 \oplus V_1$
- (3) V_1 does not have subquotients isomorphic to W
- (4) V_0 is isomorphic to a direct sum of copies of W .

Proof. Parts (1) and (2) are evident. For (3), note that by definition of formal dimension, $\varphi(\mathcal{E}_{W,K}) = \varphi(e_K)$ for any compact open subgroup $K \subset G$. This implies that if V_1 contained a subquotient isomorphic to W , then $\mathcal{E}_{W,K}$ would not act by 0 on that subquotient. However, this is false by construction.

To prove part (4), it suffices to show that V_0 is generated by its submodules that are isomorphic to W . By construction, it is generated by images of maps $W \otimes \widetilde{W} \rightarrow V_0$ defined by, for $v_0 \in V_0$, $w \otimes \widetilde{w} \mapsto \rho(m(w \otimes \widetilde{w}))v_0$. Since $W \otimes \widetilde{W}$ is isomorphic to a direct sum of copies of W (under the trivial G -action on the second factor, as in the proof of Lemma 3.2.2), the claim follows. \square

4. CUSPIDAL REPRESENTATIONS

From here onwards, $G = \text{GL}_n(F)$.

4.1. The structure of \mathcal{H}_K . Let K be an open compact subgroup of G . There is a map of sets $a : G \rightarrow H_K$ defined by $a(g) = e_K * \delta_g * e_K$. $a(g)$ is the unique K -bi-invariant distribution supported on KgK with integral 1. $a(g)$ only depends on the double coset of $g \bmod K$, and as g varies through double coset representatives, the $a(g)$ form a linear basis of \mathcal{H}_K .

Thus, to study the algebra \mathcal{H}_K we need to study double cosets. The fundamental result to this end is the *Cartan decomposition*. Consider the subgroup

$$\Lambda^+ = \begin{pmatrix} \pi^{l_1} & & \\ & \ddots & \\ & & \pi^{l_n} \end{pmatrix}, \text{ where } l_1 \geq l_2 \geq \dots \geq l_n \in \mathbb{Z}.$$

The Cartan decomposition is $G = K_0 \Lambda^+ K_0$.

Now let $K = K_i$ be a congruence subgroup, and choose representatives x_1, \dots, x_r for $K \backslash K_0$. The Cartan decomposition implies that $a(x_i \lambda x_j)$, $i, j \in \{1, \dots, r\}$, $\lambda \in \Lambda^+$ form a basis for \mathcal{H}_K . Moreover, since K_0 normalizes K , we have that $x_i K = K x_i$ and hence for any $g \in G$, $y K x_i K \cdot K g K = K x_i g K$. This implies that $a(x_i) a(g) = a(x_i g)$; similarly, $a(g x_i) = a(g) a(x_i)$.

Let \mathcal{H}_0 denote the linear span of $a(x_i)$, and C denote the linear span of $\{a(\lambda), \lambda \in \Lambda^+\}$.

Theorem 4.1.1. *C is a finitely generated commutative subalgebra of \mathcal{H}_K , and $\mathcal{H}_K = \mathcal{H}_0 C \mathcal{H}_0$.*

Before proving the theorem, note that its second part follows directly from the preceding paragraph: the $a(x_i \lambda x_j)$ form a basis for \mathcal{H}_K as $\lambda \in L^+$ and $i, j \in \{1, \dots, r\}$ vary, and since $a(x_i \lambda x_j) = a(x_i) a(\lambda) a(x_j)$, the claim follows.

The crux of the matter for the first claim lies in the fact that elements of Λ^+ do not normalize K and hence we cannot simply move the K 's around. We will divide K into subgroups that we can move around, and this will be sufficient to prove the claim.

Let U be the standard unipotent subgroup of G consisting of upper-triangular matrices with 1 on the diagonal, and let U_- be the opposite unipotent, consisting of lower-triangular matrices with 1 on the diagonal. Let $M_0 \subset G$ be the subgroup of diagonal matrices, and consider $K_+ = K \cap U, K_- = K \cap U_-, K_d = K \cap M_0$.

Proposition 4.1.2. *$K = K_+ K_d K_-$.*

Proposition 4.1.3. *For any $\lambda \in \Lambda^+$, $\lambda K_+ \lambda^{-1} \subset K_+$ and $\lambda^{-1} K_- \lambda \subset K_-$.*

Proof. For any $u = u_{i,j} \in K_+, (\lambda u \lambda^{-1})_{i,j} = \lambda_i \lambda_j^{-1} u_{i,j}$. Since $\lambda_i \lambda_j^{-1} \in \mathcal{O}$ for $j \geq i$, $\lambda u \lambda^{-1} \in K_+$ and thus $\lambda K_+ \lambda^{-1} \subset K_+$. The other case follows identically. \square

Proof of theorem 4.1.1. We need to show that for any $\lambda, \nu \in \Lambda^+, K \lambda K \mu K = K \lambda \mu K$. By Proposition 4.1.3, we have that

$$\begin{aligned} K \lambda K \mu K &= K \lambda K_+ K_d K_i \mu K = K (\lambda K_+ \lambda^{-1}) \lambda K_d \mu (\mu^{-1} K_- \mu) K \\ &\subseteq K K_+ \lambda K_d \mu K_- K \\ &= K \lambda \mu K. \end{aligned}$$

The reverse inclusion always holds, which concludes the proof. \square

Now we will use this decomposition of the algebra \mathcal{H}_K to study modules over it.

Proposition 4.1.4. *For any $V \in \text{Rep}(G)$ and $\lambda \in \Lambda^+$,*

$$\ker(a(\lambda)|_{V^K}) = \ker(e_{\lambda^{-1} K_+ \lambda}|_{V^K}).$$

Proof. We directly calculate

$$\begin{aligned} a(\lambda) &= e_K * \delta_\lambda * e_K = e_{K_+} * e_{K_d} * e_{K_-} * \delta_\lambda * e_K \\ &= e_{K_+} * e_{K_d} * \delta_\lambda * e_{\lambda^{-1} K_- \lambda} * e_K \\ &= e_{K_+} * \delta_\lambda * e_{K_d} * e_K \\ &= \delta_\lambda * e_{\lambda^{-1} K_+ \lambda} * e_K, \end{aligned}$$

where we used that $\lambda^{-1} K_- \lambda$ is contained in K . It remains to observe that e_K acts as the identity on V^K , and δ_g acts invertibly. \square

According to this theorem, to study the behaviour of the operators $a(\lambda)$, we need to understand the subgroups $\lambda^{-1} K_+ \lambda$. Consider the special case $\lambda = \text{diag}(\pi^{l_1}, \dots, \pi^{l_n})$, where $l_1 > l_2 > \dots > l_n$ (in general, we might have equalities). Then the valuation of $\lambda_i^{-n} \lambda_j^n$ gets arbitrarily small as n becomes large, and it is straightforward to show that

$$\bigcup_i \lambda^{-n} K_+ \lambda^n = U.$$

Proposition 4.1.5. $\bigcup_i \ker(e_{U_i}) = \{\pi(u)v - v\} =: V(U)$.

Proof. Follows from $\bigcup_i \lambda^{-n} K_+ \lambda^n = U$ and Proposition 4.1.4. \square

Corollary 4.1.6. $\bigcup_i \ker a(\lambda^i) \cup V^K = V(U) \cup V^K.$

For general $\lambda = \text{diag}(\pi^{l_1}, \dots, \pi^{l_n})$ with

$$l_1 = \dots = l_{a_1} > l_{a_1+1} = \dots = l_{a_1+a_2} > \dots > l_{n-a_k+1} = \dots = l_n,$$

the correct replacement for U is the unipotent radical of the parabolic P_λ :

$$P_\lambda = \begin{pmatrix} M_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & M_K \end{pmatrix}, U_\lambda = \begin{pmatrix} \text{id}_{a_1 \times a_1} & * & * \\ 0 & \ddots & * \\ 0 & 0 & \text{id}_{a_k \times a_k} \end{pmatrix}$$

where each M_i is an $a_i \times a_i$ invertible matrix; in other words, elements of P_λ and U_λ are block-upper-diagonal and block-unipotent, respectively.

Letting $K_+^P = K \cap U_P$, $K_-^P = K \cap \bar{U}_P$, one proves the following proposition similarly to the previous several propositions:

Proposition 4.1.7. *For any $V \in \text{Rep}(G)$ and $\lambda \in \Lambda^+$,*

- (1) $K = K_+^P K_d^P K_-^P$, $\lambda K_+^P \lambda^{-1} \subset K_+^P$, and $\lambda^{-1} K_-^P \lambda \subset K_-^P$.
- (2) $\bigcup_i \lambda^{-i} K_+^P \lambda^i = U_\lambda$.
- (3) $\bigcup_i \ker a(\lambda^i) \cup V^K = V(U_\lambda) \cup V^K$.

4.2. Jacquet functors and induction. Motivated by the preceding discussion, we give the following definitions.

Let $P = MU$ be any parabolic subgroup of G with its Levi decomposition.

Definition 4.2.1. *The restriction or Jacquet functor $r_{M,G} : \text{Rep}(G) \rightarrow \text{Rep}(M)$ is defined to be*

$$V \mapsto V/V(U),$$

The induction functor $i_{G,M} : \text{Rep}(M) \rightarrow \text{Rep}(G)$ is defined to be

$$W \mapsto \text{ind}_P^G(W),$$

where ind_P^G is usual functor of induction, and W is considered a P -representation by letting U acts trivially.

Note that since P is cocompact (by the Iwasawa decomposition $G = PK_0$), the two induction functors coincide. Here are some basic properties of these functors:

Proposition 4.2.2. *Let $P = MU$ be a parabolic subgroup of G .*

- (1) $i_{G,M}$ is right adjoint to $r_{M,G}$.
- (2) For any Levi subgroup N of M , $r_{N,M} \circ r_{M,G} = r_{N,G}$ and $i_{G,N} \simeq i_{G,M} \circ i_{M,N}$.
- (3) $i_{G,M}$ takes admissible representations to admissible representations.
- (4) $r_{M,G}$ takes finitely generated representations to finitely generated representations.
- (5) Both $i_{G,M}$ and $r_{M,G}$ are exact.

Proof.

- (1) For any $V \in \text{Rep}(G)$ and $W \in \text{Rep}(G)$, we have

$$\text{Hom}_G(V, i_{G,M}(W)) \simeq \text{Hom}_P(V, W) \simeq \text{Hom}_M(V/V(U), W) = \text{Hom}_M(r_{M,G}(V), W),$$

where the first isomorphism holds by Frobenius reciprocity, the second isomorphism holds since U acts trivially on W , and the last isomorphism holds by definition.

- (2) This is straightforward to check by unwinding the definitions.
(3) This holds for induction from any cocompact subgroup $H \subset G$: by definition, for any admissible representation (ρ, W) of H , $\text{ind}_P^G(W)$ is the smooth part of

$$L(W) := \{f : G \rightarrow W \mid \forall h \in H, f(hg) = \rho(h)f(g)\},$$

where g acts on the right. For any compact open subgroup $K \subset G$, by cocompactness of H there exist finitely many coset representatives g_1, \dots, g_N for $H \backslash G / K$. This implies that any $f \in L(W)^K$ is determined by its value on the g_i . Moreover, by the transformation property under action of H , we must have $f(g_i) \in V^{H \cap g_i K g_i^{-1}}$, which is finite-dimensional by assumption. This implies that there is a finite-dimensional space of choices for $f \in L(W)^K$, which proves admissibility.

- (4) Recall that $G = PK_0$, by IIlet (π, V) be any finitely generated representation of G ; let us prove that it is finitely generated as a P -representation. Letting v_1, \dots, v_N be G -generators of V , we have that $V|_P$ is generated by the vectors of the form $\pi(k)v_i$, where $k \in K_0$. Since (ρ, V) is smooth, the function $k \mapsto \pi(k)v_i$ is locally constant. Since K_0 is compact, finitely many choices of k generate the whole image, which guarantees that the finitely many $\pi(k)v_i$ generate V .
(5) The functor $i_{G,M}$ is exact because induction is exact. To prove that $r_{M,G}$ is exact, it suffices to prove that $V \mapsto V/V(U)$ is exact. This holds because any U has an exhausting filtration by compact subgroups, and for compact K , the functor $V \mapsto V/V(K)$ is exact.

□

4.3. Cuspidal representations. The restriction and induction functors are used to study representations of G using representations of its reductive groups, thus simplifying the problem. The cuspidal representations are precisely those which cannot be reduced to smaller groups in this sense:

Definition 4.3.1. $V \in \text{Rep}(G)$ is called quasi-cuspidal if for any Levi subgroup $M \subset G$, $r_{M,G}(V) = 0$. V is called cuspidal if it is quasi-cuspidal and finitely generated.

The following is a crucial technical characterization of quasi-cuspidal representations.

Theorem 4.3.2. $V \in \text{Rep}(G)$ is quasi-cuspidal if and only if for any $v \in V$ and compact open subgroup $K \subset G$, the function $D_{v,K} : G \rightarrow V$ defined by

$$g \mapsto \pi(e_K)\pi(g^{-1})v$$

has compact support modulo center (i.e. its image in $G/Z(G)$ is compact).

Proof. Let V be a quasi-cuspidal representation. We may assume that $K = K_i$ is a congruence subgroup, and, by choosing i large enough, that K fixes v .

Let x_1, \dots, x_r be representatives for $K \backslash K_0$, and recall that any $g \in G$ is expressible as $g = x_i \lambda x_j$ for $\lambda \in \Lambda^+$. It suffices to prove that the function $\Lambda^+ \rightarrow V$ defined by

$$\lambda \mapsto \pi(a(\lambda))v$$

has compact support modulo center on Λ^+ . Since V is quasi-cuspidal, for any non-central $\mu \in \Lambda^+$, we have

$$V^K \cap \bigcup_i \ker a(\mu^n) = V(U_\mu) \cup V^K = V^K.$$

This implies that for any $\mu \in \Lambda^+$, there exists a large enough $N_{\mu,v}$ such that $\pi(a(\mu^k))(v) = 0$ for any $k > N_{\mu,k}$. Letting μ_1, \dots, μ_{n-1} be a basis for Λ^+ , we find that for any $\lambda = \sum_i m_i \mu_i$ satisfying $m_i \geq N_{\mu_i,v}$, we have $\pi(\lambda)(v) = 0$. This gives us an upper bound on the m_i and

implies the required compactness mod center, as we can get a bound below mod center by multiplying by a $\text{diag}(\pi^N, \dots, \pi^N)$ for large N .

For the converse, note that the conditions imply that the function $\Lambda^+ \rightarrow V$ defined by $\lambda \mapsto \pi(a(\lambda))(v)$ has compact support modulo center on Λ^+ . For any non-central λ , the sequence $\{\lambda^n\}_{n \in \mathbb{N}}$ eventually leaves all compact-modulo-center subsets of Λ^+ and hence for large enough n , $a(\lambda^n)$ acts trivially. This implies that $V(U_\lambda) \cup V^K = V^K$ holds for all congruence subgroups K , which implies that V is quasi-cuspidal. \square

Remark 4.3.3. Identically to Proposition 3.1.2, one can prove that a representation is quasi-cuspidal if and only if its matrix coefficients are compactly supported modulo center.

This characterization of quasi-cuspidal representations highlights its relation to the compact representations studied earlier.

Consider the subgroup $G^\circ = \{g \in G \mid \det(g) \in \mathcal{O}^\times\}$. For G other than GL_n , G° is defined to be the union of the compact subgroups of G ; correspondingly, we have

Proposition 4.3.4. *G° as defined above is the union of all compact subgroups of GL_n .*

Proof. For any compact subgroup K , the image of the map $g \mapsto |\det(g)|$ is compact, which implies that $\det(g) \in \mathcal{O}^\times$ for all $g \in K$; this proves that every compact subgroup is contained in G° . We omit the proof of the converse. \square

There is an analogue of the Cartan decomposition for G° : $G^\circ = K_0 \Lambda^{+, \circ} K_0$, where $\Lambda^{+, \circ} := \Lambda^+ \cap G^\circ$. Moreover, one can directly exhibit elements ν_1, \dots, ν_l such that $\Lambda^{+, \circ} = \{\sum_i m_i \nu_i \mid m_i \geq 0\}$. We also have $G/G^\circ \simeq F^\times/\mathcal{O}^\times \simeq \mathbb{Z}$ and $[G : ZG^\circ] = n$.

The key property of G° lies in its following interaction with quasi-cuspidality:

Theorem 4.3.5 (Harish-Chandra). *$V \in \text{Rep}(G^\circ)$ is quasi-cuspidal if and only if it is compact.*

Note that the notion of quasi-cuspidality for representations of G° is naturally inherited from that of G .

Proof. Compact representations are tautologically quasi-cuspidal. Conversely, let V be a quasi-cuspidal representation. Showing that V is compact amounts to proving that for any $v \in V$, the function $\Lambda^+ \rightarrow V$ defined by

$$\lambda \mapsto \pi(a(\lambda))v$$

has compact support. As in the proof of Theorem 4.3.2, one shows that any $\lambda = \sum_i m_i \nu_i$ acts trivially if $m_i > N$ for some $N \in \mathbb{N}$, thus giving an upper bound on the support; however, unlike in the case of Λ^+ , for $\Lambda^{+, \circ}$ the lower bound is a given, thus proving that the support in $\Lambda^{+, \circ}$ is compact. \square

Corollary 4.3.6. *Any cuspidal irreducible representation of G is admissible.*

Proof. Let V be any cuspidal irreducible representation. Since $[G : ZG^\circ] < \infty$, $V|_{ZG^\circ}$ is finitely generated; since Z acts by scalars, $V|_{G^\circ}$ is finitely generated.

By Theorem 4.3.5, $V|_{G^\circ}$ is compact, which implies that it is semisimple by [quote theorem]; by [quote other theorem], it is admissible. Since G° contains all compact subgroups of G , the claim follows. \square

Lemma 4.3.7. *For any irreducible representation V of G , there exists a Levi subgroup M , a cuspidal representation W of M and an embedding $V \hookrightarrow i_{G,M}(W)$.*

Proof. Let M be a Levi subgroup minimal with respect to the condition that $W' := r_{M,G}(V) \neq 0$. Transitivity of restriction (Proposition 4.2.2(1)) implies that W' is quasi-cuspidal. Letting W be any irreducible quotient of W' , adjunction implies that

$$\mathrm{Hom}_G(V, i_{G,M}(W)) \simeq \mathrm{Hom}_M(W, W') \neq 0,$$

which implies the lemma. \square

Corollary 4.3.8. *Any irreducible representation of G is admissible.*

Theorem 4.3.9 (Uniform Admissibility). *Fix an open compact subgroup $K \subset G$. Then there exists a number $c(G, K)$ such that for any irreducible representation V of G , $\dim(V^K) \leq c(G, K)$; equivalently, all irreducible representations of \mathcal{H}_K have dimension $\leq c(G, K)$.*

In the proof we shall require the following linear-algebraic lemma, the proof of which can be found under Lemma 4.10 in [BZ]:

Lemma 4.3.10. *Let W be a k -dimensional vector space, and let $C \subset \mathrm{End}(W)$ be a commutative subalgebra, generated by l elements. Then $\dim(C) \leq k^{2 - \frac{1}{2^{l-1}}}$.*

Proof of Theorem 4.3.9. Let W be an irreducible representation of \mathcal{H}_K ; by Corollary 4.3.8, it is finite-dimensional. By Burnside's lemma, the defining map $\rho : \mathcal{H}_K \rightarrow \mathrm{End}_{\mathbb{C}}(W)$ is surjective.

Recall that $\mathcal{H}_K = \mathcal{H}_0 C \mathcal{H}_0$, where $d := \dim(\mathcal{H}_0) = [K_0 : K]$ and C is a commutative algebra generated by l elements. Letting $k = \dim(W)$, we have

$$\begin{aligned} k^2 &= \dim(\mathrm{End}(W)) = \dim(\rho(\mathcal{H}_K)) \\ &\leq d^2 \dim(\rho(C)) \\ &\leq d^2 k^{2 - \frac{1}{2^{l-1}}}, \end{aligned}$$

which implies that $k \leq d^{2^l} =: c(G, K)$. \square

4.4. Cuspidal components.

Definition 4.4.1. *An unramified character is a character $\psi : G \rightarrow \mathbb{C}^\times$ that is trivial on G° .*

Proposition 4.4.2. *Let (ρ, V) and (ρ', V') be irreducible representations of G . Then*

- (1) $V|_{G^\circ}$ is semisimple of finite length
- (2) *The following are equivalent:*
 - (a) $V|_{G^\circ} \simeq V'|_{G^\circ}$
 - (b) $V|_{G^\circ}$ and $V'|_{G^\circ}$ share a common Jordan-Holder factor
 - (c) $V' \simeq V \otimes \psi$ for some unramified character ψ .

Proof.

- (1) Since V is irreducible and $[G : ZG^\circ] < \infty$, $V|_{ZG^\circ}$ is semisimple of finite length. Since Z acts as a scalar, the claim follows.
- (2) It is clear that c) implies a) and a) implies b), so let us prove that b) implies c). Since $V|_{G^\circ}$ and $V'|_{G^\circ}$ are semisimple, the Jordan-Holder condition is equivalent to $\mathrm{Hom}_{G^\circ}(V, V') \neq 0$.

G° acts on $\mathrm{Hom}_{G^\circ}(V, V')$ by $\tau(g)f = \rho'(g)f\rho(g)^{-1}$; this action factors through $G/G^\circ \simeq \Lambda$. Since Λ is abelian, there exists an eigenfunction f with character $\psi : \Lambda \rightarrow \mathbb{C}^\times$. For any $g \in G$, the diagram below commutes:

$$\begin{array}{ccc}
V & \xrightarrow{f} & V' \\
\downarrow \rho(g)\psi(g) & & \downarrow \rho'(g) \\
V & \xrightarrow{f} & V'
\end{array}$$

This implies that f is a non-zero intertwining map $V \otimes \psi \rightarrow V'$, thus proving that $V \otimes \psi \simeq V'$. \square

The proposition implies that the group of unramified characters acts on the set of irreducible cuspidal representations by tensor product.

Definition 4.4.3. A cuspidal component is an orbit of the set of irreducible cuspidal representations under this action.

Theorem 4.4.4. Let D be a cuspidal component. Then D splits $\text{Rep}(G)$, in the sense that for any $V \in \text{Rep}(G)$, there exists a decomposition $V \simeq V_D \oplus V_D^\perp$ such that the Jordan-Holder factors of V_D are contained in D , and the Jordan-Holder factors of V_D^\perp do not intersect D .

Proof. Consider any $V \in \text{Rep}(G)$. By definition, all the representations in D become isomorphic to some ρ upon restriction to G° ; moreover, by Theorem 4.3.5, ρ is compact, hence semisimple and, since $[G : ZG^\circ] < \infty$, of finite length; in other words, $\rho \simeq \rho_1 \oplus \dots \oplus \rho_r$. By Theorem 3.3.1, there is a splitting $V \simeq V_D \oplus V_D^\perp$, as G° -representations, where the Jordan-Holder factors of V_D lie in $\{\rho_1, \dots, \rho_r\}$ and no Jordan-Holder factor of V_D^\perp is isomorphic to ρ . It remains to see that this decomposition is preserved by G , which is evident, as the G -action simply permutes the ρ_i . \square

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