PREFACE: A STUDY IN DERIVED ALGEBRAIC GEOMETRY

Кто я? Не каменщик прямои,
Не кровељщик, не корабељщик,–
Двурушник я, с двоинои душои,
Я ночи друг, я дня застрељщик.
О. Мандељштам. Грифељная ода.

Who am I? Not a straightforward mason,
Not a roofer, not a shipbuilder, –
I am a double agent, with a duplicitous soul,
I am a friend of the night, a skirmisher of the day.
O. Mandelshtam. The Graphite Ode.

Contents

1. What is the object of study in this book?

The main unifying theme of this book is the notion of *ind-coherent sheaf*, or rather, categories of such on various geometric objects. In this section we will try to explain what ind-coherent sheaves are and why we need this notion.

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1.1. **Who are we?** Let us start with a disclosure: this book is *not* about algebraic geometry.

Or, rather, in writing this book, its authors do not act as real algebraic geometers. This is because the latter are ultimately interested in geometric objets that are constrained/enriched by the algebraicity requirement.

We, however, use algebraic geometry as a tool: this book is written with the view to applications to *representation theory.*

It just so happens that algebraic geometry is a very (perhaps, even the most) convenient way to formulate representation-theoretic problems of categorical nature. This is not surprising, since, after all, algebraic groups are themselves objects of algebraic geometry.

The most basic example of how one embeds representation theory into algebraic geometry is this: take the category $\operatorname{Rep}(G)$ of algebraic representations of a linear algebraic group G. Algebraic geometry allows us to define/interpret $\operatorname{Rep}(G)$ as the category of quasi-coherent sheaves on the classifying stack BG.

The advantage of this point of view is that many natural constructions associated with the category of representations are already contained in the package of 'quasi-coherent sheaves on stacks'. For example, the functors of restriction and coinduction[1] along a group homomorphism $G' \to G$ are interpreted as the functors of inverse and direct image along the map of stacks

$$BG' \to BG.$$

But why would we want to take advantage of this advantage? Why not stick to the explicit constructions of all the required functors within representation theory?

The reason is that 'explicit constructions' involve 'explicit formulas', and once we move to the world of higher categories (which we inevitably will, in order to meet the needs of modern representation theory), we will find ourselves in trouble: constructions in higher category theory are intolerant of explicit formulas (for an example of a construction that uses formulas see point (III) in Sect. 1.5 below). Rather, there is a fairly limited package of constructions that we are allowed to perform (see [Chapter I.1, Sects. 1 and 2] where some of there constructions are listed), and algebraic geometry seems to contain a large chunk (if not all) of this package.

1.2. **A stab in the back.** Jumping a little ahead, let us now say that we want to interpret algebro-geometrically the category $\mathfrak{g}$-mod of modules over a Lie algebra.

The first question is: why would one want to do that? Namely, take the universal enveloping algebra $U(\mathfrak{g})$ and interpret $\mathfrak{g}$-mod as modules over $U(\mathfrak{g})$. Why should one mess with algebraic geometry if all we want is the category of modules over an associative algebra?

But let us say that we have already accepted the fact that we want to interpret $\operatorname{Rep}(G)$ as $\operatorname{QCoh}(BG)$. What about the restriction functor

$$\operatorname{Rep}(G) \to \mathfrak{g}\text{-mod},$$

where $\mathfrak{g}$ is the Lie algebra of G?

Now, if $\mathfrak{g}$ is a usual (=classical) Lie algebra, one can consider the associated formal group, denoted in the book $\exp(\mathfrak{g})$, and one can show (see [Chapter IV.3, Sect. 5]) that the category $\mathfrak{g}$-mod is canonically equivalent to $\operatorname{QCoh}(B(\exp(\mathfrak{g})))$, the category of quasi-coherent sheaves on the classifying stack of $\exp(\mathfrak{g})$.

[1] What we call 'coinduction' is the functor right adjoint to restriction, i.e., it is the usual representation-theoretic operation.

One can (reasonably) get somewhat uneasy from the suggestion to consider the category of quasi-coherent sheaves on the *classifying stack of a formal group*. But in fact, this is a legitimate operation.

But let us now be given a homomorphism of Lie algebras $\alpha : \mathfrak{g}' \to \mathfrak{g}$. The functor of restriction $\mathfrak{g}$-mod $\to \mathfrak{g}'$-mod still corresponds to the pullback functor along the corresponding morphism

$$B(\exp(\mathfrak{g}')) \xrightarrow{f_\alpha} B(\exp(\mathfrak{g})). \tag{1.1}$$

Note, however, that when we talk about representations of Lie algebras, the natural functor in the opposite direction is *induction*, i.e., the *left* adjoint to restriction. And being a left adjoint, it *cannot* correspond to the direct image along (1.1) (whatever the functor of direct image is, it is the *right* adjoint of pullback).

This inconsistency leads to the appearance of *ind-coherent sheaves*.

1.3. **The birth of** IndCoh.

What happens is that, although we can interpret $\mathfrak{g}$-mod as $\text{QCoh}(B(\exp(\mathfrak{g})))$, a more natural interpretation is as $\text{IndCoh}(B(\exp(\mathfrak{g})))$. The symbol 'IndCoh' will of course be explained in the sequel. It just so happens that for a classical Lie algebra, the categories $\text{QCoh}(B(\exp(\mathfrak{g})))$ and $\text{IndCoh}(B(\exp(\mathfrak{g})))$ are equivalent (as $\text{QCoh}(BG)$ is equivalent to $\text{IndCoh}(BG)$).

Now, the functor of restriction along the homomorphism α will be given by the functor

$$(f_\alpha)^! : \text{IndCoh}(B(\exp(\mathfrak{g}'))) \to \text{IndCoh}(B(\exp(\mathfrak{g})));$$

this is the !-pullback functor, which is the *raison d'être* for the theory of IndCoh.

However, the functor of induction $\mathfrak{g}'$-mod $\to \mathfrak{g}$-mod will be the functor of *IndCoh direct image*

$$(f_\alpha)_*^{\text{IndCoh}} : \text{IndCoh}(B(\exp(\mathfrak{g}'))) \to \text{IndCoh}(B(\exp(\mathfrak{g}))), \tag{1.2}$$

which is the *left* adjoint of $(f_\alpha)^!$. This adjunction is due to the fact that the morphism f_α is, in an appropriate sense, proper.

Even though, as mentioned above, for a usual Lie algebra $\mathfrak{g}$, the categories $\text{QCoh}(B(\exp(\mathfrak{g})))$ and $\text{IndCoh}(B(\exp(\mathfrak{g})))$ are equivalent, the functor $(f_\alpha)_*^{\text{IndCoh}}$ of (1.2) is as different as can be from the functor

$$(f_\alpha)_* : \text{QCoh}(B(\exp(\mathfrak{g}'))) \to \text{QCoh}(B(\exp(\mathfrak{g})))$$

(the latter is quite ill-behaved).

For an analytically minded reader let us also offer the following (albeit very loose) analogy: $\text{QCoh}(-)$ behaves more like functions on a space, while $\text{IndCoh}(-)$ behaves more like measures on the same space.

1.4. **What can we do with ind-coherent sheaves?**

As we saw in the example of Lie algebras, the kind of geometric objects on which we will want to consider IndCoh (e.g., $B(\exp(\mathfrak{g}))$) are quite a bit more general than the usual players on which we consider quasi-coherent sheaves, the latter being schemes (or algebraic stacks).

A natural class of algebro-geometric objects for which IndCoh is defined is that of *inf-schemes*, introduced and studied in Part III of the book. This class includes all schemes, but also formal schemes, as well as classifying spaces of formal groups, etc. In addition, if X is a scheme, its de Rham prestack[2] X_{dR} is an inf-scheme, and ind-coherent sheaves on X_{dR} will be the same as *crystals* (a.k.a, D-modules) on X.

Thus, for any inf-scheme $\mathcal{X}$ we have a well-defined category $\mathrm{IndCoh}(\mathcal{X})$. For any map of inf-schemes $f : \mathcal{X}' \to \mathcal{X}$ we have the functors

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}') \to \mathrm{IndCoh}(\mathcal{X})$$

and

$$f^! : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X}').$$

Moreover, if f is proper[3], then the functors $(f_*^{\mathrm{IndCoh}}, f^!)$ form an adjoint pair.

Why should we be happy to have this? The reason is that this is exactly the kind of operations one needs in geometric representation theory.

1.5. **Some examples of what we can do.**

(I) Take $\mathcal{X}'$ to be a scheme X and $\mathcal{X} = X_{\mathrm{dR}}$, with f being the canonical projection $X \to X_{\mathrm{dR}}$. Then the adjoint pair

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \rightleftarrows \mathrm{IndCoh}(X_{\mathrm{dR}}) : f^!$$

identifies with the pair

$$\mathbf{ind}_{\text{D-mod}} : \mathrm{IndCoh}(X) \rightleftarrows \text{D-mod}(X) : \mathbf{ind}_{\text{D-mod}},$$

corresponding to forgetting and inducing the (right) D-module structure (as we shall see shortly in Sect. 2.3, for a scheme X, the category $\mathrm{IndCoh}(X)$ is only slightly different from the usual category of quasi-coherent sheaves $\mathrm{QCoh}(X)$).

(II) Let us be given a morphism of schemes $g : Y \to X$ and set

$$Y_{\mathrm{dR}} \overset{f:=g_{\mathrm{dR}}}{\longrightarrow} X_{\mathrm{dR}}.$$

The corresponding functors

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Y_{\mathrm{dR}}) \to \mathrm{IndCoh}(X_{\mathrm{dR}}) \text{ and } f^! : \mathrm{IndCoh}(X_{\mathrm{dR}}) \to \mathrm{IndCoh}(Y_{\mathrm{dR}})$$

identify with the functors

$$g_{*,\mathrm{Dmod}} : \mathrm{Dmod}(Y) \to \mathrm{Dmod}(X) \text{ and } g^!_{\mathrm{Dmod}} : \mathrm{Dmod}(X) \to \mathrm{Dmod}(Y)$$

of D-module (a.k.a. de Rham) push-forward and pullback, respectively.

Note that while the operation of pullback of (right) D-modules corresponds to !-pullback on the underlying $\mathcal{O}$-module, the operation of D-module push-forward is less straightforward as it involves taking fiber-wise de Rham cohomology. So, the operation of the IndCoh direct image does something quite non-trivial in this case.

(III) Let us be given a Lie algebra $\mathfrak{g}$ that acts (by vector fields) on a scheme X. In this case we can create a diagram

$$B(\exp(\mathfrak{g})) \overset{f_1}{\longleftarrow} B_X(\exp(\mathfrak{g})) \overset{f_2}{\longrightarrow} X_{\mathrm{dR}},$$

where $B_X(\exp(\mathfrak{g}))$ is an inf-scheme, which is the quotient of X by the action of $\mathfrak{g}$.

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[2] The de Rham prestack of a given scheme X is obtained by 'modding' out X by the groupoid of its infinitesimal symmetries, see [Chapter III.4, Sect. 1.1.1] for a precise definition.

[3] Properness means the following: to every inf-scheme there corresponds its underlying reduced scheme, and a map between inf-schemes is proper if and only if the map of the underlying reduced schemes is proper in the usual sense.

Then the composed functor

$$(f_2)_*^{\mathrm{IndCoh}} \circ (f_1)^! : \mathrm{IndCoh}(B(\exp(\mathfrak{g}))) \to \mathrm{IndCoh}(X_{\mathrm{dR}})$$

identifies with the *localization functor*

$$\mathfrak{g}\text{-mod} \to \mathrm{Dmod}(X).$$

This third example should be a particularly convincing: the localization functor, which is usually defined by an explicit formula

$$M \mapsto D_X \underset{U(\mathfrak{g})}{\otimes} M,$$

is given here by the general formalism.

2. How do we do we construct the theory of IndCoh?

Whatever inf-schemes are, for one such, denoted $\mathcal{X}$, the category $\mathrm{IndCoh}(\mathcal{X})$ is bootstrapped from the corresponding categories for schemes by the following procedure:

$$\mathrm{IndCoh}(\mathcal{X}) = \lim_{Z \to \mathcal{X}} \mathrm{IndCoh}(Z). \tag{2.1}$$

Some explanations are in order.

2.1. What do we mean by limit?

(a) In formula (2.1), the symbol 'lim' appears. This is the limit of categories, but not quite. If we were to literally take the limit in the category of categories, we would obtain an utter nonsense. This is a familiar phenomenon: the (literally understood) limit of, say, triangulated categories is not well-behaved. A well-known example of this is that the derived category of sheaves on a space cannot be recovered from the corresponding categories on an open cover. However, this can be remedied if instead of the triangulated categories we consider their higher categorical enhancements, i.e., the corresponding ∞-categories.

So, what we actually mean by 'limit', is the limit taken in the *∞-category* of *∞-categories*. That is, in the preceding discussion, all our $\mathrm{IndCoh}(-)$ are actually ∞-categories. In our case, they have a bit more structure: they are k-linear over a fixed ground field k; we call them *DG categories*, and denote the ∞-category of such by DGCat.

Thus, ∞-categories inevitably appear in this book.

(b) The index (∞)-category appearing in the expression (2.1) is the (∞)-category opposite to that of schemes Z equipped with a map $Z \to \mathcal{X}$ to our inf-scheme $\mathcal{X}$. The transition functors are given by

$$(Z' \overset{f}{\to} Z) \in \mathrm{Sch}_{/\mathcal{X}} \rightsquigarrow \mathrm{IndCoh}(Z) \overset{f^!}{\longrightarrow} \mathrm{IndCoh}(Z').$$

So, in order for the expression in (2.1) to make sense we need to make the assignment

$$Z \rightsquigarrow \mathrm{IndCoh}(Z), \quad (Z' \overset{f}{\to} Z) \rightsquigarrow (\mathrm{IndCoh}(Z) \overset{f^!}{\longrightarrow} \mathrm{IndCoh}(Z')) \tag{2.2}$$

into a *functor of ∞-categories*

$$\mathrm{IndCoh}^!_{\mathrm{Sch}} : (\mathrm{Sch})^{\mathrm{op}} \to \mathrm{DGCat}\,. \tag{2.3}$$

For that end, before we proceed any further, we need to explain what the DG category $\mathrm{IndCoh}(Z)$ is for a scheme Z.

For a scheme Z, the category $\mathrm{IndCoh}(Z)$ will be nearly the same as $\mathrm{QCoh}(Z)$, yet they are different. The former is obtained from the latter by a *renormalization* procedure, whose nature we shall now explain.

2.2. **Why renormalize?** Keeping in mind the examples of $\mathrm{Rep}(G)$ and $\mathfrak{g}$-mod, it is natural to expect that the assignment (2.2) (for schemes, and then also for inf-schemes) should have the following properties:

(i) For every scheme Z, the DG category $\mathrm{IndCoh}(Z)$ should contain infinite direct sums;

(ii) For a map $Z' \xrightarrow{f} Z$, the functor $\mathrm{IndCoh}(Z) \xrightarrow{f^!} \mathrm{IndCoh}(Z')$ should preserve infinite direct sums.

This means that the functor (2.3) takes values in the subcategory of DGCat, where we allow as objects only DG categories satisfying (i)[4] and as 1-morphisms only functors that satisfy (ii)[5].

Let us first try to make this work with the usual QCoh. We refer the reader to [Chapter I.3], where the DG category $\mathrm{QCoh}(\mathcal{X})$ is introduced for an arbitrary *prestack*, and in particular a scheme. However, for a scheme Z, whatever the DG category $\mathrm{QCoh}(Z)$ is, its homotopy category (which is a triangulated category) is the usual (unbounded) derived category of quasi-coherent sheaves on Z.

Let us be given a map of schemes $Z' \xrightarrow{f} Z$. The construction of the !-pullback functor

$$f^! : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(Z')$$

is no simple business, except when f is proper. In the latter case, $f^!$, which from now on we will denote by $f^{!,\mathrm{QCoh}}$, is defined to be the *right adjoint* of

$$f_* : \mathrm{QCoh}(Z') \to \mathrm{QCoh}(Z).$$

The only trouble is that the above functor $f^{!,\mathrm{QCoh}}$ *does not* preserve infinite direct sums. The simplest example of a morphism for which this happens is

$$f : \mathrm{Spec}(k) \to \mathrm{Spec}(k[t]/t^2)$$

(or the embedding of a *singular* point into any scheme).

The reason for the failure to preserve infinite direct sums is this: the left adjoint of $f^{!,\mathrm{QCoh}}$, i.e., f_*, *does not* preserve compactness. Indeed, f_* does not necessarily send *perfect complexes* on Z' to perfect complexes on Z, unless f is of finite Tor-dimension[6].

So, our attempt with QCoh fails (ii) above.

2.3. **Ind-coherent sheaves on a scheme.** The nature of the renormalization procedure that produces $\mathrm{IndCoh}(Z)$ out of $\mathrm{QCoh}(Z)$ is to force (ii) from Sect. 2.2 ‘by hand’.

As we just saw, the problem with $f^{!,\mathrm{QCoh}}$ was that its left adjoint f_* did not send the corresponding subcategories of perfect complexes to one another. However, f_* sends the subcategory

$$\mathrm{Coh}(Z') \subset \mathrm{QCoh}(Z')$$

to

$$\mathrm{Coh}(Z) \subset \mathrm{QCoh}(Z),$$

where $\mathrm{Coh}(-)$ denotes the subcategory of bounded complexes, whose cohomology sheaves are *coherent* (as opposed to quasi-coherent).

[4]Such DG categories are called *cocomplete*.

[5]Such functors are called *continuous*.

[6]We remark that a similar phenomenon, where instead of the category $\mathrm{QCoh}(\mathrm{Spec}(k[t]/t^2)) = k[t]/t^2$-mod we have the category of representations of a finite group, leads to the notion of Tate cohomology: the trivial representation on $\mathbb{Z}$ is *not* a compact object in the category of representations.

The category $\mathrm{IndCoh}(Z)$ is defined as the *ind-completion* of $\mathrm{Coh}(Z)$ (see [Chapter I.1, Sect. 7.2] for what this means). The functor f_* gives rise to a functor $\mathrm{Coh}(Z') \to \mathrm{Coh}(Z)$, and ind-extending we obtain a functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z') \to \mathrm{IndCoh}(Z).$$

Its right adjoint, denoted $f^! : \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(Z')$ satisfies (ii) from Sect. 2.2.

Are we done? Far from it. First, we need to define the functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z') \to \mathrm{IndCoh}(Z)$$

for a morphism f that is not necessarily proper. This will not be difficult, and will be done by appealing to t-structures, see Sect. 2.4 below.

What is much more serious is to define $f^!$ for any f. More than that, we need $f^!$ not just for an individual f, but we need the whole datum of (2.2) to be a functor of ∞-categories as in (2.3). About a third of the work in this book goes into the construction of the functor (2.3); we will comment on the nature of this work in Sect. 2.5 and then in Sect. 3 below.

2.4. **In what sense is** IndCoh **a 'renormalization' of** QCoh**?** The tautological embedding $\mathrm{Coh}(Z) \hookrightarrow \mathrm{QCoh}(Z)$ induces, by ind-extension, a functor

$$\Psi_Z : \mathrm{IndCoh}(Z) \to \mathrm{QCoh}(Z).$$

The usual t-structure on the DG category $\mathrm{Coh}(Z)$ induces one on $\mathrm{IndCoh}(Z)$. The key feature of the functor Ψ_Z is that it is *t-exact*. Moreover, for every fixed n, the resulting functor

$$\mathrm{IndCoh}(Z)^{\geq -n} \to \mathrm{QCoh}(Z)^{\geq -n}$$

is an *equivalence*[7]. The reason for this is that any coherent complex can be approximated by a perfect one up to something in $\mathrm{Coh}(Z)^{<-n}$ for any given n.

In other words, the difference between $\mathrm{IndCoh}(Z)$ and $\mathrm{QCoh}(Z)$ occurs 'somewhere at $-\infty$'. So, this difference can only become tangible in the finer questions of homological algebra (such as convergence of spectral sequences).

However, we do need to address these questions adequately if we want to have a functioning theory, and for the kind of applications we have in mind (see Sect. 1.5 above), this necessitates working with IndCoh rather than QCoh.

As an illustration of how the theory of IndCoh takes something very familiar and unfolds it to something non-trivial, let is comment on the nature of the IndCoh direct image functor.

In the case of schemes, for a morphism $f : Z' \to Z$, the functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z') \to \mathrm{IndCoh}(Z)$$

does 'little new' as compared to the usual

$$f_* : \mathrm{QCoh}(Z') \to \mathrm{QCoh}(Z).$$

Namely, f_*^{IndCoh} is the unique functor that preserves infinite direct sums and makes the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(Z')^{\geq -n} & \xrightarrow[\sim]{\Psi_{Z'}} & \mathrm{QCoh}(Z')^{\geq -n} \\ {\scriptstyle f_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{IndCoh}(Z)^{\geq -n} & \xrightarrow[\sim]{\Psi_Z} & \mathrm{QCoh}(Z)^{\geq -n} \end{array}$$

[7]But the functor Ψ_Z is an equivalence on all of $\mathrm{IndCoh}(Z)$ if *and only if* Z is smooth.

commute for every n.

However, as was mentioned already, once we extend the formalism of IndCoh direct image to inf-schemes, we will in particular obtain the de Rham direct image functor. So, it is in the world of inf-schemes that IndCoh shows its full strength.

2.5. **Construction of the !-pullback functor.** As has been mentioned already, a major component of work in this book is the construction of the functor

$$\mathrm{IndCoh}^!_{\mathrm{Sch}} : (\mathrm{Sch})^{\mathrm{op}} \to \mathrm{DGCat}$$

of (2.3).

We already know what IndCoh(Z) for an individual scheme. We now need to extend it to morphisms.

For a morphism $f : Z' \to Z$, we can factor it as

$$Z' \overset{f_1}{\to} \overline{Z}' \overset{f_2}{\to} Z, \tag{2.4}$$

where f_1 is an open embedding and f_2 is proper. We then define

$$f^! : \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(Z')$$

to be

$$f_1^! \circ f_2^!,$$

where

(i) $f_2^!$ is the *right adjoint* of $(f_2)_*^{\mathrm{IndCoh}}$;

(ii) $f_1^!$ is the *left adjoint* of $(f_1)_*^{\mathrm{IndCoh}}$.

Of course, in order to have $f^!$ as a well-defined functor, we need to show that its definition is independent of the factorization of f as in (2.4). Then we will have to show that the definition is compatible with compositions of morphisms. But this is only the tip of the iceberg.

Since we want to have a functor between ∞-categories, we need to supply the assignment

$$f \rightsquigarrow f^!$$

with a *homotopy-coherent* system of compatibilities for n-fold compositions of morphisms, a task which appears unfeasible to do 'by hand'.

What we do instead is we prove an existence and uniqueness theorem... but not for (2.3), but for a more ambitious piece of structure. We refer the reader to [Chapter II.2, Proposition 2.1.4] for the precise formulation. Here we will only say that, in addition to (2.3), this piece of structure contains the datum of a functor

$$\mathrm{IndCoh} : \mathrm{Sch} \to \mathrm{DGCat}, \tag{2.5}$$

$$Z \rightsquigarrow \mathrm{IndCoh}(Z), \quad (Z' \overset{f}{\to} Z) \rightsquigarrow (\mathrm{IndCoh}(Z') \overset{f_*^{\mathrm{IndCoh}}}{\longrightarrow} \mathrm{IndCoh}(Z)),$$

as well as *compatibility* between (2.3) and (2.5).

The latter means that whenever we have a Cartesian square

$$\begin{array}{ccc} Z_1' & \xrightarrow{g'} & Z' \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ Z_1 & \xrightarrow{g} & Z \end{array} \tag{2.6}$$

there is a canonical isomorphism of functors, called base change:

$$(f_1)_*^{\mathrm{IndCoh}} \circ (g')^! \simeq g^! \circ f_*^{\mathrm{IndCoh}}. \tag{2.7}$$

2.6. **Enter DAG.** The appearance of the Cartesian square (2.6) heralds another piece of 'bad news'. Namely, Z_1' must be the fiber product

$$Z_1 \underset{Z}{\times} Z'.$$

But what category should we take this fiber product in? If we look at the example

$$\begin{array}{ccc} \mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt} & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathbb{A}^1, \end{array}$$

(here $\mathrm{pt} = \mathrm{Spec}(k)$, $\mathbb{A}^1 = \mathrm{Spec}(k[t])$), we will see that the fiber product $\mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt}$ *cannot* be taken to be the point-scheme, i.e., it cannot be the fiber product in the category of usual (=classical) schemes. Rather, we need to take

$$\mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt} = \mathrm{Spec}(k \underset{k[t]}{\otimes} k),$$

where the tensor product is understood in the *derived* sense, i.e.,

$$k \underset{k[t]}{\otimes} k = k[\epsilon], \quad \deg(\epsilon) = -1.$$

This is to say in building the theory of IndCoh, we cannot stay with classical schemes, but rather enlarge our world to that of *derived algebraic geometry.*

So, unless the reader has already guessed this, in all the previous discussion, the word 'scheme' had to be understood as 'derived scheme'[8] (although in the main body of the book we say just 'scheme', because everything is derived).

However, this is not really 'bad news'. Since we are already forced to work with ∞-categories, passing from classical algebraic geometry to DAG does not add a new level of complexity. But it does add *a lot* of new techniques, for example in anything that has to do with deformation theory (see [Chapter III.1]).

Moreover, many objects that appear in geometric representation theory naturally belong to DAG (e.g., Springer fibers, moduli of local systems on a curve, moduli of vector bundles on a surface). That is, these objects are *not* classical, i.e., we cannot ignore their derived structure if we want to study their scheme-theoretic (as opposed to topological) properties. So, we would have wanted to do DAG in any case.

Here are two particular examples:

(I) Consider the category of D-modules (resp., perverse) sheaves on the double quotient

$$I\backslash G((t))/I,$$

where G is a connected reductive group, $G((t))$ is the corresponding loop group (considered as an ind-scheme) and $I \subset G((t))$ is the Iwahori subgroup. Then Bezrukavnikov's theory (see [Bez]) identifies this category with the category of ad-equivariant ind-coherent (resp., coherent)

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[8]Technically, for whatever has to do with IndCoh, we need to add the adjective 'laft'='locally almost of finite type', see [Chapter I.2, Sect. 3.5] for what this means.

sheaves on the *Steinberg scheme* (for the Langlands dual group). But what do we mean by the Steinberg scheme? By definition, this is the fiber product

$$\widetilde{\mathcal{N}} \underset{\mathfrak{g}}{\times} \widetilde{\mathcal{N}}, \tag{2.8}$$

where $\widetilde{\mathcal{N}}$ is the Springer resolution of the nilpotent cone. However, in order for this equivalence to hold, the fiber product in (2.8) needs be understood in the *derived* sense.

(II) Let X be a smooth and complete curve. Let $\mathrm{Pic}(X)$ be the *Picard stack* of X, i.e., the stack parameterizing line bundles on X. Let $\mathrm{LocSys}(X)$ be the stack parameterizing 1-dimensional local systems on X. The Fourier-Mukaï-Laumon transform defines an equivalence

$$\mathrm{Dmod}(\mathrm{Pic}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}(X)).$$

However, in order for this equivalence to hold, we need to understand $\mathrm{LocSys}(X)$ as a *derived stack*.

2.7. **Back to inf-schemes.** The above was a long detour into the constructions of the theory of IndCoh on schemes. If now $\mathcal{X}$ is an inf-scheme, the category $\mathrm{IndCoh}(\mathcal{X})$ is defined by the formula (2.1).

Thus, informally, an object $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ is a family of assignments

$$(Z \overset{x}{\to} \mathcal{X}) \rightsquigarrow \mathcal{F}_{Z,x} \in \mathrm{IndCoh}(Z)$$

(here Z is a scheme) plus

$$(Z' \overset{f}{\to} Z) \in \mathrm{Sch}_{/\mathcal{X}} \rightsquigarrow f^!(\mathcal{F}_{Z,x}) \simeq \mathcal{F}_{Z',x'},$$

along with a homotopy-coherent compatibility data for compositions of morphisms.

For a map $g : \mathcal{X}' \to \mathcal{X}$, the functor

$$g^! : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X}')$$

is essentially built into the construction. Recall, however, that our goal is to also have the functor

$$g_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}') \to \mathrm{IndCoh}(\mathcal{X}).$$

The construction of the latter requires some work (which occupies most of Chapter III.3). What we show is that there exists a unique system of such functors so that for every commutative (but not necessarily Cartesian) diagram

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & \mathcal{X}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{i} & \mathcal{X} \end{array}$$

with Z, Z' being schemes and the morphisms i, i' proper, we have an isomorphism

$$g_*^{\mathrm{IndCoh}} \circ (i')_*^{\mathrm{IndCoh}} \simeq i_*^{\mathrm{IndCoh}} \circ f^{\mathrm{IndCoh}},$$

where i_*^{IndCoh} (resp., $(i')_*^{\mathrm{IndCoh}}$) is the left adjoint of $i^!$ (resp., $(i')^!$).

Amazingly enough, this procedure contains the de Rham push-forward functor as a particular case.

3. What is actually done in this book?

This book consists of five Parts (I-V) and an Appendix (Part A). Each part consists of several Chapters. These Chapters are designed so that they can be read independently from one another (in a sense, each Chapter is structured like a separate paper with its own introduction that explains what this particular chapter does).

Below we will describe the contents of the different Parts and Chapters from several different perspectives: (a) goals and role in the overall project; (b) nature of work; (c) practical implications; (d) logical dependence.

3.1. **The contents of the different parts.**

Part I is called 'preliminaries', and it is really preliminaries.

Part II builds the theory of IndCoh on schemes.

Part III defines the notion of *inf-scheme* and extends the formalism of IndCoh from schemes to inf-schemes, and in that it achieves one of the two main goals of this book.

Part IV consists of applications of the theory of IndCoh: we talk about formal moduli problems, Lie theory and infinitesimal differential geometry, i.e., exactly the things one needs for geometric representation theory. Making these constructions available is the second of our main goals.

Part V develops the formalism of *categories of correspondences*; it is used as a 'black box' in the key constructions in Parts II and III: this is our tool of bootstrapping the theory of IndCoh out of a much smaller amount of data.

Part A is sketch of the theory of $(\infty, 2)$-categories, which, in turn, is crucially used in Part V.

3.2. **Which chapters should a practically minded reader be interested in?** Not all the Chapters in this book make an enticing read; some are downright technical and tedious. Here is, however, a description of the 'cool' things that some of the Chapters do:

None of the material in Part I alters the pre-existing state of knowledge.

Chapters II.1 and II.2 should not be a difficult read. They construct the theory of IndCoh on schemes (the hard technical work is delegated to Chapter V.1). The reader cannot avoid reading these chapters if he/she is interested in the applications of IndCoh: one has to have an idea of what IndCoh is in order to use it.

Chapters II.3 is routine. The only really useful thing from it is the functor

$$\Upsilon_Z : \mathrm{QCoh}(Z) \to \mathrm{IndCoh}(Z),$$

given by tensoring an object of $\mathrm{QCoh}(Z)$ with the dualizing complex $\omega_Z \in \mathrm{IndCoh}(Z)$. Extract this piece of information from Sects. 3.2-3.3 and move on.

Chapter III.1 is background on deformation theory. The reason it is included in the book is that the notion of inf-scheme is based on deformation theory. However, the reader may find the material in Sects. 1-7 of this Chapter useful without any connection to the contents of the rest of the book.

Chapter III.2 introduces inf-schemes. It is quite technical. So, the practically minded reader should just understand the definition (Sect. 3.1) and move on.

Chapter III.3 bootstraps the theory of IndCoh from schemes to inf-schemes. It is not too technical, and should be read (for the same reason as Chapters II.1 and II.2). The hard technical work is delegated to Chapter V.2.

Chapter III.4 explains how the theory of crystals/D-modules follows from the theory of IndCoh on inf-schemes. Nothing in this Chapter is very exciting, but it should not be a difficult read either.

Chapter IV.1 talks about formal moduli problems. It proves a pretty strong result, namely, the equivalence of categories between formal groupoids acting on a given prestack $\mathcal{X}$ (assumed to admit deformation theory) and formal moduli problems *under* $\mathcal{X}$.

Chapter IV.2 is a digression on the general notion of Lie algebra and Koszul duality in a symmetric monoidal DG category. It gives a nice interpretation of the universal enveloping algebra of a Lie algebra of $\mathfrak{g}$ as the homological Chevalley complex of the Lie algebra obtained by *looping* $\mathfrak{g}$. The reader may find this Chapter useful and independently interesting.

Chapter IV.3 develops Lie theory in the context of inf-schemes. Namely, it establishes an equivalence of categories between group inf-schemes (over a given base $\mathcal{X}$) and Lie algebras in IndCoh($\mathcal{X}$). One can regard this result as one of the applications of the theory developed hereto.

Chapters IV.4 and IV.5 use the theory developed in the preceding Chapters for 'differential calculus' in the context of DAG. We discuss Lie algebroids and their universal envelopes, the procedure of deformation to the normal cone, etc. For example, the notion of n-th infinitesimal neighborhood developed in Chapter IV.5 gives rise to the Hodge filtration.

Chapter V.1 introduces the formalism of correspondences. The idea of the category of correspondences is definitely something worth knowing. We recommend the reader to read Sect. 1 in its entirety, then understand the universal property stated in Sect. 3, and finally get an idea about the two extension theorems, proved in Sects. 4 and 5, respectively. These extension theorems are the mechanism by means of which we construct IndCoh as a functor out of the category of correspondences in Chapter II.2.

Chapter V.2 proves a rather technical extension theorem, stated in Sect. 1; we do not believe that the reader will gain much by studying its proof. This theorem is key to the extension of IndCoh from schemes to inf-schemes in Chapter III.3.

Chapter V.3 is routine, except for one observation, contained in Sects. 2.2-2.3: the natural involution on the category of correspondences encodes *duality*. In fact, this is how we construct the Serre duality on IndCoh(Z) and Verdier duality on Dmod(Z) where Z is a scheme (or inf-scheme), see [Chapter II.2, Sect. 4.2], [Chapter III.3, Sect. 6.2], and [Chapter III.4, Sect. 2.2], respectively.

Chapter A.1 introduces the notion of $(\infty, 2)$-category and some basic constructions in the theory of $(\infty, 2)$-categories. This Chapter is not very technical (mainly because it omits most proofs) and might be of independent interest.

Chapter A.2 does a few more technical things in the theory of $(\infty, 2)$-categories. It introduces the $(\infty, 2)$-*category of* $(\infty, 2)$-*categories*, denoted 2-Cat. We then discuss the straightening/unstraightening procedure in the $(\infty, 2)$-categorical context and the $(\infty, 2)$-categorical Yoneda. The statements of the results from this Chapter may be of independent interest.

Chapter A.3 discusses the notion of adjunction in the context of $(\infty, 2)$-categories. The main theorem in this Chapter explicitly constructs the *universal adjointable functor* (and its variants), and we do believe that this is of interest beyond the particular goals of this book.

3.3. **The nature of the technical work.** The substance of mathematical thought in this book can be roughly split into three modes of cerebral activity: (a) making constructions; (b) overcoming difficulties of homotopy-theoretic nature; (c) dealing with issues of *convergence*.

Mode (a) is hard to categorize or describe in general terms. This is what one calls 'the fun part'.

Mode (b) is something much better defined: there are certain constructions that are obvious or easy for ordinary categories (e.g., define categories or functors by an explicit procedure), but require some ingenuity in the setting of higher categories. For many readers that would be the least fun part: after all it is clear that the thing should work, the only question is how to make it work without spending another 100 pages.

Mode (c) can be characterized as follows. In low-tech terms it consists of showing that certain spectral sequences converge. In a language better adapted for our needs, it consists of proving that in some given situation we can swap a limit and a colimit (the very idea of IndCoh was born from this mode of thinking). One can say that mode (c) is a sort of analysis within algebra. Some people find it fun.

Here is where the different Chapters stand from the point of view of the above classification:

Chapter I.1 is (b) and a little of (c).

Chapter I.2 is (a) and a little of (c).

Chapter I.3 is (c).

Chapter II.1 is (a) and (c).

Chapter II.2 is (a).

Chapter II.3 is (b).

Chapter III.1 is (a) and a little of (c).

Chapter III.2 is (a) and a little of (c).

Chapter III.3 is (a).

Chapter III.4 is (a).

Chapter IV.1 is (a).

Chapter IV.2 is (c) and a little of (b).

Chapter IV.3 is (c) and a little of (a).

Chapters IV.4 and IV.5 are (a).

Chapters V.1-V.3 are (b).

Chapters A.1-A.3 are (b).

3.4. **Logical dependence of chapters.** This book is structured so that the most straightforward way to read is the linear one. However, below is a scheme of the logical dependence of chapters, where we allow a 5% *skip margin* (by which we mean that the reader skips certain things[9] and comes back to them when needed).

[9]These are things that can be taken on faith without compromising the overall understanding of the material.

3.4.1. Chapter I.1 reviews ∞-categories and higher algebra. Read it only if you have zero prior knowledge of these subjects. In the latter case, here is what you will need in order to access the constructions in the main body of the book:

Read Sects. 1-2 to get an idea of how to operate with ∞-categories (this is a basis for everything else in the book).

Read Sects. 5-7 for a summary of *stable* ∞-categories: this is what our QCoh$(-)$ and IndCoh$(-)$ are; forget on the first pass about the additional structure of k-linear DG category (the latter is discussed in Sect. 10).

Read Sects. 3-4 for a summary of monoidal structures and duality in the context of higher category theory. You will need it for this discussion of Serre duality and for Chapter II.3.

Sects. 8-9 are about algebra in (symmetric) monoidal stable ∞-categories. You will need it for Part IV of the book.

Chapter I.2 introduces DAG proper. If you have not seen any of it before, read Sect. 1 for the (shockingly general, yet useful) notion of prestack. Whatever type of geometric objects we will encounter in this book (e.g., (derived) schemes, Artin stacks, inf-schemes, etc.) it will be a full subcategory in the ∞-category of prestacks. Proceed to Sect. 3.1 for the definition of derived schemes. Skip all the rest.

Chapter I.3 introduces QCoh on prestacks. Even though the main focus of this book is the theory of ind-coherent sheaves, the latter theory takes a significant input and interacts with that of quasi-coherent sheaves. If you have not seen this before, read Sect. 1 and then Sects. 3.1-3.2.

3.4.2. In Chapter II.1 we develop the elementary aspects of the theory of IndCoh on schemes: we define the DG category IndCoh(Z) for an individual scheme Z, construct the IndCoh direct image functor, and also the !-pullback functor for proper morphisms. This Chapter uses the material from Part I mentioned above. You will need the material from this chapter in order to proceed with the reading of the book.

Chapter II.2 builds on Chapter II.1, and accomplishes (modulo the material delegated to Chapter V.1) one of the main goals of this book. We construct IndCoh as a *functor out of the category of correspondences*. In particular, we construct the functor (2.3). The material from this Chapter is also needed for the rest of the book.

In Chapter II.3 we study the interaction between IndCoh and QCoh. For an individual scheme Z we have an action of QCoh(Z) (viewed as a monoidal category) on IndCoh(Z). We study how this action interacts with the formalism of correspondences from Chapter II.2, and in particular with the operation of !-pullback. The material in this Chapter uses the formalism of monoidal categories and modules over them from Chapter I.1, as well as the material from Chapter II.2. Skipping Chapter II.3 will not impede your understanding of the rest of the book, so it might be a good idea to do so on the first pass.

3.4.3. Chapter III.1 introduces deformation theory. It is needed for the definition of inf-schemes and, therefore, for proofs of any results about inf-schemes (that is, for Chapter III.2). We will also need it for the discussion of formal moduli problems in Chapter IV.1. The prerequisites for Chapter III.1 are Chapters I.2 and I.3, so it is (almost)[10] independent of the material from Part II.

[10]Whenever we want to talk about *tangent* (as opposed to *cotangent*) spaces, we have to use IndCoh rather than QCoh, and these parts in Chapter III.1 use the material from Chapter II.2.

In Chapter III.2 we introduce inf-schemes and some related notions (ind-schemes, ind-inf-schemes). The material here relies in that of Chapter III.1, and will be needed in Chapter III.3.

In Chapter III.3 we construct the theory of IndCoh on inf-schemes. The material here relies on that from Chapters II.2 and III.2 (and also a tedious general result about correspondences from Chapter V.2). Thus, Chapter III.3 achieves one of our goals, the later being making the theory of IndCoh on inf-schemes available. The material from Chapter III.3 will (of course) be used when will apply the theory of IndCoh, in Chapters III.4 and Chapters IV.3-IV.5.

In Chapter III.4 we apply the material from Chapter III.3 in order to develop a proper framework for crystals (=D-modules), together with the forgetful/induction functors that related D-modules to $\mathcal{O}$-modules. The material from this Chapter will not be used later, except for the hyper-useful notion of the de Rham prestack construction $\mathcal{X} \rightsquigarrow \mathcal{X}_{\mathrm{dR}}$.

3.4.4. In Chapter IV.1 we prove a key result that says that in the category of prestacks that admit deformation theory, the operation of taking the quotient with respect to a formal groupoid is well-defined. The material here relies on that from Chapter III.1 (at some point we appeal to a proposition from Chapter III.3, but that can be avoided). So, the main result from Chapter IV.1 is independent of the discussion of IndCoh.

Chapter IV.2 is about Lie algebras (or more general operad algebras) in symmetric monoidal DG categories. It only relies on the material from Chapter I.1, and that is independent of the preceding Chapters of the book (no DAG, no IndCoh). The material from this Chapter will be used for the subsequent Chapters in Part IV.

3.4.5. *A shortcut.* As has been mentioned earlier, Chapters IV.3-IV.5 are devoted to applications of IndCoh to 'differential calculus'. This 'differential calculus' occurs on prestacks that admit deformation theory.

If one really wants to use arbitrary such prestacks, one needs the entire machinery of IndCoh provided by Chapter III.3. However, if one is content with working with inf-schemes (which would suffice for the majority of applications), much less machinery would suffice:

The cofinality result from [Chapter III.3, Sect. 4.3] implies that we can bypass the entire discussion of correspondences, and only use the material from Chapter II.1, i.e., IndCoh on schemes and !-pullbacks for proper (in fact, finite) morphisms.

3.4.6. Chapters IV.3-IV.5 form a logical succession. As in input from the preceding chapters they use Chapter III.3 (resp., Chapter II.2 (see Sect. 3.4.5 above)), Chapter III.1 and Chapters IV.1-IV.2.

3.4.7. Part V develops the theory of categories of correspondences. It plays a service role for Chapters II.2 and III.3, and relies on the theory of $(\infty, 2)$-categories, developed in Part A.

Chapters V.2 and V.3 rely on Chapter V.1, but can be read independently of one another.

3.4.8. Part A develops the theory of $(\infty, 2)$-categories. It plays a service role for Part V.

INTRODUCTION TO PART I: PRELIMINARIES

Why do we need these preliminaries?

0.1. None of the contents of Part I is original mathematics.

Chapter I.1 is a review of higher category theory and higher algebra, mostly following [Lu1] and [Lu4].

Chapter I.2 is a review of the basic definitions of derived algebraic geometry (derived schemes, Artin stack and general prestacks), mostly following [TV1, TV2].

Chapter I.3 is a review of the basics of quasi-coherent sheaves (there are no deep theorems there, so one can say that it is mostly folklore).

0.2. We wish to emphasize that by no means do these chapters supply a self-contained exposition of elements of the theory required for the rest of the book. Our goal is rather to give the reader a concise account of the most basic pieces of structure, in order to enable him/her to start reading the subsequent chapters.

Our hope is that once he/she gets started, he/she will gradually acquire the ability to look up or reconstruct the necessary bits of foundational material.

1. ∞-categories and higher algebra

1.1. Let us accept the inevitable: when we talk about algebraic geometry, we need to talk in the language of categories.

For one thing, geometric objects (such as schemes and their generalizations) form a category. But event more importantly, the flora to be found on these geometric objects (sheaves of various sorts) is a category. And there is no way to develop the theory of sheaves without using categories.

Since the introduction of the categorical language to the study of algebraic geometry by Grothendieck in the 1950's, and up until the late 2000's, the methods of *usual* (=ordinary) category theory sufficed for most purposes. People used either the *abelian* category of quasi-coherent sheaves or its derived category, which is a *triangulated category.*

However, there are some instances where triangulated categories are not enough. Perhaps the main example of this is the failure of gluing: one cannot glue the derived category of quasi-coherent sheaves on a scheme from just knowing it on an open cover.

Now, the problem of inadequacy of triangulated categories becomes even more acute in the context of *derived algebraic geometry* (DAG). So, having accepted the inevitability of categories for usual algebraic geometry, we now have no choice but accept the inevitability of ∞-categories if we want to work in DAG. This is further reinforced by the fact that the geometric objects themselves (derived schemes or, more generally, prestacks) now form an ∞-category.

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1.2. In [Chapter I.1, Sects. 1 and 2] we give a concise review of the basics of ∞-categories.

We mostly focus on the syntax: how to use the language of ∞-categories. In other words, the reader does not have to be familiar with a particular model for ∞-categories, be it topological categories, simplicial categories, or the model that finally won the day–Joyal's quasi-categories, put into action by Lurie in [Lu1].

We introduce the key notions of Cartesian/coCartesian fibration, Yoneda, limit/colimit, cofinality, left/right Kan extension, adjunction for functors.

1.3. In [Chapter I.1, Sects. 3 and 4] we give the first taste of *higher algebra*. We introduce the notions of *monoidal* ∞*-category* and of associative algebra inside a monoidal ∞-category. We also introduce the corresponding commutative notions.

We also introduce the corresponding notions of module (that is, a module category for a given monoidal ∞-category, and the notion of module for an associative algebra).

We then proceed to the discussion of duality. We discuss the notion of left/right dualizability of an object in a monoidal ∞-category, and the related notion of dualizability of left/right module over an algebra.

1.4. In [Chapter I.1, Sects. 5, 6, 7] we discuss the notion of *stable* ∞*-category.*

Stable ∞-categories are the higher categorical replacement of triangulated categories, i.e., this is where we really do algebra.

An operation that will play a key role in the book is that of *Lurie tensor product* of (co-complete) stable ∞-categories, that gives the totality of the latter, denoted, $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$ a structure of symmetric monoidal ∞-category.

1.5. In [Chapter I.1, Sects. 8 and 9] we supply a framework for "really doing algebra":

We talk about (symmetric) monoidal stable ∞-categories, i.e., associative (resp., commutative) algebra objects in the symmetric monoidal category $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$.

1.6. Finally, in [Chapter I.1, Sect. 10] we introduce the notion of *DG category*, i.e., a DG category that is equipped with a linear structure over a given ground field k.

2. Basics of derived algebraic geometry

In [Chapter I.2] we start our discussion of derived algebraic geometry proper, i.e., we introduce the ∞-category of the corresponding geometric objects.

2.1. We start with the category of (derived)[1] affine schemes over k, denoted Sch^{aff}, which is, by definition, the category opposite to that of *connective commutative DG algebras over* k.

In [Chapter I.2, Sect. 1] we introduce the most general class of geometric objects: prestacks. The ∞-category of the latter is simply that of functors

$$(\text{Sch}^{\text{aff}})^{\text{op}} \to \text{Spc},$$

where Spc is the ∞-category of spaces (a.k.a. ∞-groupoids). I.e., a prestack is just something that has a Grothendieck functor of points.

All other geometric objects that we will consider (schemes, Artin stacks, etc.) will be prestacks. That is, for example, a scheme (resp., Artin stack) will be a prestack with certain properties (as opposed to additional pieces of structure).

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[1]Henceforth the adjective 'derived' will be dropped, because everything will be derived.

In later Chapters of the book we will be interested in yet another class of prestacks–indschemes.

2.2. In [Chapter I.2, Sect. 2] we will introduce the descent condition with respect to the Zariski, étale or faithfully flat topology. We call prestacks that satisfy the descent condition *stacks*.

We study how the descent condition interacts with the basic properties that a prestack can possess (such as being locally of finite type).

2.3. In [Chapter I.2, Sect. 3] we introduce what is, arguably, the main object of study in derived algebraic geometry: (derived) schemes.

According to what was said above, we *do not* introduce schemes as locally ringed spaces. Rather, we define schemes as prestacks that admit an open covering by affine schemes.

2.4. In [Chapter I.2, Sect. 4] we introduce the hierarchy of k-Artin stacks, $k \geq 0$. We should say that we call a k-Artin stack for a particular k may diverge from elsewhere in the literature (for example, for us, a 0-Artin stack is a stack that is a (possibly infinite) disjoint of affine schemes). However, the union over all k produces the same class of objects. The advantage of our particular system of definitions is that it makes inductive proofs of various properties of k-Artin stacks very simple.

We should also point out that from the point of view of our hierarchy of k-Artin stacks, schemes are a red herring. They are more general than 0-Artin stacks, but are a tiny particular case of 1-Artin stacks.

3. Quasi-coherent sheaves

In [Chapter I.3] we introduce what is perhaps the main object of study of (derived) algebraic geometry: quasi-coherent sheaves.

3.1. In [Chapter I.3, Sect. 1] we start with the functor

$$\mathrm{QCoh}^*_{\mathrm{Sch}^{\mathrm{aff}}} : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}, \quad S = \mathrm{Spec}(A) \rightsquigarrow A\text{-mod}, \quad (S' \xrightarrow{f} S) \rightsquigarrow f^*.$$

We apply the procedure of *right Kan extension* along the (Yoneda) embedding $\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk}$ and thus obtain a functor

$$\mathrm{QCoh}^*_{\mathrm{PreStk}} : (\mathrm{PreStk})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

Thus, for any prestack $\mathcal{Y}$ we have a well-defined DG category $\mathrm{QCoh}(\mathcal{Y})$ and for a morphism $f : \mathcal{Y}' \to \mathcal{Y}$ we have a pullback functor $f^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}')$.

3.2. Note, in particular, that if Z is a scheme, we obtain a category $\mathrm{QCoh}(Z)$. This definition of QCoh of a scheme is equivalent to any other (correct) definition. However, we note that we *do not* approach it via first considering *all sheaves of $\mathcal{O}$-modules in Zariski topology*, and then passing to a subcategory. Instead, we directly glue $\mathrm{QCoh}(Z)$ from affines.

A similar feature of our definition is also present in the case of Artin stacks.

3.3. In [Chapter I.3, Sect. 2] we study the functor of *direct image* for quasi-coherent sheaves

$$f_* : \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y})$$

for a morphism $f : \mathcal{Y}' \to \mathcal{Y}$. By definition, f_* is the right adjoint of f^*, and exists for abstract reasons (the Adjoint Functor Theorem).

For a general morphism f, the functor f_* is very badly behaved. For example, it fails to satisfy the base change formula. However, by imposing some additional assumptions on f one can ensure that it is reasonable. One such assumption is that f should be schematic quasi-compact.

3.4. In [Chapter I.3, Sect. 3] we study the natural *right lax* symmetric monoidal structure on the functor $\mathrm{QCoh}^*_{\mathrm{PreStk}}$. Concretely, this structure amounts to (a compatible family of) functors

$$\mathrm{QCoh}(\mathcal{Y}_1) \boxtimes \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2), \quad \mathcal{Y}_1, \mathcal{Y}_2 \in \mathrm{PreStk}\,.$$

We study the question of when the above functor is an equivalence.

The symmetric monoidal structure on $\mathrm{QCoh}^*_{\mathrm{PreStk}}$ induces a symmetric monoidal structure on the category $\mathrm{QCoh}(\mathcal{Y})$ for an individual prestack $\mathcal{Y} \in \mathrm{PreStk}$.

We study how various properties of a prestack $\mathcal{Y}$ reflect in properties of $\mathrm{QCoh}(\mathcal{Y})$ (compact generation, dualizability, rigidity).