Chiral Algebra of Differential Operators

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March 19, 2018

1 Definition of CADO and Basic Properties

The chiral algebra of differential operators $\mathcal{D}_{G,\kappa}$ was introduced as a substitute for $\mathrm{D}(G(K))$ (which we know doesn't exist), in the sense that D-modules over the loop group were defined to be chiral modules over $\mathcal{D}_{G,\kappa}$ supported at a point. Conjecturally, modules over CADO are the KM-nondegenerate part of $\mathrm{DMod}(G(K))$. (Details on this?)

Here is the definition of CADO, paralleling that of a CDO. Throughout this note, fix X a curve, $x \in X$ a point on it, $\mathcal{C} = \mathcal{O}_G \otimes \mathcal{O}_X$ for G a reductive group. If M is a right D-module, M^l is its corresponding left D-module given by $M \otimes \Omega_X^{-1}$. All levels will be centered around the critical level.

Recall that $J(\mathcal{C})^l = \mathrm{Sym}_{O_X}(D_X \otimes_{O_X} \mathcal{C})/\ker(\mathrm{Sym}(\mathcal{C}) \to \mathcal{C})$ is the function of the jet of $\mathcal{C}$, and $\Theta_\mathcal{C} = T_\mathcal{C} \otimes_\mathcal{C} (J(\mathcal{C})^l \otimes_{O_X} D_X)$ is the Lie-* algebra of vector fields, where $T_\mathcal{C}$ denotes vertical vector fields on $\mathrm{Spec}(\mathcal{C})$. It is a commutative module over the chiral algebra $J(\mathcal{C})$, and it itself also admits a Lie-* algebra structure over which $J(\mathcal{C})$ is a module. In fact we have $J(\mathcal{C})^l \otimes L\mathfrak{g} \cong \Theta_\mathcal{C}$.

Chiral Envelopes The obvious forgetful functor from chiral algebra to Lie-* algebra admits a left adjoint called the chiral envelope or the universal enveloping algebra, and is denoted $L \mapsto U(L)$. I mentioned a few basic properties of this construction during the talk, more precisely, its fiber, its PBW filtration and the correspondence between continuous modules over $\mathrm{DR}(D_x^*, L)$ and chiral modules over $U(L)$ supported at x; readers can refer to [AG02] for a quick summary.

Definition 1. *A CADO $\mathcal{D}$ over X is a chiral algebra endowed with a filtration $\mathcal{D}_i, i \geq 0$ such that:*

1. $\{\cdot,\cdot\} : j_*j^*(\mathcal{D}_i \boxtimes \mathcal{D}_j) \to \Delta_!(\mathcal{D}_{i+j})$, $\{\cdot,\cdot\} : \mathcal{D}_i \boxtimes \mathcal{D}_j \to \Delta_!(\mathcal{D}_{i+j-1})$;
2. $\mathcal{D}_0 = J(\mathcal{C})$;
3. *There exists a map $\Theta_\mathcal{C} \to \mathcal{D}_1/\mathcal{D}_0$ that is simutaneously a Lie-* algebra map and a chiral $J(\mathcal{C})$-module map, and intertwines the Lie-* module structure of $J(\mathcal{C})$ on both sides.*
4. *The map $Sym_{J(\mathcal{C})^l}\Theta_\mathcal{C}^l \to gr(\mathcal{D})^l$ is an isomorphism.*

A distinguished property of CADO is that it admits both a left and a right embedding of the chiral Kac-Moody algebra into it. Let us first define that: write $L\mathfrak{g} = \mathfrak{g} \otimes D_X$, $\tilde{L}_{\mathfrak{g},\kappa} = \Omega_X \oplus L\mathfrak{g}$. The Lie-* algebra structure on the latter is given by extending that of the former by the map $L_\mathfrak{g} \boxtimes L_\mathfrak{g} \to \Delta_!(\Omega_X)$ induced by the map $\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\kappa} \mathbb{C} \xrightarrow{1 \mapsto \frac{dx \wedge dy}{(x-y)^2}} \Gamma(X \times X, \Delta_!(\Omega_X))$. Let $\mathcal{A}_{\mathfrak{g},\kappa}$ denote the chiral envelope of $\tilde{L}_{\mathfrak{g},\kappa}$ with the two copies of Ω_X (that of the unit and that from $\tilde{L}_{\mathfrak{g},\kappa}$) identified, then this is our chiral Kac-Moody algebra. In particular, $(\mathcal{A}_{\mathfrak{g},\kappa})_x = \mathbb{V}_{\mathfrak{g},\kappa}$ is our usual Vacuum Weyl module.

Roughly speaking this is the chiral version of the usual construction: replace Ω_X with $\mathbb{C}$, $J(\mathcal{C})$ with O_G, $L_\mathfrak{g}$ with $\mathfrak{g}((t))$ and you get the usual construction.[1] In the same spirit, recall that the usual ring of differential operators D_G can be expressed as $U(O_G \oplus \mathfrak{g})/\ker(\mathrm{Sym}(O_G) \to O_G)$ is given the obvious Lie algebra structure. The chiral version essentially copies it: we consider $J(\mathcal{C}) \oplus L\mathfrak{g}$, with the following Lie-* algebra structure:

[1] How do I make this a precise statement?

1. $L\mathfrak{g} \boxtimes L_{\mathfrak{g}} \to \Delta_!(L\mathfrak{g})$ is inherited;

2. $L\mathfrak{g} \boxtimes L_{\mathfrak{g}} \to \Delta_!(J(\mathcal{C}))$ is given by $L\mathfrak{g} \boxtimes L_{\mathfrak{g}} \to \Delta_!(\Omega_X)$ followed by the unit;

3. $L\mathfrak{g} \boxtimes J(\mathcal{C}) \to \Delta_!(J(\mathcal{C}))$ is the Lie-* algebra structure of $J(\mathcal{C})$ over $L\mathfrak{g} \subset \Theta_{\mathcal{C}}$.

Then we define $\mathcal{D}_{G,\kappa}$ to be the chiral envelope of this Lie-* algebra mod out the ideal generated by $\ker \mathrm{Sym}(J(\mathcal{C})) \to J(\mathcal{C})$. The filtration is given by the image of $j_*j^*(J(\mathcal{C}) \boxtimes U(J(\mathcal{C}) \oplus L_{\mathfrak{g}})_i)$ under the chiral bracket.

Proposition 1 (Theorem 3.4 of [AG02])**.** *$\mathcal{D}_{\mathfrak{g},\kappa}$ is a CADO.*

Proof. Requirement 1 and 2 are basically by design. 3 comes from the factorization of $j_*j^*(J(\mathcal{C}) \boxtimes L\mathfrak{g}) \to \Delta_!(\mathcal{D}^1_{G,\kappa}) \to \Delta_!(\mathcal{D}^1_{G,\kappa}/\mathcal{D}^0_{G,\kappa})$ through $\Delta_!(J(\mathcal{C}) \otimes^! L\}) = \Delta_!(\Theta_{\mathcal{C}})$. The map in 4 is surjective by design and both sides are flat on X so it suffices to check on the fiber. In particular, let us denote by $\mathbb{B}_{\mathfrak{g},\kappa}$ the fiber (vacuum module) of $\mathcal{D}_{\mathfrak{g},\kappa}$ at x; it equals $\mathrm{Ind}^{\hat{\mathfrak{g}_\kappa}}_{\mathfrak{g}[[t]]} O_{G[[t]]}$. Thus we have LHS $\simeq \mathrm{Sym}_{O_{G[[t]]}}((\mathfrak{g}((t))/\mathfrak{g}[[t]]) \otimes O_{G[[t]]}) \simeq O_{G[[t]]} \otimes (\mathfrak{g}((t))/\mathfrak{g}[[t]]) = \mathrm{gr}(\mathcal{D}_{\mathfrak{g},\kappa})_x$. □

Note that by construction we have an embedding $\mathcal{A}_{\mathfrak{g},\kappa}$ into $\mathcal{D}_{\mathfrak{g},\kappa}$, which we'll denote by $\mathfrak{l}$ and call the left embedding (because it corresponds to the embedding of $\mathfrak{g}$ as left invariant vector fields). The main result of [AG02] is the following:

Proposition 2. *There is another embedding $\mathfrak{r} : \mathcal{A}_{\mathfrak{g},-\kappa} \to \mathcal{D}_{\mathfrak{g},\kappa}$ that chiral-commutes with $\mathfrak{l}$ and corresponds to the embedding of $\mathfrak{g}$ as right-invariant vector fields.*

The construction is technical and is not terribly enlightening, and all conditions are checked explicitly via chiral algebra computations. We note that these two embeddings admit a double centralizer theorem:

Proposition 3 (Lemma 5.2 of [FG04])**.** *The image of $\mathfrak{l}(\mathcal{A}_{\mathfrak{g},\kappa})$ and that of $\mathfrak{r}(\mathcal{A}_{\mathfrak{g},-\kappa})$ are centralizers of each other in $\mathcal{D}_{G,\kappa}$.*

Proof. By symmetry it suffices to show $\mathrm{im}(\mathfrak{l})$ and $Z(\mathrm{im}(\mathfrak{r}))$ are the same. One inclusion is just chiral commutativity. To show the other inclusion it suffices to check on fibers; more precisely, the left embedding on the fiber level corresponds to $\mathrm{Ind}^{\hat{\mathfrak{g}}_\kappa}_{\mathfrak{g}[[t]]\oplus\mathbb{C}}(\mathbb{C}) \hookrightarrow \mathrm{Ind}^{\hat{\mathfrak{g}}_\kappa}_{\mathfrak{g}[[t]]\oplus\mathbb{C}}(O_{G(O)})$ coming from $\mathbb{C} \hookrightarrow O_{G(O)}$ and the right $\mathfrak{g}[[t]] \subset \hat{\mathfrak{g}}$ structure comes from acting on $O_{G(O)}$ i.e. as right $\mathfrak{g}[[t]]$ module it is free over $O_{G(O)}$. It suffices to check that the vacuum is in $\mathcal{D}_x^{\mathfrak{g}[[t]]} \supset \mathcal{D}_x^{\hat{\mathfrak{g}}}$ (because the action is adjoint action) which is obvious from this description. □

Now let M be a module over $\mathcal{D}_{G,\kappa}$ supported at x. Here is a more concrete description of what it is:

Proposition 4 (Prop 6.2 of [AG02])**.** *This data is the same as continuous actions of $O_{G(K)}$ and $\hat{\mathfrak{g}}_\kappa$ on the vector space $V = i_x^!(M)[1]$, such that for any $\zeta \in \hat{\mathfrak{g}}_\kappa, f \in O_{G(K)}$, we have $[\zeta, f] = Lie_\zeta(f)$ (where we use left embedding) as operators on V.*

By the right embedding described above, we see that any such module automatically comes with a $\hat{\mathfrak{g}}_{-\kappa}$ action from the right that is compatible with $O_{G(K)}$ action via right embedding. This can be explained more succinctly with the Tate extension, as was done in Section 21 of [FG06] which we reproduce below.

1.1 Non-Chiral Version of the Two Embeddings

For the next part, write $\mathfrak{g}$ for $\mathfrak{g}((t))$ and G for $G((t))$ for sanity. There are some topological vector space issues regarding different completions which I'm going to ignore.

In particular, let us fix a module V with the said action above at the zero level. Then we'd expect an $\hat{\mathfrak{g}}_{2\kappa_0}$ action from the right on the same module V.

Proposition 5. *$\hat{\mathfrak{g}}_{2\kappa_0}$ is the Baer negative of the canonical Tate extension.*

Recall that $\mathrm{Cl} = \mathrm{Cl}(\mathfrak{g}, \mathfrak{g}^*)$ is the (filtered, graded) Clifford algebra associated to $\mathfrak{g} \oplus \mathfrak{g}^*$, and Cl_0^2 fits into a central extension:

$$0 \to \mathbb{C} \to \mathrm{Cl}_0^2 \to \mathfrak{g} \otimes \mathfrak{g}^* = \mathfrak{gl}(\mathfrak{g}) \to 0$$

Pull back along $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ (by adjoint on itself) we obtain the Tate extension. It is clear that the 2-cocycle for the short exact sequence above is the Killing form.

Fix any S a representation of Cl and let $M = V \otimes S$. First we have Cl acting on M by acting on S, which we'll denote by $i_r : \mathrm{Cl} \to \mathrm{End}\, M$. This action extends to an action of arbitrary elements of $T(G)$ and $T^*(G)$ by e.g. seeing $T(G) \simeq O_G \otimes \mathfrak{g}$ using left-invariant vector fields and let it act on V and S respectively. This action we'll write as $i : T(G), T^*(G) \to \mathrm{End}\, M$. Now use $\mathfrak{g} \xrightarrow{-\text{adjoint}} O_G \otimes \mathfrak{g} \xrightarrow{\text{inv} \otimes 1} O_G \otimes \mathfrak{g}$ and let it act on V and S respectively, we get another map $i_l : \mathrm{Cl} \to \mathrm{End}\, M$.

Let us define another action $T(G) \simeq O_G \otimes \mathfrak{g}$ on M (in reality need to define it for RHS first them see it lifts to completion), which we'll write as $\mathrm{Lie} : T(G) \to \mathrm{End}\, M$. It ought to extend (in the sense of $x \mapsto 1 \otimes x$) the left Lie algebra action $\mathrm{Lie} : \mathfrak{g} \to \mathrm{End}\, M$ by acting on V. In particular, it is given by $(f \otimes x)v = f \cdot \mathrm{Lie}(x) \cdot v + i(df) \cdot i_l(x) \cdot v$ for $f \in O_G, x \in \mathfrak{g}, v \in M$. Now take $\mathfrak{g} \xrightarrow{-\text{adjoint}} O_G \otimes \mathfrak{g}$ followed by Lie we get another action $\mathrm{Lie}_r : \mathfrak{g} \to \mathrm{End}\, M$.

Finally, define $\hat{\mathfrak{g}}_{2\kappa_0} \to \mathrm{End}\, M$ as $\hat{\mathfrak{g}}_{2\kappa_0} \to \mathfrak{g} \xrightarrow{\mathrm{Lie}_r} \mathrm{End}\, M$ minus $\hat{\mathfrak{g}}_{2\kappa_0} \simeq \mathfrak{g}^{\mathrm{Tate}} \to \mathrm{Cl}_0^2 \to \mathrm{End}\, S \to \mathrm{End}\, M$ (where the isomorphism is as vector spaces). This can be checked to be a Lie algebra action, and has the following properties (all as endomorphisms on M):

1. Its commutator with $f \in O_G$ is the right Lie derivative of f;
2. It commutes with $\mathrm{Lie}(x) \in \mathrm{End}\, V \to \mathrm{End}\, M$ for all $x \in \mathfrak{g}$;
3. It commutes with i_r.

The last part means that this action preserves V, so this is the expected $\hat{\mathfrak{g}}_{\mathrm{Tate}}$ action from the right.

1.2 Regularity

Now recall that there is a canonical pairing $\langle \cdot, \cdot \rangle : \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(O)} \times \hat{\mathfrak{g}}_{-\kappa}\text{-mod}^{G(O)} \to \mathrm{Vect}$ given by $(M, N) \mapsto H^{\frac{\infty}{2}}(\mathfrak{g}((t)), G[[t]], M \otimes N)$. The main point is that the vacuum module of CADO is an universal object among the $G(O)$-integrable objects with respect to this pairing:

Proposition 6. *$\langle M, \mathbb{B}_{\mathfrak{g},\kappa} \rangle = M$ for all $M \in \hat{\mathfrak{g}}_\kappa$-mod$^{G(O)}$.*

Let $\mathbb{V}^\lambda_{\mathfrak{g},\kappa}$ be the $\hat{\mathfrak{g}}_\kappa$ module induced from the Weyl module of highest weight λ, for $\lambda \in \Lambda^+$. It is $G(O)$-integrable. At a generic level, we can say more explicitly what $\mathbb{B}_{\mathfrak{g},\kappa}$ looks like:

Proposition 7. *At a generic level κ, we have* $\mathbb{B}_{\mathfrak{g},\kappa} = \bigoplus_{\lambda \in \Lambda^+} \mathbb{V}^\lambda_{\mathfrak{g},\kappa} \otimes \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},-\kappa}$.

Proof. We have $V^\lambda \otimes V^{\tau(\lambda)} \to O_G \to O_{G(O)}$, which induces a $\hat{\mathfrak{g}}_\kappa \otimes \hat{\mathfrak{g}}_{-\kappa}$ module homomorphism $\mathbb{V}^\lambda_{\mathfrak{g},\kappa} \otimes \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},-\kappa} \to \mathbb{B}_{\mathfrak{g},\kappa}$. Affine Weyls are generically irreducible, so this map is injective. Then it suffices to conclude by comparing characters. $\square$

It was also checked in [AG] that for any normal subgroup $K \subset G(O)$ of finite codimension, the category of K-integrable modules over the CADO coincide with what we expect, i.e. the category of D-modules on $K \backslash G((t))$.

2 Structure at Critical Level

Now let us specialize to the critical level. At the critical level, CADO becomes a $\hat{\mathfrak{g}}_{\kappa_0}$-bimodule, and $\mathfrak{l}, \mathfrak{r}$ become two embeddings of $\mathcal{A}_{\mathfrak{g},\kappa_0}$. Let $\mathfrak{z}_\mathfrak{g}$ denote the chiral center of $\mathcal{A}_{\mathfrak{g},\kappa_0}$, then we get two embeddings of it into CADO. By the double centralizer theorem, their images coincide and is the intersection of the images of $\mathfrak{l}$ and $\mathfrak{r}$. By general chiral algebra, $\mathfrak{z}_{\mathfrak{g},x}$, the fiber at x, is isomorphic to $\mathrm{End}(\mathbb{V}_{\mathfrak{g},\kappa_0})$ and is therefore isomorphic to $\mathrm{Fun}(\mathrm{Op}_{^LG}(D_x))$ by Feigin-Frenkel.

Further, let $\mathfrak{Z}_x$ denote the completed version of $\mathfrak{z}_x$, which corresponds to functions of opers on the punctured disc; let $\mathfrak{Z}_x^{\mathrm{unr}}$ correspond to functions on $\mathrm{Op}_{^LG}^{\mathrm{unr}} = \bigcup_{\lambda\in\Lambda^+} \mathrm{Op}_{^LG}^{\mathrm{unr},\lambda}$, the ind-scheme of unramified opers; and let $\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}$ correspond to the closed subscheme $\mathrm{Op}_{^LG}^{\mathrm{reg},\lambda}$, which is the reduced part of $\mathrm{Op}_{^LG}^{\mathrm{unr},\lambda}$. The main result of [FG10] is that $\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda} = \mathrm{End}(\mathbb{V}^\lambda_{\mathfrak{g},\kappa_0})$. Now back to our main story.

Proposition 8 (Theorem 5.4 of [FG04])**.** *The two embeddings of $\mathfrak{z}_{\mathfrak{g}}$ into $\mathcal{D}_{\mathfrak{g},\kappa_0}$ differ by a Cartan involution.*

The structure of CADO at the critical level is more complicated. First note that $\mathbb{B}_{\mathfrak{g},\kappa_0}$ is $G(O)$ integrable as a $\hat{\mathfrak{g}}_{\kappa_0}$-bimodule. By what we saw last time, its support is contained in unramified opers, and we obtain a decomposition $\mathbb{B}_{\mathfrak{g},\kappa_0} = \bigoplus_{\lambda\in\Lambda^+} \mathbb{B}^\lambda_{\mathfrak{g},\kappa_0}$ by support. (By moving the point, this gives a D-module decomposition $\mathcal{D}_{\mathfrak{g},\kappa_0} = \bigoplus_{\lambda\in\Lambda^+} \mathcal{D}^\lambda_{\mathfrak{g},\kappa_0}$.) The map $\mathbb{V}^\lambda_{\mathfrak{g},\kappa_0} \otimes \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\kappa_0} \to \mathbb{B}_{\mathfrak{g},\kappa_0}$ obviously factors through $\mathbb{B}^\lambda_{\mathfrak{g},\kappa_0}$; moreover, by the proposition above, it also factors through $\mathbb{V}^\lambda_{\mathfrak{g},\kappa_0} \otimes_{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg}}} \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\kappa_0}$. (I believe the action translates to $\mathfrak{z}_{\mathfrak{g}}$ action on the right–haven't actually checked.) In other words, we lose some degree of freedom because the left and right actions of automorphisms now coincide. It turns out this was restored by the infinitesimal data of $\mathrm{Op}_{^LG}^{\mathrm{reg},\lambda} \subset \mathrm{Op}_{^LG}^{\mathrm{unr},\lambda}$, which is unsurprising from a geometric point of view.

The precise statement is as follows. Let I be the ideal of $\mathrm{Op}_{^LG}^{\mathrm{reg},\lambda} \subset \mathrm{Op}_{^LG}^{\mathrm{unr},\lambda}$, and let $N = (I/I^2)^*$ be the normal bundle. Filtrate $\mathbb{B}^\lambda_{\mathfrak{g},\kappa_0}$ by the kernel of $I^{k+1}, k \geq 0$,

Proposition 9. $\mathbb{V}^\lambda_{\mathfrak{g},\kappa_0} \otimes_{\mathfrak{z}_{\mathfrak{g}}^{reg}} \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\kappa_0} \cong F^0(\mathbb{B}^\lambda_{\mathfrak{g},\kappa_0})$, *and* $gr^n(\mathbb{B}^\lambda_{\mathfrak{g},\kappa_0}) \cong F^0(\mathbb{B}^\lambda_{\mathfrak{g},\kappa_0}) \otimes_{\mathfrak{z}_{\mathfrak{g}}^{reg,\lambda}} Sym^n_{\mathfrak{z}_{\mathfrak{g}}^{reg,\lambda}}(N)$.

The proof of this uses renormalized chiral algebroid $\mathcal{A}_{\mathfrak{g}}^{\mathrm{ren},\tau}$. Let me briefly summarize why this was involved.

Some Mumbles about Renormalized Algebra The proof of Beilinson-Bernstein localization of affine Grassmannian at negative / irrational level uses the crucial fact that the vacuum module is projective in $\hat{\mathfrak{g}}_\kappa$ module; at critical level this is false, and it is only projective in the "regular part". So the exactness statement boils down to check that the forget functor from CADO module to $\hat{\mathfrak{g}}$-module, followed by the !-pullback functor to the regular support, is exact (everything is $G(O)$-integrable). The key fact here are the following:

1. That chiral modules over $\mathcal{A}_{\mathfrak{g}}^{\mathrm{ren},\tau}$ admit a Kashiwara-style theorem (Theorem 6.15 of [FG04]);
2. That the two embeddings of $\mathcal{A}_{\mathfrak{g},\kappa_0}$ factor through $\mathcal{A}_{\mathfrak{g}}^{\mathrm{ren},\tau}$ (Theorem 5.4 of [FG04]);
3. and that the universal enveloping algebra of $\mathcal{A}_{\mathfrak{g}}^{\mathrm{ren},\tau}$ is $\mathcal{D}^0_{\mathfrak{g},\kappa_0}$ (Prop 9.7 of [FG04]).

Then the exactness statement comes from pulling back on the renormalized algebroid module level.

Back to our description of CADO: $\mathcal{D}^0_{\mathfrak{g},\kappa_0}$ carries a PBW-type filtration, whose zeroth graded piece is $\mathcal{A}_{\mathfrak{g},\kappa_0} \otimes_{\mathfrak{z}_{\mathfrak{g}}} \mathcal{A}_{\mathfrak{g},\kappa_0}$ and first graded piece is $\Omega^1(\mathfrak{z}_{\mathfrak{g}})$, which is the chiral version of the conormal bundle. By the Kashiwara-style theorem, $\mathbb{B}^\lambda_{\mathfrak{g},\kappa_0}$ is the induced chiral module from $F^0(\mathbb{B}^\lambda_{\mathfrak{g},\kappa_0})$ by $\mathcal{D}^0_{\mathfrak{g},\kappa_0}$, and the PBW filtration induces another filtration $\mathbf{F}$ on $\mathcal{B}^\lambda_{\mathfrak{g},\kappa_0}$, for which it is immediate that $\mathbf{gr}^n(\mathbb{B}^\lambda_{\mathfrak{g},\kappa_0}) \cong F^0(\mathbb{B}^\lambda_{\mathfrak{g},\kappa_0}) \otimes_{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}} \mathrm{Sym}^n_{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}}(N)$, so it remains to check that $\mathbf{F}$ and F are the same filtration. This comes down to the fact that the coisson structure on $\mathfrak{z}_{\mathfrak{g}}$ is elliptic (more precisely that its anchor map is injective)—anyone want to explain this?

It remains to check the first part of the description above. First check the map is injective: if it were not, then the kernel would admit a $\hat{\mathfrak{g}}_{\kappa_0}$ map from the vacuum module. We contradict this by observing that $\mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(\mathbb{V}^\lambda, \mathbb{V}^\lambda \otimes_{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}} \mathbb{V}^{\tau(\lambda)}) \to \mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(\mathbb{V}^\lambda, \mathbb{B}^\lambda)$ is an iso. This follows from the following: first, $\mathbb{V}^{\tau(\lambda)} \to \mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(\mathbb{V}^\lambda, \mathbb{V}^\lambda \otimes_{\mathfrak{z}_{\mathfrak{g}}^{\mathrm{reg},\lambda}} \mathbb{V}^{\tau(\lambda)})$ is an iso because the latter is $\mathrm{End}(\mathbb{V}^\lambda) \otimes \mathbb{V}^{\tau(\lambda)}$ (using flatness) which is simply $\mathbb{V}^{\tau(\lambda)}$ by [FG10]'s main result. Next, composition of those two functions is the following iso:

Proposition 10. $\mathrm{Hom}_{\hat{\mathfrak{g}}_{\kappa_0}}(\mathbb{V}^\lambda_{\mathfrak{g},\kappa_0}, \mathbb{B}_{\mathfrak{g},\kappa_0}) = \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\kappa_0}$ *(in the derived sense).*

Proof.

$$\begin{aligned}&\mathrm{Hom}_{\hat{\mathfrak{g}}_{\kappa_0}}(\mathbb{V}^{\lambda}_{\mathfrak{g},\kappa_0}, \mathbb{B}_{\mathfrak{g},\kappa_0}) = \\ &\mathrm{Hom}_{G(O)}(V^{\lambda}, \mathbb{B}_{\mathfrak{g},\kappa_0}) = ((V^{\lambda})^* \otimes \mathbb{B}_{\mathfrak{g},\kappa_0})^{G(O)} = (V^{\tau(\lambda)} \otimes \mathbb{B}_{\mathfrak{g},\kappa_0})^{G(O)} = \\ &\mathrm{Hom}_{G(O)}(k, V^{\tau(\lambda)} \otimes \mathbb{B}_{\mathfrak{g},\kappa_0}) = \langle \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\kappa_0}, \mathbb{B}_{\mathfrak{g},\kappa_0}\rangle = \mathbb{V}^{\tau(\lambda)}_{\mathfrak{g},\kappa_0}.\end{aligned}$$

□

To conclude the theorem, we consider $G \times G \times \mathbb{G}_m$ action on $\mathbb{B}_{\mathfrak{g},\kappa_0}$, coming from the two $G(O)$ actions and the "rotation operator" $L_0 = -t\dfrac{d}{dt}$. It suffices to count the multiplicity of any rep $V^{\mu_1} \otimes V^{\mu_2} \otimes \mathbb{C}^d$ within $\mathbb{B}_{\mathfrak{g},\kappa_0}$ is the same as that within $\mathbb{V}^{\lambda} \otimes_{\mathfrak{z}^{\mathrm{reg}}} \mathbb{V}^{\tau(\lambda)} \otimes \mathrm{Sym}_{\mathfrak{z}^{\mathrm{reg}}}(N) = \mathbb{V}^{\lambda} \otimes_{\mathfrak{z}^{\mathrm{reg}}} \mathbb{V}^{\tau(\lambda)} \otimes \mathrm{Sym}_{\mathfrak{z}^{\mathrm{reg}}}(\Omega^1(\mathfrak{z}^{\mathrm{reg}}))$. We know that $\mathfrak{z}^{\mathrm{reg}}$ is a polynomial algebra so this is same multiplicity as $\mathbb{V}^{\lambda} \otimes \mathbb{V}^{\tau(\lambda)}$. Since we have a flat family based on κ for vacuum modules, the statement follows from deforming away from the critical level.

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