MODEL THEORETIC CHARACTERIZATIONS OF LARGE CARDINALS

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ABSTRACT. We consider model-theoretic characterizations of large cardinals, especially in terms of the compactness for omitting types (previously used by Benda [Ben78] to characterize super compacts).

1. Introduction

Some large cardinals have well-known model theoretic characterizations. Chief among these are the weakly and strongly compact cardinals, which were first isolated with generalizations of the compactness theorem to infinitary languages.

Fact 1.1.

(1) *$\kappa > \omega$ is weakly compact iff every $< \kappa$-satisfiable theory of size κ in $\mathbb{L}_{\kappa,\kappa}$ is satisfiable.*
(2) *$\kappa > \omega$ is strongly compact iff every $< \kappa$-satisfiable theory in $\mathbb{L}_{\kappa,\kappa}$ is satisfiable.*

Measurable cardinals also have such a characterization, this time in terms of *chain* compactness of $\mathbb{L}_{\kappa,\kappa}$. This result is interesting because it seems have been well-known in the past (evidenced by the fact that it appears as an exercise in Chang and Keisler's **Model Theory** [CK12, Exercise 4.2.6]), but seems to have fallen out of common knowledge even among researchers working in the intersection of set theory and model theory (at least among the younger generation)[1].

Fact 1.2. *κ is measurable iff every theory $T \subset \mathbb{L}_{\kappa,\kappa}$ that can be written as a union of an increasing sequence of satisfiable theories is itself satisfiable.*

There are a few other known characterizations. Looking at second-order logic, Magidor [Mag71] characterizes supercompacts as the Löwenheim-Skolem-Tarski numbers for second-order and extendibles as thestrong compactness cardinals for second-order. Magidor's paper focuses on finitary $\mathbb{L}^2$ in a finite language, but the arguments can easily be adapted to show the following:

Fact 1.3 ([Mag71, Theorems 2 and 4])**.**

(1) *κ is supercompact iff for every language τ of size $< \kappa$, every $\alpha < \kappa$, every τ-structure N, and every $\phi \in \mathbb{L}^2_{\alpha,\alpha}(\tau)$ satisfied by N, there is $M \subset N$ of size $< \kappa$ such that $M \vDash \phi$*[2].
(2) *κ is extendible iff every $< \kappa$-satisfiable $\mathbb{L}^2_{\kappa,\kappa}$-theory T is satisfiable.*

Magidor and Väänänen [MV11] explore the possibilities surrounding the Löwenheim-Skolem-Tarski numbers of various fragments of second-order logic and Bagaria and Väänänen [BV16] connect structural reflection properties and Löwenheim-Skolem-Tarski numbers through Väänänen's notion of symbiosis.

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[1]Although the statement seems forgotten, the proof is standard: if $T = \cup_{\alpha<\kappa} T_\alpha$ and $M_\alpha \vDash T_\alpha$, then set $M_\kappa := \prod M_\alpha / U$ for any κ-complete, nonprincipal ultrafilter U on κ. Łoś' Theorem for $\mathbb{L}_{\kappa,\kappa}$ implies $M_\kappa \vDash T$.
[2]A cardinal satisfying this property is called the Löwenheim-Skolem-Tarski of the logic.

Returning to $\mathbb{L}_{\kappa,\kappa}$, Benda [Ben78] characterized supercompact cardinals on a level-by-level basis by how compact $\mathbb{L}_{\kappa,\kappa}$ is *with regards to omitting types.* Tragically, Benda's fantastic result is even less well-known than the characterization of measurable cardinals. In the almost 40 years since it's publication, the publisher reports zero citations of Benda's paper.

Fact 1.4 ([Ben78, Theorem 1]). *Let $\kappa \leq \lambda$. κ is λ-supercompact iff for every $\mathbb{L}_{\kappa,\kappa}$-theory T and type $p(x, y) = \{\phi_i(x, y) \mid i < \lambda\}$, if there are club-many $s \in \mathcal{P}_\kappa\lambda$ such that there is a model of*

$$T \cup \left\{ \exists x \left(\bigwedge_{i \in s} \exists y \phi(x, y) \wedge \neg \exists y \bigwedge_{i \in s} \phi(x, y) \right) \right\}$$

then there is a model of

$$T \cup \left\{ \exists x \left(\bigwedge_{i < \lambda} \exists y \phi(x, y) \wedge \neg \exists y \bigwedge_{i < \lambda} \phi(x, y) \right) \right\}$$

There are a few other results in this direction. Makowsky [Mak85] characterizes Vopenka's Principle by the existence of a strong compactness cardinals for every logic (see Fact 3.10 below) and Stavi gave a similar characterization of Vopenka's Principle by the existence of Lowenheim-Skolem-Tarski numbers (unpublished, but see [MV11, Theorem 6]).

Inspired by these results, this paper attempts to add to the characterizations of large cardinals based on model-theoretic properties. The start of most of these results is Benda's idea of compactness for omitting types. We can give Benda's result a more modern gloss (and a slight extension):

Let T be an $\mathbb{L}_{\kappa,\kappa}$-theory that can be written as an increasing union of $\{T_s \mid s \in \mathcal{P}_\kappa\lambda\}$ and $p(x) = \{\phi_i(x) \mid i < \lambda\}$ a type. If there are club-many[3] $s \in \mathcal{P}_\kappa\lambda$ such that T_s has a model omitting $p_s(x) := \{\phi_i(x) \mid i \in s\}$, then T has a model omitting p.

In Definition 3.3, we generalize this notion to arbitrary subsets of $\mathcal{P}(\lambda)$ on which there might be a κ-complete, fine, normal ultrafilter. In that language, Benda's result deals with $\mathbb{L}_{\kappa,\kappa}$ being $\mathcal{P}_\kappa\lambda$-$\kappa$-compact for type omission.

Section 3 gives a characterization (Theorem 3.4) of when $\mathbb{L}_{\kappa,\kappa}$ satisfies variously parameterized compactness for type omissions in terms of the existence of certain ultrafilters and the existence of elementary $j : V \to \mathcal{M}$ with $\mathcal{M}$ satisfying certain closure conditions (these of course being equivalent to one another). In this way, *any* large cardinal property that can be characterized by the existence of a normal ultrafilter on some $\mathcal{P}(\lambda)$ (or concentrating on a subset of it) can be given a model-theoretic characterization along the lines of Benda's result. We then discuss various consequences of these characterizations, including a definable class version and a method for going directly between model-theoretic and elementary embedding characterizations of large cardinals *without* appealing to an intermediary ultrafilter.

Section 4 extends this characterization to the compactness for type omission of second-order logic (Theorem 4.1) and sort logic (Theorem 4.9). In addition to discussing consequences of these characterizations, we also discuss other large cardinal notions (strong, rank-into-rank, etc.) and ways to characterize them by model-theoretic means. One outgrowth of this is that, in a sense, the "identity crisis" that exists for strongly compact (situated somewhere between measurable and supercompact, see Magidor [Mag76]) disappears when moving to the 'second-order version' of these cardinals, as they are all equivalent to being extendible (Theorem 4.3), or $C^{(n)+}$-extendible cardinals for the sort version (Proposition 4.13).

There are some implications of this work that might be of interest even if the reader is uninterested in model-theoretic characterizations or compactness for type omission. This mode

[3]Clubs are used because they are guaranteed to be in any normal, fine, κ-complete ultrafilter on $\mathcal{P}_\kappa\lambda$.

of analyzing large cardinals provides a fertile ground for isolating new large cardinal concepts; if a useful large cardinal property can be characterized by imposing some structure on a logic $\mathcal{L}$, then it stands to reason that varying the logic will give a candidate large cardinal property that might be useful. We do this, e. g., in introducing the notion of weakly compact-for-$\mathbb{L}^2_{\kappa,\kappa}$; see Theorem 4.5 (and Definition 4.2 for this naming convention). In particular, Gabriel Goldberg reports that huge-for-$\mathbb{L}^2_{\kappa,\kappa}$ has a formulation in terms of hyperhuge cardinals, which have recently become important for set-theoretic geology due to results of Usuba [Usu] (see the discussion after Proposition 4.4). Explorations of sort logic and the $C^{(n)}$-hierarchy lead to an equivalent formulation of $C^{(n)+}$-extendible cardinals by a weaker seeming definition (Proposition 4.11). Finally, Section 4.3 combines Magidor's result on supercompacts and Henkin models to define a notion of elementary substructure for second-order. While still model-theoretic, this leads to a purely combinatorial notion of a superclub on $\mathcal{P}_\kappa\lambda$ (that has the same relation to second-order elementary substructure as clubs has to first-order) and a new characterization of supercompacts (Corollary 4.28).

Most of the outline of the paper has been given about. Section 2 gives the necessary model-theoretic preliminaries. Section 5 discusses extenders and type omission around the following question: we give characterizations of large cardinals with various logics, from $\mathbb{L}_{\kappa,\kappa}$ to the all-powerful $\mathbb{L}^{s,\Sigma_n}$. However, $\mathbb{L}_{\kappa,\kappa}$ is already able to work it's way up the large cardinal hierarchy, including n-hugeness (Corollary 3.5) and rank-into-rank (Section 4.2). Are these more powerful logics necessary? In other words, can we characterize all model-theoreticaly characaterizable large cardinals (extendible, etc.) by some property of $\mathbb{L}_{\kappa,\kappa}$, or are the use of stronger logics necessary to pin down certain cardinals. We focus on strong cardinals, and give a theory and collection of types such that the ability to find a model of the theory omitting the types is equivalent κ being λ strong. As close as this seems to an $\mathbb{L}_{\kappa,\kappa}$ (or rather $\mathbb{L}_{\kappa,\omega}(Q^{WF})$) characterization of strong cardinals, we still lack a general compactness for type omission for this case.

Preliminary results along these lines were first presented at the Workshop on Set-Theoretical Aspects of the Model Theory of Strong Logics hosted by the Centre de Recerca Matemàtica in 2016, and I'd like to thank many of the the participants for helpful conversations, especially Jouko Väänänen for discussions about sort logic. I'd also like to thank Gabriel Goldberg for helpful discussions regarding the strength of huge-for-$\mathbb{L}^2_{\kappa,\kappa}$ cardinals.

2. Preliminaries

We begin with an informal introduction to the logics used. The large cardinals notions are standard; consult Kanamori [Kan08] or the locally given citation for detail. We introduce some new large cardinal notions, typically naming them and defining them in the statement of a result: see Corollary 3.6, Proposition 4.4, and Theorems 4.5 and 4.8.

$\mathbb{L}_{\omega,\omega}$ is the standard, elementary first-order logic.

$\mathbb{L}_{\lambda,\kappa}$ augments by allowing

- conjunctions of $< \lambda$-many formulas that together contain $< \kappa$-many free variables;
- $< \kappa$-ary functions and relations in the language.
- universal and existential quantification over $< \kappa$-many variables at once.

We typically restrict to $\lambda \geq \kappa$, both regular.

$\mathbb{L}^2 = \mathbb{L}^2_{\omega,\omega}$ is second-order logic which extends $\mathbb{L}_{\omega,\omega}$ by allowing quantification over subsets of cartesian powers of the universe and has an atomic 'membership' relation. We can also introduce higher-order variants $\mathbb{L}^n$, but these are all codeable in $\mathbb{L}^2$ (or $\mathbb{L}^2_{|\alpha|^+,\omega}$ for $\mathbb{L}^\alpha$ when α is infinite), so we focus on $\mathbb{L}^2$. The standard interpretation of the quantification is over all subsets, but an important concept is the nonstandard Henkin models (M, P, E), where M is τ-structure,

$E \subset M \times P$ is an extensional relation, and P represents a collection of subsets that the second-order quantifiers can range over. The class of Henkin models of a second-order theory reduces to the models of a sorted $\mathbb{L}_{\omega,\omega}$-theory, but we will still find use for this in Definition 4.6 and the characterization of strong cardinals.

Additionally, when dealing with second-order logic, we allow the language to include functions and relations whose domain and range include the second-order part of the model. Given such a second-order language τ, we describe it as consisting of a first-order part and a strictly second-order part.

$\mathbb{L}(Q^{WF})$ is $\mathbb{L}_{\omega,\omega}$ augmented by the quantifier Q^{WF} that takes in two free variables and so $Q^{WF}xy\phi(x, y, \mathbf{z})$ is true iff there is no infinite sequence $\{x_n \mid n < \omega\}$ such that $\phi(x_{n+1}, x_n, \mathbf{z})$ holds for all $n < \omega$; that is, $\phi(x, y, \mathbf{z})$ defines a well-founded relation. Note that, in models of some choice, Q^{WF} is both $\mathbb{L}_{\omega_1,\omega_1}$ and $\mathbb{L}^2$ expressible. However, it will be useful to have, e. g., in Theorem 4.7.

Finally, sort logic $\mathbb{L}^s$ is a logic introduced by Väänänen [Vää79]. This augments second-order logic by adding sort quantifiers $\exists^\sim, \forall^\sim$ where $\exists^\sim X\phi(X, \mathbf{x})$ is true iff there is a set X (any set, not just a subset of the universe) such that $\phi(X, \mathbf{x})$ is true. Sort logic is very powerful because it allows one to access a large range of information regardless of the language of the initial structure. For instance, one can easily write down a formula Φ whose truth in any structure implies the existence of an inaccessible cardinal. Väänänen discusses its use as a foundation of mathematics in [Vää14]. Since sort logic involves satisfaction of formulas in V, for definability of truth reasons, we must restrict to the logics $\mathbb{L}^{s,\Sigma_n}$ that are Σ_n when looking at the quantifiers over sorts.

Finally, all of these logics can be combined in the expected way, e. g., $\mathbb{L}^2_{\kappa,\kappa}$. We often take the union of two logics, e. g., $\mathbb{L}^2 \cup \mathbb{L}_{\kappa,\omega}$ is the logic whose formulas are in $\mathbb{L}^2$ or $\mathbb{L}^{\kappa,\omega}$; however, no second-order quantifier or variable can appear in any formula with an infinite conjunction, which separates it from $\mathbb{L}^2_{\kappa,\omega}$. We typically use boldface $\mathbb{L}$ when discussing a particular logic and script $\mathcal{L}$ when discussing an abstract logic.

For a logic $\mathcal{L}$ and a language τ, an $\mathcal{L}(\tau)$-theory T is a collection of sentences (formulas with no free variables) of $\mathcal{L}(\tau)$. An $\mathcal{L}(\tau)$-type $p(x)$ in x is a collection of formulas from $\mathcal{L}(\tau)$ all of whose free variables are at most x[4]. A type $p(x)$ is realized in a τ-structure M iff there is an element of the model that satisfies every formula in it and a type is *omitted* precisely when it is not realized. Note that the "monotonicity of type omission" works the opposite way as theories: if $p(x) \subset q(x)$ are both types, then it is easier to omit q than p. We will often refer to filtrations of a theory T. This means there is some ambient partial order $(I, \subset)$ and a collection of theories $\{T_s \mid s \in I\}$ such that $T = \cup_{s \in I} T_s$ and $s \subset t$ implies $T_s \subset T_t$.

In general, we are agnostic about how one codes these logics as sets, except to insist that it is done in a reasonable way, e. g., τ is coded as a set of rank $|\tau| + \omega$, $\mathbb{L}_{\kappa,\kappa}(\tau) \subset V_{\kappa+|\tau|}$, etc. This gives us two nice facts about the interaction between languages τ and elementary embeddings $j : V \to \mathcal{M}$ (or $V_\alpha \to V_\beta$, etc.) with crit $j = \kappa$:

- if τ is made up of $< \kappa$-ary functions and relations, then j"τ and τ are just renamings of each other; and
- if $\phi \in \mathbb{L}^{s,\Sigma_n}_{\kappa,\kappa}(\tau)$, then $j(\phi) \in \mathbb{L}^{s,\Sigma_n}_{\kappa,\kappa}(\tau)$.

This means that, when searching for a model of T, it will suffice to find a model of j"T, which is a theory in the same logic and an isomorphic language.

[4]Types in single variables suffice for the various characterizations in the paper, but they also extend to types of arity $< \kappa$.

Given an inner model[5] $\mathcal{M}$ (or some V_α), we collect some facts about when $\mathcal{M}$ is correct about various logics. That it, the statement "M is a τ-structure" is absolute from $\mathcal{M}$ to V and we want to know when the same holds of "ϕ is an $\mathcal{L}(\tau)$-formula and $M \vDash_{\mathcal{L}} \phi$."

- $\mathcal{M}$ is correct about the logic $\mathbb{L}_{ON^{\mathcal{M}},\omega}$.
- If ${}^{<\kappa}\mathcal{M} \subset \mathcal{M}$, then $\mathcal{M}$ is correct about the logic $\mathbb{L}_{\kappa,\kappa}$.
- If $\mathcal{P}(A) \in \mathcal{M}$, then $\mathcal{M}$ is correct about $\mathbb{L}^2$ for structures with universe A.
- If $\mathcal{M} \prec_{\Sigma_n} V$, then $\mathcal{M}$ is correct about $\mathbb{L}^{s,\Sigma_n}$.

3. Type omitting compactness in $\mathbb{L}_{\infty,\infty}$

We introduce some basic definitions that will be used in each of our characterizations.

Definition 3.1. *Let $\kappa \leq \lambda$ and $I \subset \mathcal{P}(\lambda)$.*

(1) *I is* κ-robust *iff it is closed under $< \kappa$-sized unions; for every $\alpha < \lambda$, $[\alpha]^I := \{s \in I \mid \alpha \in s\} \neq \emptyset$; and $I \subset \{s : |s \cap \kappa| < \kappa\}$.*
(2) *$C \subset I$ is a club iff*
   - *if $s \in I$, then there is $s' \in C$ such that $s \subset s'$; and*
   - *if $\{s_\alpha \in C \mid \alpha < \mu\}$ is increasing with $\cup s_\alpha \in I$, then $\cup s_\alpha \in C$.*

Let U be an ultrafilter on I.

(1) *U is* fine *iff for all $\alpha \in \lambda$, $[\alpha]^I \in U$.*
(2) *U is* normal *iff for all $F : I \to \lambda$ such that $\{s \in I \mid F(s) \in s\} \in U$, there is $\alpha_0 < \lambda$ such that $\{s \in I \mid F(s) = \alpha_0\} \in U$.*

The conditions of κ-robustness are intended to make sure that I includes enough sets that the notion of a "κ-complete, normal, fine ultrafilter on I" makes sense and is possible. In particular, any such ultrafilter U will be characterized by an elementary embedding j_U with crit $j_U = \kappa$; this implies that $\{s \in I : |s \cap \kappa| < \kappa\} \in U$.

We are interested in model-theoretic conditions that guarantee the existence of a fine, normal, κ-complete ultrafilter on some κ-robust I. Recall that, from Kunen's proof of the inconsistency of Reinhardt cardinals, every such ultrafilter must concentrate on $\mathcal{P}_\mu\lambda$ for $\mu \leq \lambda$. In case $\mu > \kappa$, we will need the following technical condition. In practice, the set X will always be a theory.

Definition 3.2. *Let $I \subset \mathcal{P}(\lambda)$ and X be a set that is filtrated as an increasing union of $\{X_s \mid s \in I\}$. Then we say this filtration* respects the index *iff there is a collection $\{X^t \mid t \in \mathcal{P}_\kappa\lambda\}$ such that, for each $s \in I$, $X_s = \cup_{t \subset s} X^t$.*

This condition says that the filtration at $s \in I$ is just determined by the elements of s. Note this condition is trivially satisfied when $I \subset \mathcal{P}_\kappa\lambda$, but will be important in certain cases, e. g., to characterize huge cardinals (see Corollary 3.5.(3)). There, it will guarantee that if $\phi \in \cup_{s \in [\lambda]^\kappa} T_s$, then there are a large number of $s \in I$ such that $\phi \in T_s$.

The main concept of this section is the following:

Definition 3.3. *Let $\mathcal{L}$ be a logic (in the sense of Barwise [Bar74] or taken without formal definition), $\kappa \leq \lambda$, and $I \subset \mathcal{P}(\lambda)$ be κ-robust. Then we say that $\mathcal{L}$ is* I-κ-compact for type omission *iff for any language τ, any $\mathcal{L}(\tau)$-theory T that can be written as an increasing union of T_s for $s \in I$ that respects the index, and collection of $\mathcal{L}(\tau)$-types $\{p^a(x) \mid a \in A\}$, if there are club many $s \in I$ such that T_s has a model omitting each type in $\{p^a_s(x) \mid a \in A\}$, then there is a model of T omitting each type in $\{p^a(x) \mid a \in A\}$.*

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[5] In an unfortunate collision of notation, M is commonly used for both inner models in set theory and for τ-structures in model theory. Owing to my model-theoretic roots, this paper uses standard M for τ-structures and script $\mathcal{M}$ for inner models.

We can link this to other notions as follows:

Theorem 3.4. *Let $\kappa \leq \lambda$, and $I \subset \mathcal{P}(\lambda)$ be κ-robust. The following are equivalent:*

(1) $\mathbb{L}_{\kappa,\omega}$ *is I-κ-compact for type omission.*
(2) $\mathbb{L}_{\kappa,\kappa}$ *is I-κ-compact for type omission.*
(3) *There is a fine, normal, κ-complete ultrafilter on $\mathcal{P}(\lambda)$ concentrating on I.*
(4) *There is an elementary $j : V \to \mathcal{M}$ with crit $j = \kappa$ and $j``\lambda \in \mathcal{M} \cap j(I)$.*

Moreover, the first μ such that $\mathbb{L}_{\mu,\omega}$ is I-κ-compact for type omission is the first μ with a fine, normal, μ-complete ultrafilter on I.

Proof: The equivalence of (3) and (4) is straightforward from standard methods and (2) implies (1) is obvious.
(4) $\to$ (2): Suppose we have a set-up for I-κ-compact type omission. Let $C \subset I$ be the witnessing club and, for $s \in C$, let $M_s \vDash T_s$ and omit $\{p^a_s \mid a \in A\}$. Writing $\bar{M}$ for the function taking s to M_s, $j(\bar{M})$ is a function with domain $j(C)$. By standard arguments, $j``\lambda \in j(C)$. Then $\mathcal{M}$ thinks that $j(\bar{M})(j``\lambda)$ is a $j(\tau)$-structure that models $j(\bar{T})(j``\lambda)$ that omits each $j(\bar{p}^a)(j``\lambda)$. Then $\mathcal{M}$ is correct about this, so some canonical renaming of $j(\bar{M})(j``\lambda)$ models T and omits each p^a.
(1) $\to$ (3): Set $\tau = \{P, Q, E, c_X\}_{X \subset I}$ with P and Q unary predicates, $E \subset P \times Q$ a binary relation, and c_s a constant. We look at the standard structure $M = \langle I, \mathcal{P}(I), \in, X\rangle_{X \subset I}$. Set $T_0 = Th_{\mathbb{L}_{\kappa,\omega}}(M)$ (although much less is necessary). Set $T^\alpha := T_0 \cup \{d \in c_{[\alpha]^I}\}$ for $\alpha < \lambda$; $T_s := \cup_{\alpha \in s} T^\alpha$ for $s \in I$; and $T = \cup_{s \in I} T_s$. For a function $F : I \to \lambda$, define

$$\begin{aligned} X_F &:= \{s \in I \mid F(s) \in s\} \\ X_{F,\alpha} &:= \{s \in I \mid F(s) = \alpha\} \\ p_F(x) &:= \left\{x = d \wedge xEc_{X_F} \wedge \neg\left(xEc_{X_{F,\alpha}}\right) \mid \alpha < \lambda\right\} \\ \Gamma &:= \{p_F \mid F : I \to \lambda\} \end{aligned}$$

Now we have a set-up for compactness for type omission.
Claim 1: If there is a model of T omitting Γ, then there is a fine, normal, κ-complete ultrafilter on I.
Let N be this model. Define U on I by

$$X \in U \iff N \vDash dEc_X$$

It is straightforward to check that U is a κ-complete ultrafilter on I. For instance, given $\langle X_\alpha \in U \mid \alpha < \mu < \kappa\rangle$, we know that the following sentence is in T:

$$\forall x \left(\bigwedge_{\alpha < \mu} xEc_{X_\alpha} \to xEc_{\cap_{\alpha<\mu} X_\alpha}\right)$$

Thus, $N \vDash dEc_{\cap_{\alpha<\mu} X_\alpha}$. Given $\alpha < \lambda$, by κ-robustness, there is some $s \in I$ such that $\alpha \in s$. Thus T entails $dEc_{[\alpha]^I}$, so U is fine. For normality, if $F : I \to \lambda$ is regressive on a U-large set, then $N \vDash dEc_{X_F}$. Since N omits p_F, there is $\alpha < \lambda$ such that $X_{F,\alpha} \in U$, so U is normal.
Claim 2: For each $s \in I$, there is a model of T_s omitting $\{p_{F,s} \mid F : I \to \lambda\}$.
Expand M to M_s by interpreting $d^{M_s} = s$. This models T_s since $s \in [\alpha]^I$ for each $\alpha \in s$ by definition. Moreover, if there is $x \in M_s$ such that

$$M_s \vDash x = d \wedge xEX_F$$

for some $F : I \to \lambda$, then $x = s$ and $F(s) \in s$, so there is $\alpha \in s$ such that $F(s) = \alpha$. Thus

$$M_s \vDash xEX_{F,\alpha}$$

So M_s omits each $p_{F,s}$.
By the I-κ-compactness for type omission, we are done.

The proof of the "moreover" follows similarly. †

As an example of the moreover, $\mathbb{L}_{\omega,\omega}$ satisfies Benda's supercompactness theorem iff $\mathcal{P}_\kappa\lambda$ carries a fine, normal measure that need not even be countably complete. Note that $\mathbb{L}_{\omega,\omega}$ can never be $\mathcal{P}_\omega\lambda$-$\omega$-compact for type omission: the max function shows that no fine ultrafilter on $\mathcal{P}_\omega\lambda$ can be normal.

This general framework directly gives model theoretic characterizations of large cardinals that are characterized by normal ultrafilters.

Corollary 3.5. *For each numbered item below, all of it's subitems are equivalent:*

(1) (a) *κ is measurable.*
  (b) *$\mathbb{L}_{\kappa,\kappa}$ is $\mathcal{P}_\kappa\kappa$-κ-compact for type omission.*
(2) (a) *κ is λ-supercompact.*
  (b) *$\mathbb{L}_{\kappa,\kappa}$ is $\mathcal{P}_\kappa\lambda$-compact for type omission.*
(3) (a) *κ is huge at λ*
  (b) *$\mathbb{L}_{\kappa,\kappa}$ is $[\lambda]^\kappa$-κ-compact for type omission.*
(4) (a) *κ is n-huge at $\lambda_1, \ldots, \lambda_n$.*
  (b) *$\mathbb{L}_{\kappa,\kappa}$ is $\{s \subset \lambda : \forall i < n, |s \cap \lambda_{i+1}| = \lambda_i\}$-$\kappa$-compact for type omission.*

Item (2) is Benda's supercompactness theorem (Fact 1.4). Item (1) can be reformulated along the lines of Fact 1.2:

If $T = \cup_{\alpha<\kappa} T_\alpha$ is an $\mathbb{L}_{\kappa,\kappa}(\tau)$-theory and $p(x) = \{\phi_i(x) \mid i < \kappa\}$ is a type such that for every $\alpha < \kappa$, there is a model of T_α omitting $\{\phi_i(x) \mid i < \alpha\}$, then there is a model of T omitting p.

This helps highlight the impact that the normality of an ultrafilter has on the resulting ultraproduct: if U is a normal ultrafilter on $I \subset \mathcal{P}(\lambda)$ and $\{M_s \mid s \in I\}$ are τ-structures, then $\prod M_s/U$ omits any type $p = \{\phi_\alpha(x) \mid \alpha < \lambda\}$ such that $\{s \in I \mid M_s \text{ omits } p_s\} \in U$.

As mentioned above, any ultrafilter on $\mathcal{P}(\lambda)$ concentrates on a $\mathcal{P}_\mu\lambda$. We can characterize when an ultrafilter exists on some $\mathcal{P}_\mu\lambda$ with the following large cardinal notion:

Corollary 3.6. *Fix $\kappa \leq \mu \leq \lambda$. The following are equivalent:*

(1) *κ is λ-supercompact with μ clearing: there is $j : V \to \mathcal{M}$ with $j``\lambda \in \mathcal{M}$ and $j(\mu) > \lambda$.*
(2) *$\mathbb{L}_{\kappa,\kappa}$ is $\mathcal{P}_\mu\lambda$-κ-compact for type omission.*

Proof: This follows from Theorem 3.4: $j(\mu) > \lambda$ ensures that $j``\lambda \in j(\mathcal{P}_\mu\lambda)$. †

One feature of the compactness schema are that the theories are not required to have a specific size, but rather be filtrated by a particular index set. Note this is also true for strongly compact cardinals; that is, rather than characterizing λ-strongly compact cardinals as compactness cardinals for λ-sized theories in $\mathbb{L}_{\kappa,\kappa}$, we can give the following.

Proposition 3.7. *κ is λ-strongly compact iff any $\mathbb{L}_{\kappa,\kappa}$-theory T that can be filtrated as an increasing union of satisfiable theories indexed by $\mathcal{P}_\kappa\lambda$ is itself satisfiable.*

This can even be extended to theories of proper class size. Each item of Corollary 3.5 remains true if compactness for type omission is generalized to allow for the T and the T_s in Definition 3.3 to be definable proper classes. The same proof of Theorem 3.4 goes through with this generalization.

Remark 3.8. *$ED_{\mathcal{L}}(V, \in, x)_{x\in V}$ is used to denote the $\mathcal{L}$-elementary diagram of the structure with universe V, a single binary relation $\in$, and a constant for each x that is interpreted as x. Formally, from Tarski's undefinability of truth, this is not a definable class when $\mathcal{L}$ extends $\mathbb{L}_{\omega,\omega}$. Similarly, the statement that $j : V \to \mathcal{M}$ is elementary is not definable.*

However, e. g., [Kan08, Proposition 5.1.(c)] shows that, for embeddings between inner models, Σ_1-elementarity implies Σ_n-elementarity for every $n < \omega$. Thus, mentions of elementary embeddings with domain V can be replaced by Σ_1-elementarity. Similarly, the full elementary diagram of V could be replaced by its Σ_1-counterpart. However, following set-theoretic convention, we continue to refer to the full elementary diagram.

Armed with a class version of omitting types compactness, we can show equivalences directly between the model-theoretic characterizations and the elementary embedding characterizations without working through an ultrafilter characterization. At each stage, we find a model $\mathcal{N}$ of $ED_{\mathbb{L}_{\kappa,\kappa}}(V, \in, x)_{x \in V}$ along with the sentences $\{c_\alpha < c < c_\kappa \mid \alpha < \kappa\}$ where c is a new constant and some other sentences. Such a model is well-founded because it models the $\mathbb{L}_{\omega_1,\omega_1}$-sentence asserting well-foundedness, so we can take the Mostowski collapse $\pi : \mathcal{N} \cong \mathcal{M}$ with $\mathcal{M}$ transitive. Then $x \mapsto \pi(c_x)$ is an $\mathbb{L}_{\kappa,\kappa}$-elementary embedding that necessarily sends $\alpha < \kappa$ to itself. Moreover, the interpretation of c guarantees that the critical point is at most κ and the use of $\mathbb{L}_{\kappa,\omega}$ guarantees the critical point is at least κ. This is enough to show κ measurable and extra sentences to be satisfied and types to be omitted can be added to characterize the above large cardinal notions. The following observation is straightforward.

Proposition 3.9. *For each numbered item below, all of it's subitems are equivalent:*

(1) (a) *κ is measurable.*
 (b) *There is a model of*

$$ED_{\mathbb{L}_{\kappa,\kappa}}(V, \in, x)_{x \in V} \cup \{c_\alpha < c < c_\kappa \mid \alpha < \kappa\}$$

(2) (a) *κ is λ-strongly compact.*
 (b) *There is a model of*

$$ED_{\mathbb{L}_{\kappa,\kappa}}(V, \in, x)_{x \in V} \cup \{c_\alpha < c < c_\kappa \mid \alpha < \kappa\} \cup \{c_\alpha \in d \wedge |d| < c_\kappa \mid \alpha < \lambda\}$$

(3) (a) *κ is λ-supercompact.*
 (b) *There is a model of*

$$ED_{\mathbb{L}_{\kappa,\kappa}}(V, \in, x)_{x \in V} \cup \{c_\alpha < c < c_\kappa \mid \alpha < \kappa\} \cup \{c_\alpha \in d \wedge |d| < c_\kappa \mid \alpha < \lambda\}$$

 that omits

$$p(x) = \{xEd \wedge x \neq c_\alpha \mid \alpha < \lambda\}$$

(4) (a) *κ is n-huge at $\lambda_1, \dots, \lambda_n$.*
 (b) *There is a model of*

$$ED_{\mathbb{L}_{\kappa,\kappa}}(V, \in, x)_{x \in V} \cup \{c_\alpha < c < c_\kappa \mid \alpha < \kappa\} \cup \{c_\alpha \in d_{i+1} \wedge |d_{i+1}| = c_{\lambda_i} \mid \alpha < \lambda_{i+1}, i < n\}$$

 that omits, for $i < n$,

$$p_i(x) = \{xEd_{i+1} \wedge x \neq c_\alpha \mid \alpha < \lambda_{i+1}\}$$

Also, in each case the theory has a natural filtration by the appropriate partial order that is easily seen to be locally consistent while omitting the necessary type. For instance, in the case of κ being λ-supercompact, for $s \in \mathcal{P}_\kappa\lambda$, set $\alpha_s := otp(s)$.

$$\begin{aligned} T_s &:= ED_{\mathbb{L}_{\kappa,\kappa}}(V, \in, x)_{x \in (V_{\geq\kappa} \cup V_{\alpha_s})} \cup \{c_i < c < c_\kappa \mid i < \alpha\} \cup \{c_i Ed \wedge |d| < c_\kappa \mid i \in s\} \\ p_s(x) &:= \{xEd \wedge x \neq c_i \mid i \in s\} \end{aligned}$$

Then V is a model of this theory by interpreting every constant in the language by its index, c as α_s, and d as s. This gives a way to go directly between model-theoretic and elementary embedding characterizations. It also shows that it is enough to omit a single[6] type to obtain the I-κ-type omission for any number of types.

[6] In the case of n-huge, recall that the omission of finitely many types can be coded by a single type.

The ability to characterize cardinals at the level of huge and above shows that the addition of type omission to attempts to characterize large cardinals is a real necessity. Measurable and supercompact cardinals have known model-theoretic characaterizations without type omission, so one might wonder if type omission is necessary to characterize huge cardinals. From the following theorem of Makowsky, we can deduce that it is necessary.

Fact 3.10 ([Mak85, Theorem 2]). *The following are equivalent:*

(1) *Every logic $\mathcal{L}$ has a strong compactness cardinal; that is, for every logic $\mathcal{L}$, there is a cardinal $\mu_{\mathcal{L}}$ such that for any language τ and $\mathcal{L}(\tau)$-theory T, if every $T_0 \in \mathcal{P}_{\mu_{\mathcal{L}}} T$ has a model then so does T.*
(2) *Vopenka's Principle.*

Thus, Vopenka's Principle "rallies at last to force a veritable Götterdammerung" for compactness cardinals for logics[7]. Nonetheless, κ being almost huge implies that V_κ satisfies Vopenka's Principle. Thus, if κ is the first huge cardinal, then V_κ is a model of "Every logic is compact, but there are no $\mu \leq \lambda$ such that μ is $[\lambda]^\mu$-μ-compact for type omission." Indeed, other approaches to model-theoretic characterizations of large cardinals focused solely on compactness or reflection principles have yet to characterize huge cardinals.

4. Second-order logic and beyond

We now turn to characterizations based on logics beyond (or orthogonal to) $\mathbb{L}_{\infty,\infty}$. In the spirit of Theorem 3.4, we can characterize compactness for omitting types in second with a similar theorem.

Theorem 4.1. *Let $\kappa \leq \lambda$ and $I \subset \mathcal{P}(\lambda)$ be κ-robust. The following are equivalent:*

(1) *$\mathbb{L}^2 \cup \mathbb{L}_{\kappa,\omega}$ is I-κ-compact for type omission.*
(2) *$\mathbb{L}^2_{\kappa,\kappa}$ is I-κ-compact for type omission.*
(3) *For every $\alpha > \lambda$, there is some $j : V_\alpha \to V_\beta$ such that crit $j = \kappa$ and $j``\lambda \in j(I)$.*
(4) *For every $\alpha > \lambda$, there is some $j : V \to \mathcal{M}$ such that crit $j = \kappa$, $j``\lambda \in j(I)$, and $V_{j(\alpha)} \subset \mathcal{M}$.*

Moreover, the first μ such that $\mathbb{L}^2$ is I-κ-compact for type omission is the first μ that satisfies (3) except with crit $j = \mu$.

Proof: (4) implies (3) and (2) implies (1) are immediate. We show that (1) implies (3) implies (4) implies (2).

For (1) implies (3), fix $\alpha \geq \lambda$ and consider the $\mathbb{L}^2 \cup \mathbb{L}_{\kappa,\omega}$-theory and type

$$\begin{aligned} T &= ED_{\mathbb{L}_{\kappa,\omega}}(V_\alpha, \in, x)_{x \in V_\alpha} \cup \{c_i < c < c_\kappa \mid i < \kappa\} \cup \{\Phi\} \\ p(x) &= \{xEd \wedge x \neq c_i \mid i < \lambda\} \end{aligned}$$

where $\Phi \in \mathbb{L}^2$ is a sentence that asserts the universe is isomorphic to a rank-initial segment of V (see the proof of [Mag71, Theorem 2]). Then, we can filtrate this theory as

$$\begin{aligned} T_s &= ED_{\mathbb{L}_{\kappa,\omega}}(V_\alpha, \in, x)_{x \in V_{\sup s \cup [\kappa,\alpha)}} \cup \{c_i < c < c_\kappa \mid i < \sup s\} \cup \{\Phi\} \\ p_s(x) &= \{xEd \wedge x \neq c_i \mid i \in s\} \end{aligned}$$

For each $s \in I$, we have that the natural expansion $(V_\alpha, \in, x, s)_{x \in V_{\sup s \cup [\kappa,\alpha)}}$ models T_s and omits p_s. Thus, our compactness principle tells us there is a model of T omitting p, which, after taking the transitive collapse, gives the desired $j : V_\alpha \to V_\beta$.

For (3) implies (4), fix $\alpha \geq \lambda$ and let α' be the next strong limit cardinal above α. Then there is $j : V_{\alpha'} \to V_\beta$ with crit $j = \kappa$ and $j``\lambda \in j(I)$. Then derive the extender E of length $\beth_{j(\alpha)}$ to

[7] With apologies to Kanamori [Kan08, p. 324].

capture this embedding. Forming the extender power of V and taking the transitive collapse, we get $j_E : V \to \mathcal{M}_E$ with the desired properties.

For (4) implies (2), let $\bar{T} = \{T_s \mid s \in I\}$ be an increasing filtration of the $\mathbb{L}^2_{\kappa,\kappa}$-theory T that respects the index and $\{p^a(x) \mid a \in A\}$ be a collection of types indexed as $p^a(x) = \{\phi^a_i(x) \mid i < \lambda\}$ such that there are a club of $s \in I$ with a model M_s that models T_s and omits each p^a_s. Fix strong limit $\alpha \geq \lambda$ to be greater than the rank of these models, their power sets, and the function f that takes each of these s to M_s; form $j : V \to \mathcal{M}$ with crit $j = \kappa$, $j"\lambda \in j(I) \cap \mathcal{M}$, and $V_{j(\alpha)} \subset \mathcal{M}$. Since the domain of f contains a club, it includes $j"\lambda$. Set $M_* := j(f)(j"\lambda)$. By the elementarity of j, inside of $\mathcal{M}$ we have that $M_* \vDash j(\bar{T})_{j"\lambda}$ and, for each $a \in A$, M_* omits $j(p^a)_{j"\lambda} = \{j(\phi^a_i) \mid i < \lambda\} = j"p^a$. Since $V_{j(\alpha)} \subset \mathcal{M}$ and rank $M_* < j(\alpha)$, $\mathcal{M}$ is correct about this satisfaction. Finally, $j"T \subset j(\bar{T})_{j"\lambda}$ because the filtration respects the index. Thus, after a suitable renaming, we have found a model of T omitting $\{p^a(x) \mid a \in A\}$. †

To aid in the discussion of the implication of this theorem, we introduce the following ad hoc naming convention for large cardinal properties.

Definition 4.2. *Suppose a large cardinal property P is characterized by begin an I-κ-compactness cardinal for $\mathbb{L}_{\kappa,\kappa}$. Given a logic $\mathcal{L}$, we say that κ is P-*for*-$\mathcal{L}$ iff $\mathcal{L}$ is I-κ-class compact for type omission.*

For instance, Corollary 3.5.(3) characterizes huge as the existence of a $\lambda > \kappa$ such that $\mathbb{L}_{\kappa,\kappa}$ is $[\lambda]^\kappa$-κ-compact for type omission, so saying that κ is huge-for-$\mathbb{L}^2_{\kappa,\kappa}$ means that there is a $\lambda > \kappa$ so $\mathbb{L}^2_{\kappa,\kappa}$ is $[\lambda]^\kappa$-κ-compact for type omission.

Comparing Theorems 3.4 and 4.1, a large difference is that the first-order characterizations are witnessed by a single embedding, while the second-order characterizations require class many embeddings. The reason for this is that a single model $\mathcal{M}$ can be right about $\mathbb{L}_{\kappa,\kappa}$ everywhere, but cannot be right about $\mathbb{L}^2$ everywhere; otherwise, it would compute the power set of every set correctly and would be V. Similarly, the type omitting compactness does *not* hold for definable class theories for second-order as it does for first. If it did, one could easily derive an nontrivial embedding $j : V \to V$.

The first consequence of Theorem 4.1 regards the identity crisis. In the language of Definition 4.2, Magidor has shown that extendible cardinals are exactly those that are strong compact-for-$\mathbb{L}^2_{\kappa,\kappa}$ [Mag71, Theorem 4] and additionally shown that the first strongly compact cardinal could be the first measurable or the first supercompact [Mag76]. This second result means that various compactness notions for $\mathbb{L}_{\kappa,\kappa}$ have an imprecise relation to one another: chain compactnes could coincide with compactness, or there could be many chain compact cardinals below the first compactness cardinal. Surprisingly, when moving to $\mathbb{L}^2$, these notions coincide and the identity crisis disappears!

Theorem 4.3. *The following are equivalent.*

1. *κ is measurable-for-$\mathbb{L}^2 \cup \mathbb{L}_{\kappa,\omega}$.*
2. *κ is strongly compact-for-$\mathbb{L}^2_{\kappa,\kappa}$.*
3. *κ is supercompact-for-$\mathbb{L}^2_{\kappa,\kappa}$.*

In particular, all three of these statements characterize extendible cardinals.

Here we take 'κ is measurable-for-$\mathcal{L}$' as in Fact 1.2. That is, we don't incorporate any type omission; however, the type omission characterization holds as a result of the above.

Proof: Clearly, (3) implies (2) implies (1) using (for the first implication) the trivially omitted type $\{x \neq x \mid i < \lambda\}$.

The condition Theorem 4.1.(3) is clearly stronger than extendability, so any compactness for $\mathbb{L}^2_{\kappa,\kappa}$ (including chain compactness) gives extendability. In particular, $j"\kappa = \kappa \in j(\mathcal{P}_\kappa \kappa)$. So measurable-for-$\mathbb{L}^2 \cup \mathbb{L}_{\kappa,\omega}$ implies extendable.

Similarly, the definition of extendability includes that $j(\kappa) > \alpha$. In this case, $j"\lambda$ has size $\lambda \leq \alpha$, so $j"\lambda \in j(\mathcal{P}_\kappa \lambda)$. Thus extendibility implies supercompact-for-$\mathbb{L}^2_\kappa, \kappa$. †

The key to these equivalences is that the condition about $j"\lambda$ in Theorem 3.4.(4) often had more to do with the closure of the target model (i. e., is $j"\lambda$ in $\mathcal{M}$?), rather than the nature of the relations ship between $j(I)$ and $j"\lambda$. When we have extendible-like embeddings, $j"\lambda$ is always in the target model, so many of the type omitting compactness principles (or even just compactness one's become trivial.

A possible metamathematical reason for the collapse of the identitiy crisis is that type omission in $\mathbb{L}^n_{\kappa,\kappa}$ is expressible[8] in $\mathbb{L}^{n+1}_{\kappa,\kappa}$, which is again codeable in $\mathbb{L}^n_{\kappa,\kappa}$. Thus, one might expect no difference between strong compact- and supercompact-for-$\mathbb{L}^2_{\kappa,\kappa}$. However, this does not explain why measurability coincides with these notions, and the below proposition shows that some notions of type-omitting compactness for $\mathbb{L}^2$ are strictly stronger than extendibility (in consistency strength).

Proposition 4.4.

(1) *κ is huge at λ-for-$\mathbb{L}^2_{\kappa,\kappa}$ iff for every $\alpha \geq \lambda$, there is $j : V_\alpha \to V_\beta$ such that crit $j = \kappa$ and $j(\kappa) = \lambda$.*

(2) *If κ is almost 2-huge at λ_1, λ_2, then there is a κ-complete, normal ultrafilter on κ containing*
$$\{\alpha < \kappa \mid V_{\lambda_2} \vDash \text{“}\alpha \textit{ is huge-for-}\mathbb{L}^2_{\kappa,\kappa}\text{”}\}$$

(3) *If κ is huge-for-$\mathbb{L}^2_{\kappa,\kappa}$, then there is a κ-complete, normal ultrafilter on κ containing*
$$\{\alpha < \kappa \mid \alpha \textit{ is huge}\}$$

Proof: The first item is just a restatement of Theorem 4.1 with $I = [\lambda]^\kappa$.

Suppose κ is 2-huge at λ_1, λ_2 and $j : V \to \mathcal{M}$ witnesses this. Fixing $\alpha \in [\lambda_1, \lambda_2)$, $j \restriction V_\alpha$ is an embedding from V_α to V_β with $j(\kappa) = \lambda_1$ that is in $\left(V_{j(\lambda_2)}\right)^{\mathcal{M}}$. So
$$\left(V_{j(\lambda_2)}\right)^{\mathcal{M}} \vDash \text{“}\exists j_0 : V_\alpha \to V_\beta \text{ such that crit } j_0 = \kappa \text{ and } j_0(\kappa) = \lambda_1\text{”}$$
Recall that, $V_{\lambda_2} = (V_{\lambda_2})^{\mathcal{M}} \prec \left(V_{j(\lambda_2)}\right)^{\mathcal{M}}$. Thus,
$$V_{\lambda_2} \vDash \text{“}\exists j_0 : V_\alpha \to V_\beta \text{ such that crit } j_0 = \kappa \text{ and } j_0(\kappa) = \lambda_1\text{”}$$
Since α was arbitrary,
$$\begin{aligned} V_{\lambda_2} &\vDash \text{“}\forall \alpha \geq \lambda_1, \exists j_0 : V_\alpha \to V_\beta \text{ such that crit } j_0 = \kappa \text{ and } j_0(\kappa) = \lambda_1\text{”} \\ V_{\lambda_2} &\vDash \text{“}\kappa \text{ is huge at } \lambda_1\text{-for-}\mathbb{L}^2_{\kappa,\kappa}\text{”} \end{aligned}$$
Thus, $\{\alpha < \kappa \mid V_{\lambda_2} \vDash \text{“}\alpha \text{ is huge-for-}\mathbb{L}^2_{\kappa,\kappa}\text{”}\}$ is in the normal ultrafilter on κ derived from j.

Suppose κ is huge at λ-for-$\mathbb{L}^2_{\kappa,\kappa}$. Picking α large enough and getting the corresponding $j : V_\alpha \to V_\beta$ with $j(\kappa) = \lambda$, we can derive a normal, κ-complete, fine ultrafilter U on $[\lambda]^\kappa$. Then $U \in V_\beta$, so $V_\beta \vDash$ “κ is huge.” Thus, $\{\alpha < \kappa \mid \alpha \text{ is huge}\}$ is in the normal, κ-complete ultrafilter on U generated from j. †

[8]This is immediate for ϕ-type omission for fixed ϕ, and any type omission can be coded as ϕ-type omission in an expansion.

Similar results show that n-huge-for-$\mathbb{L}^2_{\kappa,\kappa}$ lie strictly between n-huge and almost $n+1$-huge. The preceding argument is due to Gabriel Goldberg, who also reports that he can show that huge-for-$\mathbb{L}^2_{\kappa,\kappa}$ can be characterized in terms of hyperhugeness.

Recall that κ is λ-hyperhuge iff there is $j : V \to \mathcal{M}$ with crit $j = \kappa$ and ${}^{j(\lambda)}\mathcal{M} \subset \mathcal{M}$ and κ is hyperhuge iff it is λ-hyperhuge for every λ. Hyperhuge cardinals have recently been shown to imply the existence of a minimal inner model of V that can reach V by set-forcing extensions by Usuba [Usu]. Goldberg can prove that κ being hyperhuge is equivalent to the existence of a $\kappa_0 < \kappa$ such that κ_0 is huge at κ-for-$\mathbb{L}^2_{\kappa,\kappa}$. Additionally, κ being λ-hyperhuge is equivalent to the existence of $\mu > \lambda$ and a normal, fine, κ-complete ultrafilter on

$$[\mu]^\lambda_{*\kappa} := \{s \subset \mu \mid |s| = \lambda, |s \cap \kappa| \in \kappa, \mathrm{otp}(s \cap \lambda) < \kappa\}$$

, which is equivalent to $\mathbb{L}_{\kappa,\kappa}$ being $[\mu]^\lambda_{*\kappa}$-κ-compact for type omission by Theorem 3.4.

Examining the proof of Theorem 4.1, we see that a level-by-level characterization of, e. g., α-extendibility is harder due to the tricky nature of the Löwenheim-Skolem number for second-order logics. In first-order, the Löwenheim-Skolem number of $\mathbb{L}_{\lambda,\kappa}$ for theories of size μ is $((\lambda+\mu)^{<\kappa})^+$, which is also its Löwenheim-Skolem-Tarski number. For second-order logic, $LS(\mathbb{L}^2)$ (for sentences) is the supremum of all Π_2-definable ordinals (Väänänen [Vää79, Corollary 4.7]) and $LST(\mathbb{L}^2)$ is the first supercompact, if one exists [Mag71, Theorem 2] . However, weak compactness restricts the size of the theory, so admits a more local characterization. Denote the Löwenheim-Skolem number of sentences of $\mathbb{L}^2_{\kappa,\kappa}$ by ℓ^2_κ.

Theorem 4.5. *The following are equivalent for κ.*

1. *κ is weakly compact-for-$\mathbb{L}^2 \cup \mathbb{L}_{\kappa,\omega}$*
2. *κ is weakly compact-for-$\mathbb{L}^2_{\kappa,\kappa}$.*
3. *Given any $\kappa+1 \subset \mathcal{M} \subset V_{\ell^2_\kappa}$ of size κ, there is a partial elementary embedding $j : V_{\ell^2_\kappa} \to V_\beta$ for some β with dom $j = \mathcal{M}$ and crit $j = \kappa$.*

Proof: Clearly, (2) implies (1). We show (1) implies (3) implies (2).

Suppose κ is weakly compact-for-$\mathbb{L}^2 \cup \mathbb{L}_{\kappa,\omega}$ and let $\kappa+1 \subset \mathcal{M} \subset V_{\ell^2_\kappa}$. Let T be the $\mathbb{L}^2$ theory consisting of

1. The $\mathbb{L}_{\kappa,\omega}$-elementary diagram of $\mathcal{M}$ in $V_{\ell^2_\kappa}$
2. $c_i < c < c_\kappa$ for $i < \kappa$
3. Φ from [Mag71]

Then every $<\kappa$-sized subset of T is satisfiable as witnessed by an expansion of $V_{\ell^2_\kappa}$. By weak compactness, we get a model of T, which must be some V_β. This induces a partial function $j : V_{\ell^2_\kappa} \to V_\beta$ with dom $j = \mathcal{M}$. Moreover, the elements of T make this a partial elementary embedding with crit $j = \kappa$.

Suppose that κ satisfies the embedding property. Let $T = \{\phi_i \mid i < \kappa\}$ be a $\mathbb{L}^2_{\kappa,\kappa}(\tau)$-theory that is $<\kappa$-satisfiable with $|\tau| \leq \kappa$. Then, there is a function f with domain κ such that $f(\alpha) \vDash \{\phi_i \mid i < \alpha\}$ for every $\alpha < \kappa$; moreover, by the definition of ℓ^2_κ, we can assume that $f(\alpha) \in V_{\ell^2_\kappa}$. Let $\mathcal{M} \subset V_{\ell^2_\kappa}$ contain all of this information and be of size κ. Then, there is partial elementary $j : V_{\ell^2_\kappa} \to V_\beta$ with dom $j = \mathcal{M}$ and crit $j = \kappa$. In particular, we have that

1. $j(\kappa) > \kappa$
2. $V_\beta \vDash$ "$j(f)(\kappa) \vDash j$"T" and V_β is correct about this
3. j"T and T are just renamings of the same theory

Thus, the suitably renamed $j(f)(\kappa)$ witnesses that T is satisfiable. †

A key piece in translating weak compactness for second-order into an embedding characterization is the ability to axiomatize well-foundedness. If we look at a fragment of $\mathbb{L}^2_{\kappa,\kappa}$ that includes an expression of well-foundedness, then weak compactness for this fragment is characterizable in a similar with, replacing ℓ^2_κ with the Löwenheim-Skolem number of that fragment. However, if the fragment cannot express well-foundedness, then this characterization is harder.

Similar results can be proved by restricting the size of the theories under consideration. In the general scheme, the theory T is allowed to be as large as one wants, as are the pieces T_s of the filtration. If one restricts these pieces to be of size $\leq \mu$ and wants to characterize $\mathbb{L}^2_{\kappa,\kappa}$ being I-κ-compact for type omission, then it suffices to look at an embedding as in Theorem 4.1.(2) for α equal to the Löwenheim-Skolem number of $\mathbb{L}^2_{\kappa,\kappa}$ for μ-sized theories.

For the characterizations of strong and its variants, we need the concept of a Henkin second-order structure that is full up to some rank. Recall the notion of a Henkin model.

Definition 4.6. *Let $M_* = (M, P, E)$ be a Henkin structure and A a transitive set.*

(1) *M_* is* full to A *iff every $X \in \mathcal{P}(M) \cap A$ is represented in P; this means that there is $c_X \in P$ such that, for all $y \in M$,*
$$y \in X \iff M_* \vDash yEc_X$$
(2) *M_* is* full up to rank α *iff it is full to V_α.*

While a Henkin structure has a nonstandard interpretation of second-order quantifiers, other additions to the logic must be interpreted standardly. In particular, the next theorem discusses Henkin models of $\mathbb{L}^2(Q^{WF})$-theories; while any second-order assertions of well-foundedness–i. e., "$\forall X \exists y \forall z (y \in X \wedge z \in X \to yRz)$"–can be satisfied non-standardly, any Q^{WF} assertions of well-foundedness–i. e., $Q^{WF}xy(xRY)$–must be correctly interpreted (and so R is well-founded.

Theorem 4.7. *The following are equivalent for $\kappa \leq \lambda$.*

(1) *κ is λ-strong.*
(2) *If $T \subset \mathbb{L}^2_{\kappa,\omega}(Q^{WF})(\tau)$ is a theory that can be written as an increasing union $T = \cup_{\alpha<\kappa} T_\alpha$ such that every T_α has a (full) model, then T has a Henkin model whose universe is an ordinal and is full up to rank λ.*
(3) *Same as (2), but there is also a type $p = \{\phi_i(x) \mid i < \kappa\}$ such that T_α has a (full) model omitting p_α, and the resulting model omits p.*

Note that we add the condition on the universe of the model in (2) to remove the possibility that the "full up to rank λ" condition is vacuous; if the universe of M just consist of elements of rank bigger than λ, then M is trivially full up to rank λ.

Proof: First, suppose that κ is λ-strong and let T be a theory and p a type as in (3). We produce a model of T in the standard way: let f be a function with domain κ such that $f(\alpha)$ is a model of T_α. WLOG, $f(s)$ is a full Henkin structure. Then, in $\mathcal{M}$, $j(f)(\kappa)$ is a model of (a theory containing) j"T and $\mathcal{M} \vDash$ "$j(f)(\kappa)$is a full Henkin structure that omits $j(p)_\kappa = j$"p". $\mathcal{M}$ is incorrect about second-order satisfaction above rank λ; however, since $V_\lambda \subset \mathcal{M}$, it is correct about second-order satisfaction up to rank λ.

Second, suppose we have compactness. Then we wish to build an embedding witnessing strength. By the normal arguments, e. g. [Kan08, Section 26] or see Proposition 5.2, it is enough to derive a $(\kappa, \beth_\lambda)$-extender from an embedding $j : V_{\kappa+2} \to \mathcal{M}$ with crit $j = \kappa$, $V_\lambda \subset \mathcal{M}$, and $\mathcal{M}$ well-founded. We can find such a model by considering the theory
$$ED_{\mathbb{L}_{\kappa,\omega}}(V_{\kappa+2}, \in, x)_{x \in V_{\kappa+2}} \cup \{c_i < c < c_\kappa \mid i < \kappa\} \cup \{\Phi\} \cup \{Q^{WF}xy(xEy)\}$$
This can be written as an increasing κ-length union of satisfiable theories in the standard way and any model leads to, after taking transitive collapse, the necessary $j : V_{\kappa+2} \to \mathcal{M}$. †

We could ask for a variation of (2) that allows for arbitrary κ-satisfiable theories or, equivalently, theories indexed by some $\mathcal{P}_\kappa\mu$. This would be equivalent to a jointly λ-strong and

μ-strongly compact cardinal: there is a $j : V \to \mathcal{M}$ such that crit $j = \kappa$, $V_\lambda \subset \mathcal{M}$, and there is $Y \in j(\mathcal{P}_\kappa \mu)$ such that $j"\mu \subset Y$.

If we drop the Q^{WF}, then we can characterize a weakening of λ-strong. In the following theorem and proof, we break the convention that $\mathcal{M}$ always denotes some transitive model of a fragment of ZFC. In particular, we allow it to be ill-founded. For such models, $wfp(\mathcal{M})$ denotes the well-founded part of $\mathcal{M}$.

Theorem 4.8. *The following are equivalent for $\kappa \leq \lambda$.*

(1) *κ is non-standardly λ-strong: there is an elementary embedding $j : V \to \mathcal{M}$* with $\mathcal{M}$ not necessarily transitive *such that crit $j = \kappa$ and $V_\lambda \subset wfp(\mathcal{M})$.*
(2) *If τ is a language and $T \subset \mathbb{L}^2_{\kappa,\omega}$ is a theory that can be written as an increasing, continuous union $T = \cup_{\alpha<\kappa} T_\alpha$ such that every T_α has a (full) model, then T has a Henkin model whose universe is an ordinal and is full up to rank λ.*

Proof: The proof is the same as Theorem 4.7, with the changes exactly that we no longer insist on being correct about statements about well-foundedness. †

An argument of Goldberg shows that the level-by-level notions of non-standardly λ-strong and λ-strong are inequivalent, but full non-standard strong is equivalent to strong.

4.1. $C^{(n)}$ **and sort logic.** Moving to sort logic, we can prove a metatheorem along the lines of Theorems 3.4 and 4.1 by introducing the notion of a $C^{(n)}$-cardinal. The $C^{(n)}$ variants of large cardinals were introduced by Bagaria [Bag12]. Briefly, set $C^{(n)} = \{\alpha \in \mathbf{ON} \mid V_\alpha \prec_{\Sigma_n} V\}$, where $\prec_{\Sigma_n}$ is elementarity for Σ_n formulas in the Levy hierarchy (in the language of set theory). For a large cardinal notion P witnessed by a certain type of elementary embedding, κ is $C^{(n)}$-P iff there is an elementary embedding j witnessing that κ is P and so $j(\kappa) \in C^{(n)}$. In the context of extendible cardinals, there is also the notion of $C^{(n)+}$-extendible cardinals [Bag12, Between Theorems 4.11 and 4.12]: κ is $C^{(n)+}$-extendible iff for all $\alpha > \kappa$ in $C^{(n)}$, there $j : V_\alpha \to V_\beta$ with crit $j = \kappa$ and $j(\kappa), \beta \in C^{(n)}$.

For some large cardinal notions, there is no increase of strength from moving to the $C^{(n)}$-versions (measurable, strong [Bag12, Propositions 1.1 and 1.2], strongly compact [Tsa14, Theorem 3.6]), but several other notions give an increasing hierarchy of strength.

Theorem 4.9. *Let $\kappa \leq \lambda$, $n < \omega$, and $I \subset \mathcal{P}(\lambda)$ be κ-robust. The following are equivalent:*

(1) *$\mathbb{L}^{s,\Sigma_n} \cup \mathbb{L}_{\kappa,\omega}$ is I-κ-compact for type omission.*
(2) *$\mathbb{L}^{s,\Sigma_n}_{\kappa,\kappa}$ is I-κ-compact for type omission.*
(3) *For every $\alpha \geq \lambda$ in $C^{(n)}$, there is some $j : V_\alpha \to V_\beta$ such that crit $j = \kappa$, $j"\lambda \in j(I)$, and $\beta \in C^{(n)}$.*
(4) *For every $\alpha \geq \lambda$, there is some $j : V \to \mathcal{M}$ such that crit $j = \kappa$, $j"\lambda \in j(I) \cap \mathcal{M}$,$V_{j(\alpha)} \subset \mathcal{M}$, and $j(\alpha) \in C^{(n)}$.*

The proof of Theorem 4.9 follows the structure of Theorems 3.4 and 4.1. To make the necessary changes, we introduce the following notion and lemma. Given a Σ_n formula $\phi(\mathbf{x})$ (in the Levy hierarchy), let $\phi^\sim(\mathbf{x}) \in \mathbb{L}^{s,\Sigma_n}$ be the same formula where unbounded quantifiers are replaced with the corresponding sort quantifiers. This allows us to characterize $C^{(n)}$ as follows.

Lemma 4.10. *Let α be an ordinal. $\alpha \in C^{(n)}$ iff V_α models*

$$\{\forall \mathbf{x}\, (\phi(\mathbf{x}) \leftrightarrow \phi^\sim(\mathbf{x})) \mid \phi \text{ is } \Sigma_n\}$$

Proof: For $\mathbf{a} \in V_\alpha$, we always have $\phi(\mathbf{a})$ holds in V iff $V_\alpha \vDash \phi^\sim(\mathbf{a})$. The above theory makes this equivalent to $V_\alpha \vDash \phi(\mathbf{a})$. †

Proof of 4.9: We sketch the proof and highlight the changes from the proof of Theorem 4.1. Given the compactness, we prove (3) by considering the theory and type

$$\begin{aligned} T &= ED_{\mathbb{L}_{\kappa,\omega}}(V_\alpha, \in, x)_{x\in V_\alpha} \cup \{c_i < c < c_\kappa \mid i < \kappa\} \cup \{\Phi\} \\ &\quad \cup \{\forall \mathbf{x}\, (\phi(\mathbf{x}) \leftrightarrow \phi^{\sim}(\mathbf{x})) \mid \phi \text{ is } \Sigma_n\} \\ p(x) &= \{xEd \wedge x \neq c_i \mid i < \lambda\} \end{aligned}$$

We filtrate this according to I in the standard way and use expansion of V_α to provide witness models; here it is crucial that we started with $\alpha \in C^{(n)}$. The model of T omitting p gives the desired j. We can adjust this proof to get a proof of (4) by finding strong limit $\alpha' > \alpha$, also in $C^{(n)}$, and relativizing the appropriate parts of the theory to ensure that $j(\alpha) \in C^{(n)}$. Then, derive the extender E from this model, and $j_E : V \to \mathcal{M}_E$ that retains the desired properties.

Given (3) or (4), we prove the compactness by starting with a filtration $\bar{T} = \{T_s \mid s \in I\}$ of an $\mathbb{L}^{s,\Sigma_n}_{\kappa,\kappa}$-theory and types $\{p^a(x) = \{\phi^a_i(x) \mid i < \lambda\} \mid a \in A\}$, find strong limit $\alpha \in C^{(n)}$ above the rank of these objects and the function f that takes s to the model of T_s omitting each p^a_s. V_α reflects these properties since $\alpha \in C^{(n)}$, so by elementarity the target model thinks that $j(f)(j``\lambda)$ models $j``T$ and omits $\{j``p^a(x) \mid a \in A\}$. Since V_β or $V_{j(\alpha)}$ are Σ_n-elementary in V, the target model is correct. †

Similar to second-order logic, the identity crisis disappears in sort logic and $C^{(n)-}$-extendible cardinals witness a wide range of type omitting compactness.

First, we show that $C^{(n)+}$-extendible cardinals are equivalent to an apparent weakening: the requirement that $j(\kappa) \in C^{(n)}$ is redundant.

Proposition 4.11. *The following are equivalent:*

(1) *κ is $C^{(n)+}$-extendible.*
(2) *For every $\alpha > \kappa$ in $C^{(n)}$, there is $j : V_\alpha \to V_\beta$ such that crit $j = \kappa$ and $\beta \in C^{(n)}$.*

We use the following lemma which will also be useful when examining Löwenheim-Skolem-Tarski numbers. This is similar to Magidor's characterization of supercompacts.

Lemma 4.12. *Let κ be as in Proposition 4.11.(2). Then for all $\alpha > \kappa$ in $C^{(n)}$ and $R \subset V_\alpha$, there are cofinally many $\gamma < \kappa$ such that there are $\bar{\alpha} < \kappa$ in $C^{(n)}$ and $S \subset V_{\bar{\alpha}}$ with elementary $j : (V_{\bar{\alpha}}, \in, S) \to (V_\alpha, \in, R)$, crit $j = \gamma$, and $j(\gamma) = \kappa$.*

Proof: Fix $\alpha \in C^{(n)}$ above κ, $R \subset V_\alpha$, and $\beta < \kappa$. Find $\alpha' > \alpha$ in $C^{(n)}$. By assumption, there is $j : V_{\alpha'} \to V_{\beta'}$ with crit $j = \kappa$, $j(\kappa) > \alpha'$, and $\beta' \in C^{(n)}$. Given a transitive model $\mathcal{M}$ of a fragment of ZFC, write $C^{(n),\mathcal{M}}$ for $\mathcal{M}$'s version of $C^{(n)}$. Since $\alpha, \alpha' \in C^{(n)}$, $\alpha \in C^{(n),V_{\alpha'}}$. By elementarity, $j(\alpha) \in C^{(n),V_{\beta'}}$. Thus,

$$\begin{aligned} V_{\beta'} \vDash \text{``}\exists \bar{\alpha} < j(\kappa) \text{ and } S \subset V_{\bar{\alpha}}, j_0 : (V_{\bar{\alpha}}, \in, S) \to \big(V_{j(\alpha)}, \in, j(R)\big) \text{ such that} \\ j_0(\text{crit } j_0) = j(\kappa), \text{crit } j_0 > j(\beta), \text{ and } \bar{\alpha} \in C^{(n)}\text{''} \end{aligned}$$

This is witnessed by $j \upharpoonright V_\alpha$. By elementarity,

$$\begin{aligned} V_{\alpha'} \vDash \text{``}\exists \bar{\alpha} < \kappa \text{ and } S \subset V_{\bar{\alpha}}, j_0 : (V_{\bar{\alpha}}, \in, S) \to (V_\alpha, \in, R) \text{ such that} \\ j_0(\text{crit } j_0) = \kappa, \text{crit } j_0 > \beta, \text{ and } \bar{\alpha} \in C^{(n)}\text{''} \end{aligned}$$

This is the desired result; note that it implies $\bar{\alpha} \in C^{(n)}$ because α' is. †

Proof of 4.11: Suppose (2) holds. By Lemma 4.12, the class of $C^{(n)}$-ordinals are cofinal in κ. Since this class is club, $\kappa \in C^{(n)}$. Fix $\alpha > \kappa$ in $C^{(n)}$ and find j as in (2); it suffices to show $j(\kappa) \in C^{(n)}$. Since $\kappa, \alpha \in C^{(n)}$, we have $\kappa \in C^{(n),V_\alpha}$. By elementarity, $j(\kappa) \in C^{(n),V_\beta}$. Since $\beta \in C^{(n)}$, this gives $j(\kappa) \in C^{(n)}$. †

Proposition 4.13. *The following are equivalent for every $n < \omega$.*

(1) *κ is $C^{(n)^+}$-extendible.*
(2) *κ is measurable-for-$\mathbb{L}^{s,\Sigma_n} \cup \mathbb{L}_{\kappa,\omega}$.*
(3) *κ is strong compact-for-$\mathbb{L}^{s,\Sigma_n}_{\kappa,\kappa}$.*
(4) *κ is supercompact-for-$\mathbb{L}^{s,\Sigma_n}_{\kappa,\kappa}$.*

Proof: This follows the same argument as Theorem 4.3. †

However, the notion of a huge-for-$\mathbb{L}^{s,\Sigma_n}_{\kappa,\kappa}$ cardinal would be similarly stronger in consistency strength.

While the Löwenheim-Skolem-Tarski number for second order was determined by Magidor in [Mag71] and Magidor and Väänänen have explored the Löwenheim-Skolem-Tarski numbers of various fragments of $\mathbb{L}^2$ in [MV11], the Löwenheim-Skolem-Tarski number of sort logic seems unknown. We give a characterization of these cardinals in terms of a $C^{(n)+}$-version of Magidor's characterization of supercompacts. We work with Löwenheim-Skolem-Tarski numbers for strictly first-order languages to avoid the technicalities around trying to develop a notion of elementary substructure for sort logic. See Section 4.3 for this in second-order logic.

Theorem 4.14. *The following are equivalent for κ:*

(1) *The conclusion of Lemma 4.12.*
(2) *For all $\alpha < \kappa$, if N is a structure in a strictly first-order language of size $< \kappa$, then there is $M \prec_{\mathbb{L}_{\alpha,\alpha}} N$ of size $< \kappa$ such that M and N have the same $\mathbb{L}^{s,\Sigma_n}_{\alpha,\alpha}$-theory.*

Proof: (1) implies (2): Find $\gamma < \kappa$ above α and $|\tau|$ and find $\alpha' > \kappa$ such that $N \in V_{\alpha'}$. Code the structure N into a relation $R \subset V_{\alpha'}$ By assumption, there is $\bar{\alpha} \in (\gamma, \kappa)$ in $C^{(n)}$ and $j : (V_{\bar{\alpha}}, \in, S) \to (V_\alpha, \in, R)$ with crit $j > \gamma$. Then S codes a structure M in $V_{\bar{\alpha}}$ that, by elementarity, models $Th_{\mathbb{L}^{s,\Sigma_n}_{\alpha,\alpha}}(N)$. Moreover, $j \upharpoonright M$ is $\mathbb{L}_{\alpha,\alpha}$-elementary, so the range of $j \upharpoonright M$ is the desired model.

(2) implies (1): Fix $\alpha > \kappa$ in $C^{(n)}$, $R \subset V_\alpha$, and $\beta < \kappa$. Apply the assumption to the structure $(V_\alpha, \in, R, \delta)_{\delta<\beta}$ to get $M \prec_{\mathbb{L}^{\beta,\beta}} N$ with the same $\mathbb{L}^{s,\Sigma_n}$-theory. This includes Magidor's Φ, so after taking the transitive collapse, we get $\prec_{\mathbb{L}_{\beta,\beta}}$-elementary $j : (V_{\bar{\alpha}}, \in, S, \delta)_{\delta<\beta} \to (V_\alpha, \in, R, \delta)_{\delta<\beta}$. The constants for the elements of β force the critical point of j above β. †

4.2. **Rank-into-rank.** We now turn to the strongest large cardinal principles, the rank-into-rank embeddings. For an excellent survey of these, see Dimonte [Dim].

We can cast $I1$, $I2$, and $I3$ in a uniform way by saying, for $n < \omega$, $I2_n(\kappa, \delta)$ is the assertion

There is $j : V_\delta \to V_\delta$, $\kappa = \text{crit } j$, δ is the supremum of the critical sequence, and j is $\Sigma^1_{2n}(\mathbb{L}_{\kappa,\omega})$-elementary.

Laver [Lav97, Theorem 2.3] proved[9] that Σ^1_{2n+1}-elementarity of such a j implies its Σ^1_{2n+2}-elementarity, so it suffices to consider the even levels. Then $I3$ is $I2_0$, $I2$ is $I2_1$[10], and $I1$ is $I2_{<\omega}$.

Note that second-order elementary embeddings should be understood in the context of Section 4.3. Given $j : V_\delta \to V_\delta$, we can naturally extend this to $j^+ : \mathcal{P}(V_\delta) \to \mathcal{P}(V_\delta)$ by $j^+(R) = \cup_{\alpha<\delta} j(R \cap V_\delta)$. Then set $A^{V_\delta} = j^+(A)$. Note that j^+ is the only possible extension of j to $V_{\delta+1}$ that could be elementary and it's Σ^1_0-elementarity follows from its first-order elementarity.

These principles have natural characterizations in terms of extendibility criteria. Recall that κ is weakly compact iff every κ-sized structure has a proper $\mathbb{L}_{\kappa,\kappa}$-elementary extension [Kan08,

[9]Laver's paper attributes this result to Martin without citation, but other sources attribute it to Laver
[10]Which is in turn equivalent to being elementary about statements of well-foundedness [Dim, Lemma 6.13].

Theorem 4.5] and κ is measurable iff every $\geq \kappa$-sized structure has a proper $\mathbb{L}_{\kappa,\kappa}$-elementary extension.

Proposition 4.15. *For each $n \leq \omega$, the following are equivalent.*

(1) $I2_n(\kappa, \delta)$
(2) *Every δ-sized structure has a proper $\Sigma^1_{2n}(\mathbb{L}_{\kappa,\omega})$-elementary extension isomorphic to itself.*

Proof: For one direction, take $(V_\delta, \in)$ as the δ-sized structure. After collapse, δ is necessarily strong limit and the sup of the ciritical sequence by the Kunen inconsistency.

For the other direction, if M is a structure of size δ, we can arrange $|M| = \delta$. Then $j \restriction \delta$ lifts to $j^- : M \to M$, which inherits the elementarity of J. †

$I3$ and $I2$ also have standard characterizations in terms of coherent ω-sequences of normal ultrafilters. This allows us to prove the following type omitting compactness from them. Unfortunately, this does not give an equivalence, but does allow us to sandwich these properties between two compactness for omitting types statements. Recall that any type can be trivially extended to an equivalent, larger one by adding instances of "$x \neq x$."

Theorem 4.16.

(1) *If $I2_0(\kappa, \delta)$, then for any theory $T = \cup_{s \in \mathcal{P}_\kappa \kappa} T_s \subset \mathbb{L}_{\kappa,\omega}(\tau)$ and set of types $\{p^\beta = \{\phi^\beta_i(x) \mid i < \kappa_{n_\beta+1}\} \mid \beta < \mu\}$, if there are club many $s \in \mathcal{P}_\kappa \kappa$ such that T_s has a model M_s with the property*
$$\text{for } \beta < \mu, \left\{t \in [\kappa_{n_\beta+1}]^{\kappa_{n_\beta}} \mid M_{t \cap \kappa} \text{ omits } p^\beta_t = \{\phi^\beta_i(x) \mid i \in t\}\right\} \text{ contains a club.}$$
Then T has a model omitting $\{p^\beta \mid \beta < \mu\}$.
(2) *$I2_1(\kappa, \delta)$ implies the same for the logic $\mathbb{L}_{\kappa,\omega}(Q^{WF})$.*

We will use the ultrafilter characterizations of these cardinals, in part because $I3$ doesn't have a characterization in terms of $j : V \to \mathcal{M}$ and we don't want to restrict to models of size $\leq \delta$.

Proof: We prove the second item. The first follows by the same argument, just removing the mentions of well-foundedness.

$I2_1(\kappa, \delta)$ is equivalent to the existence of κ-complete, normal ultrafilters U_n on $[\kappa_{n+1}]^{\kappa_n}$ (where $\{\kappa_n \mid n < \omega\}$ is the critical sequence of the witnessing embedding) such that

(1) *coherence:* For any $X \subset [\kappa_{n+1}]^{\kappa_n}$ and $m > n$,
$$X \in U_n \iff \{s \in [\kappa_{m+1}]^{\kappa_m} \mid s \cap \kappa_{n+1} \in X\} \in U_m$$
(2) *well-founded:* For any $\{n_i < \omega \mid i < \omega\}$ and $\{X_i \in U_{n_i} \mid i < \omega\}$, there is $s \subset \delta$ such that, for all $i < \omega$, $s \cap \kappa_{n_i+1} \in X_i$.

Let M_s be the desired models. Since they exists for club many s and U_0 contains this club, we can form the direct limit of the ultrapowers as standard: $M^*_n = \prod_{s \in [\kappa_{n+1}]^{\kappa_n}} M_{s \cap \kappa}/U$ and there is a coherent system of $\mathbb{L}_{\kappa,\kappa}$-embeddings $f_{n,m} : M^*_n \to M^*_m$ that takes $[f]_{U_n}$ to $[s \mapsto f(s \cap \kappa_{n+1})]_{U_{n+1}}$. Then M^* is the direct limit of these models. Standard arguments (see Proposition 5.3 for a more general case) show that Łoś' Theorem holds for formulas of $\mathbb{L}_{\kappa,\omega}(Q^{WF})$. This guarantees that $M^* \vDash T$. To show the type omission, let $\beta < \mu$ and $[n, f]_{\bar{U}} \in M^*$. Setting $n' = \max n, n_\beta$, this implies
$$X := \left\{t \in [\kappa_{n'+1}]^{\kappa_{n'}} \mid M_{t \cap \kappa} \text{ omits } p^\beta_{t \cap \kappa_{n_\beta+1}}\right\} \in U_{n'}$$
Define a function h on X by setting, for $t \in X$, $h(t) \in t$ such that $M_{t \cap \kappa} \vDash \neg\phi^\beta_{h(t)}(f(t \cap \kappa_{n+1}))$; this is possible exactly because of the type omission. Then, h is regressive on a $U_{n'}$-large set, so there is $\alpha_0 < \kappa_{n'+1}$ such that
$$Y := \left\{t \in [\kappa_{n'+1}]^{\kappa_{n'}} \mid M_{t \cap \kappa} \vDash \neg\phi^\beta_{\alpha_0}(f(t \cap \kappa_{n+1}))\right\} \in U_{n'}$$

Then, by Łoś' Theorem, $[n,f]_{\bar{U}} = [n', f(s \cap \kappa_{n+1})]_{\bar{U}}$ omits p^β in M^* as

$$M^* \vDash \neg\phi^\beta_{\alpha_0}([n,f]_{\bar{U}})$$

Since $[n,f]_{\bar{U}}$ and β we arbitrary, we are done, †

We can also isolate a type omitting compactness stronger than these rank-into-rank axioms. Note that, unlike previous theorems, the types in the following don't shrink in the hypothesis.

Theorem 4.17. *Fix a cardinal $\delta = \beth_\delta$.*

(1) *Suppose we have the following for some κ:*

For any $\mathbb{L}_{\omega,\omega}(\tau)$-theory T with a filtration $\{T_\alpha \mid \alpha < \kappa\}$ and $|\tau| = \delta$ and any types $\{p_i(x) \mid i < \delta\}$, if every T_α has a model omitting Γ, then T has a model omitting Γ.

Then there is $\kappa_0 \leq \kappa$ such that $I2_0(\kappa_0, \delta)$.

(2) *The above implies $I2_n(\kappa, \delta)$ after replacing the logic with $\Sigma^1_{2n}(\mathbb{L}_{\omega,\omega})$.*

Proof: Fix a bijection $f : \delta \to V_\delta$. Consider the theory $T = ED_{\mathbb{L}_{\omega,\omega}}(V_\delta, \in, x)_{x \in V_\delta} \cup \{c_i < c < c_\kappa\} \cup \{d_i E d_j \mid i, j < \delta, h(i) \in h(j)\}$ and, for $i < \delta$, the types $p_i(x) = \{x E d_i \mid x \neq d_j \mid j < \delta, h(j) \in h(i)\}$ and $p_\delta(x) = \{x \neq d_i \mid i < \delta\}$. Then this theory is chain compact by considering the structure V_δ, which omits all of the types. Then any model of T omitting the types gives rise to the desired elementary embedding in the standard way. †

4.3. **Elementary substructure for $\mathbb{L}^2$.** In contrast with first-order logic[11], the notion of elementary substructure in *second-order* logic has not been well-studied. As evidence to this, in Väänänen's paper on second-order logic and set theory [Vää01], the word 'structure' appears 210 times, but 'substructure,' 'sub-structure,' or even 'sub structure' never appear. Similarly, Shapiro's book [Sha91] never discusses the matter[12]. A guess at the cause for this is that second-order logic is often employed to find categorical theories, whereas first-order logic attempts only to axiomatize classes of structures. In such contexts, there is no reason to talk about a models relation with others.

A first-attempt at second-order elementary substructure would be to work in analogy with first-order and say M is a $\mathbb{L}_2$-elementary substructure of N iff every formula holds of parameters from M in M iff it holds in N. However, this notion doesn't allow for any proper elementary extensions as there are definable sets that must grow. Concretely, the formula $\phi(X) :=$ "$\forall x\,(x \in X)$" must be satisfied by the entire universe, so no extension of M can think $\phi(M)$ holds. A more modest generalization is used by Magidor and Väänänen in [MV11, Between Definitions 3 and 4]. They say that M is an $\mathbb{L}^2$-elementary substructure of N iff the above property holds *restricted* to formulas with only first-order free variables. This works (in the sense that proper extensions can exist), and they observe that Magidor's theorem on the Löwenheim-Skolem-Tarski number of second-order extends to this notion of elementarity. However, it seems lacking as there's no discussion of free variables in second order.

Our definition includes a new notion of second-order substructure in addition to second-order elementary substructure. Before giving the formal definitions, we give a motivation. Any second-order structure M can be viewed as a Henkin structure $M^+ = (M, P[M], E)$. Then every second-order statement about M can be transferred to a first-order statement about M^+, and the fact that M^+ is a full structure (isomorphic to $(M, \mathcal{P}(M), \in)$) can be captured by a single second statement Ψ asserting every subset of M is represented by a member of E.

[11]Here meant as sublogics of $\mathbb{L}_{\infty,\infty}$.

[12]The notion of "elementary substructure" does appear here, but always in reference to its first-order version.

Now that we have moved to a more familiar first-order setting, we can ask what relation between the original structures M and N characterizes when M^+ is an elementary substructure of N^+. The key is that, given $s \subset M$ and $m_s \in P[M]$ representing it, N^+ thinks the same facts about m_s as M does when the parameters come from M, but N also thinks new things about m_s. Crucially, there might be elements of $N - M$ that N thinks are in m_s. Then, setting $s^N := \{n \in N \mid N \vDash nEm_s\}$, N thinks all the same facts about s^N that M does about s. This notion of extending subsets is key to defining second-order elementary substructure. This leads to the following definitions[13]:

Definition 4.18. *Given τ-structures M and N, we say M is a* second-order substructure *of N, written $M \subset_2 N$ iff $|M| \subset |N|$ and, for every $s \subset M^n$, there is $s^N \subset N^n$ such that for every atomic $\phi(x_1, \ldots, x_n; X_1, \ldots, X_{n'})$, $m_i \in M$, and $s_i \subset M^{n_i}$, we have*

$$M \vDash \phi(m_1, \ldots, m_n; s_1, \ldots, s_{n'}) \iff N \vDash \phi(m_1, \ldots, m_n; s_1^N, \ldots, s_{n'}^N)$$

Note that this condition implies that $s^N \cap |M|^n = s$.

Definition 4.19. *Given τ-structures M and N, we say M is an $\mathbb{L}^2$-elementary substructure of N, written $M \prec_{\mathbb{L}^2} N$ iff $M \subset_2 N$ and there is an extension $s \mapsto s^N$ witnessing this such that for every $\phi(x_1, \ldots, x_n; X_1, \ldots, X_m) \in \mathbb{L}^2$, $m_i \in M$, and $s_i \subset M^{n_i}$, we have*

$$M \vDash \phi(m_1, \ldots, m_n; s_1, \ldots, s_{n'}) \iff N \vDash \phi(m_1, \ldots, m_n; s_1^N, \ldots, s_{n'}^N)$$

Thus, $M \prec_{\mathbb{L}^2} N$ comes with a choice of extensions for each $s \subset M$. This avoids the issue with the "first-attempt" notion of elementary substructure from (**??**) above. Indeed, for any definable subset A of M, A^N must be the collection of elements satisfying that definition in N if $M \prec_{\mathbb{L}^2} N$.

Note that if τ is a strictly first-order language (as is often the case), then $M \subset N$ is equivalent to $M \subset_2 N$ for any collection of extensions. This means that comments about substructures in first-order languages in the context of second-order can be given the normal interpretation. Also, for such a language, $\prec_{\mathbb{L}^2}$ will imply elementarity in the sense of Magidor and Väänänen.

For a general notion of $\leq$, a $\leq$-elementary embedding is $f : M \to N$ such that f is a τ-isomorphism onto its range and $f(M) \leq N$ (a set theorist might prefer to call this model $f"M$). Specializing to $\prec_{\mathbb{L}^2}$, f is a map on elements of M and, given $s \subset M$, $f"s \subset f(M)$. Then $f(M) \prec_{\mathbb{L}^2} N$ implies there is an extension $(f"s)^N \subset N$ that satisfies the definition of $\prec_{\mathbb{L}^2}$. To avoid this unfortunate notation, we say that $f : M \to N$ is $\mathbb{L}^2$-elementary means that f is a map from $M \cup \mathcal{P}^\omega(M)$ to $N \cup \mathcal{P}^\omega(N)$ such that, for all $\phi(\mathbf{x}, \mathbf{X})$, $\mathbf{a} \in M$, and $\mathbf{s} \in \mathcal{P}^\omega(M)$,

$$M \vDash \phi(\mathbf{a}, \mathbf{s}) \iff N \vDash \phi(f(\mathbf{a}), f(\mathbf{s}))$$

One should be skeptical that this is the "right" notion of $\prec_{\mathbb{L}^2}$ as it adds a strange new condition about global choices of extensions of subsets. In addition to the heuristic with Henkin models above, we offer two "sanity" checks that this is the right notion.

The first check is that Magidor's Theorem on the LST number of second order logic holds with this notion.

Corollary 4.20. *Let κ be supercompact. For any τ of size $< \kappa$ and $\alpha < \kappa$ and τ-structure N, there is $M \prec_{\mathbb{L}^2_{\alpha,\alpha}} N$ of size $< \kappa$.*

This follows from the argument given in our heuristic: take a model N, turn it into a full Henkin structure N^+ with Skolem functions, and apply Magidor's version along with the sentence $\forall X \exists x\, (X` = 'x)$. This quotational equality is an abbreviation of the statement that X and x have the same elements.

[13] We focus on $\prec_{\mathbb{L}^2}$ for ease, but these definitions could easily be changed to accommodate $\prec_{\mathbb{L}^2_{\alpha,\alpha}}$ for $\alpha < \kappa$.

The second check is the natural notion of an elementary diagram characterizes $\mathbb{L}^2$-elementary embedability. Given a τ-structure M, we define it's $\mathbb{L}^2$ elementary diagram by first adding a first-order constant c_a for each $a \in M$ and a second-order constant d_s for each $s \in \mathcal{P}^\omega(M)$ and setting

$$ED_{\mathbb{L}^2}(M) = \{\phi(c_{a_1}, \dots, c_{a_n}, d_{s_1}, \dots, d_{s_k}) \mid M \vDash \phi(a_1, \dots, a_n, s_1, \dots, s_k)\}$$

Proposition 4.21. *Let M, N be τ-structures. The following are equivalent.*

(1) *N has an expansion that models $ED_{\mathbb{L}^2}(M)$.*
(2) *There is a $\mathbb{L}^2$-elementary $f : M \to N$.*

Proof: The proof follows as the first-order one. Given $\mathbb{L}^2$-elementary $f : M \to N$, expand N by $c_a^N = f(a)$ and $d_s^N = f(s)$. Similarly, if $N^* \vDash ED_{\mathbb{L}^2}(M)$, define $f : M \to N$ by the same formula. †

Some of the basic results of first-order elementary substructure transfer, while others do not. For instance, the Tarski-Vaught test goes through with the same proof, although a slightly modified statement.

Proposition 4.22 (Tarski-Vaught Test for $\prec_{\mathbb{L}^2}$). *Given $M \subset_2 N$ (with a specified extension map $s \mapsto s^N$), we have that $M \prec_{\mathbb{L}^2} N$ iff both of the following hold:*

(1) *$\forall m_i \in M, s_j \subset |M|^{n_j}$ and $\phi(x, \mathbf{x}, \mathbf{X})$, if $N \vDash \exists x \phi(x, \mathbf{m}, \mathbf{s}^N)$, then there is $m \in M$ such that $N \vDash \phi(m, \mathbf{m}, \mathbf{s}^N)$; and*
(2) *$\forall m_i \in M, s_j \subset |M|^{n_j}$ and $\psi(X, \mathbf{x}, \mathbf{X})$, if $N \vDash \exists X \phi(X, \mathbf{m}, \mathbf{s}^N)$, then there is $s \subset M^n$ such that $N \vDash \phi(s^N, \mathbf{m}, \mathbf{s}^N)$.*

However, properties such as coherence and smoothness under unions of chains fail, as witnessed by Silver's example (the bane of many a nonelementary model theorist's hope).

We can also define Skolem function for second-order in the same way as first-order. To do so, we must add strictly second-order functions to the language, so that even if τ started out strictly first-order, its Skolemization τ_{sk} won't be.

In the context of first-order, the notions of elementary substructure and that of club sets are very closely intertwined. For instance, given a structure M and cardinal $\kappa < \|M\|$,

$$\{s \in \mathcal{P}_\kappa \lambda : s \text{ is the universe of an elementary substructure of } M\}$$

is club, and, conversely, given a club $\mathcal{C} \subset \mathcal{P}_\kappa \lambda$, we can find a structure M with universe λ such that the above set is contained in $\mathcal{C}$. This connection is mediated by the fact that both notions can be characterized by being the closure sets of certain functions (Skolem functions, in the case of elementary substructure).

With a definition for second-order elementary substructure in hand, we can define a notion of club, which we call superclubs. Recall our focus on $\mathbb{L}^2$.

Definition 4.23. (1) *Set $\mathcal{P}^\omega(X) = \cup_{n<\omega} \mathcal{P}(X^n)$.*

(2) *Given $s \subset \lambda$, an* extension function *if $f : \mathcal{P}^\omega(s) \to \mathcal{P}^\omega(\lambda)$ such that, for all $x \subset s^n$, $f(x) \cap s^n = x$ and $f(x) \subset X^n$ and f fixes any finite set.*
(3) *Fix $\mu < \kappa$ and let $F_i : [\lambda]^{<\omega} \times [\mathcal{P}^\omega(\lambda)]^{<\omega} \to \mathcal{P}^\omega(\lambda)$ for $i < \mu$. Then*

$$C(\bar{F}) := \{s \in \mathcal{P}_\kappa \lambda \mid \exists \text{ extension } f : \mathcal{P}^\omega(s) \to \mathcal{P}^\omega(\lambda) \text{ such that } \forall \mathbf{a} \in s, \mathbf{x} \subset s^{\mathbf{n}}, i < \mu, \text{ we have } F_i(\mathbf{a}, f(\mathbf{x})) = f(y) \text{ for some } y \in \mathcal{P}^\omega(s)\}$$

(4) *We call the collection $\mathcal{F}$ of all sets containing some $C(\bar{F})$ the* superclub filter.

Although we defined it with a combinatorial characterization in the spirit of Definition 3.1, our interest in superclubs comes from the following model theoretic characterization.

Lemma 4.24. *The superclub filter is generated by sets of the form*

$$D(M) := \{s \in \mathcal{P}_\kappa\lambda \mid M \upharpoonright s \prec_{\mathbb{L}^2} M\}$$

for M a τ-structure with universe λ and $|\tau| < \kappa$.

Proof: First, suppose we are given $\{F_\alpha \mid \alpha < \mu\}$. Set $\tau = \{F_\alpha^{n;m_1,\dots,m_k} \mid \alpha < \mu; k, n, m_i < \omega\}$ to be a functional language so the domain of $F_\alpha^{n;m_1,\dots,m_k}$ is $M^n \times \mathcal{P}(M^{m_1}) \times \cdots \times \mathcal{P}(M^{m_k})$. Expand λ to a τ-structure by interpreting $F_\alpha^{n;m_1,\dots,m_k}$ as F_α restricted to the appropriate arity. Then $D(M) \subset C(\bar{F})$.

Second, suppose we are given M with universe λ. WLOG, we can assume that τ has (second-order) Skolem functions. Let $\bar{F} = \{F^M \mid F \in \tau\}$ be the functions of M (interpreted as projection on other arities). Then, since we have Skolem functions, $C(\bar{F}) \subset D(M)$. †

This model-theoretic characterization tends to be more useful to show the things that we want.

Corollary 4.25. *The superclub filter extends the club filter.*

Proof: The club filter can be characterized in the same way using first-order elementary substructure. †

The next proposition shows that calling this a filter is justified.

Proposition 4.26. *The superclub filter is a κ-complete, possibly non-proper filter.*

Proof: It is a filter by definition. Given $D(M_\alpha)$ with M_α a τ_α structure, set M_* to be the $\cup\tau_\alpha$-structure that is simultaneously an expansion of each M_α. †

Proposition 4.26 leaves open the possibility that the superclub filter is trivial, i. e., contains the empty set. In fact, the emptiness of the superclub filter characterizes supercompact cardinals.

Theorem 4.27. *Let $\kappa \leq \lambda$.*

(1) *If κ* **is not** *λ-supercompact, then there is an empty superclub. Moreover, there is a uniform definition of the empty superclub.*
(2) *If κ* **is** *λ-supercompact, then every normal, fine, κ-complete ultrafilter on $\mathcal{P}_\kappa\lambda$ extends the superclub filter.*

Proof: If κ is not λ-supercompact, then by Corollary 4.20, there is a structure M of size λ in a langauge τ of size $< \kappa$ such that M has no $\prec_{\mathbb{L}^2}$-substructures; looking at Magidor's proof, we can take M to be some $(V_\beta, \in, \alpha)_{\alpha\in\mu}$ for some $\beta < \lambda$ and $\mu < \beta$. WLOG $|M|$ is λ by expanding the universe by a trivial sort. Then $D(M) = \emptyset$ is in the superclub filter.

On the other hand, let U be a normal, κ-complete ultrafilter on $\mathcal{P}_\kappa\lambda$ and derive $j_U : V \to \mathcal{M}_U$. Let M be a τ-structure with universe λ and, WLOG, it has Skolem functions. By Magidor's result and its extensions, j_U"$M \subset_{\mathbb{L}^2} j_U(M)$. Since M has Skolem functions, j_U"$M \prec_{\mathbb{L}^2} j_U(M)$. Thus, j"$\lambda \in (D(j_U(M)))^{\mathcal{M}_U} = j\,(D(M))$. So $D(M) \in U$. †

In particular, we get the following:

Corollary 4.28. *Given $\kappa \leq \lambda$, κ is λ-supercompact iff the superclub filter on $\mathcal{P}_\kappa\lambda$ is proper.*

While superclubs give a characterization of supercompact cardinals, they lack a nice characterization in the spirit of "closed unbounded sets" that clubs have. Such a characterization would shed light on properties of the $\prec_{\mathbb{L}^2}$ relations and permit the exploration of superstationary sets.

5. Extenders and omitting types

A key distinction between Theorem 3.4 and Theorems 4.1 and 4.9 is the lack of an analogue of Theorem 3.4.(3) in the results of Section 4. The large cardinals of Section 4 are typically characterized by the existence of certain kinds of extenders, but the modifier "certain" is typically characterized in a way that nakedly concerns embeddings between models of set theory–e. g., [Kan08, Exercise 26.7] characterizes κ being λ-strong by the existence of an extender E such that $j_E : V \to \mathcal{M}_E$ witnesses λ-strength–rather than any combinatorial feature of the extender.

We begin with a combinatorial characterization, although it still references the V_α's and may be of limited interest. However, we use this as a starting point to investigate a larger question: to what extent is second-order logic necessary to characterize the large cardinals in Section 4?

We should say a few words about our definition of extenders. We say that a (κ, λ)-extender E is $\{E_{\mathbf{a}} \mid \mathbf{a} \in [\lambda]^{<\omega}\}$, where each $E_{\mathbf{a}}$ is a κ-complete ultrafilter on ${}^{\mathbf{a}}\lambda$ satisfying coherence, normality, and well-foundedness (see [Kan08, Section 26] for a statement of these conditions in a slightly different formalism). Note that this is a compromise between the original definition of Martin-Steel [MS89]–which took $\mathbf{a} \in [V_\lambda]^{<\omega}$ and required $E_{\mathbf{a}}$ to be on ${}^{\mathbf{a}}V_\kappa$–and more modern presentations like [Kan08]–which takes $\mathbf{a} \in [\lambda]^{<\omega}$ but requires $E_{\mathbf{a}}$ to be on $[\kappa]^{|\mathbf{a}|}$. Crucially, we also depart from both of these and don't require that $\{s \in {}^{\mathbf{a}}\kappa \mid \forall a_1, a_2 \in \mathbf{a}\, (a_1 < a_2 \leftrightarrow s(a_1) < s(a_2))\} \in E_{\mathbf{a}}$.

Now we are ready to give a combinatorial characterization.

Definition 5.1. *We say a bijection $h : \beth_\alpha \to V_\alpha$ is* rank-layering *iff for every $\beta < \alpha$, $h``\beth_\beta = V_\beta$.*

Note this condition is equivalent to $x \in y \in V_\alpha$ implies $h^{-1}(x) \in h^{-1}(y)$.

Proposition 5.2. *The following are equivalent.*

1. *κ is λ-strong.*
2. *For* some *rank-layering bijection $h : \beth_\lambda \to V_\lambda$, there is a $(\kappa, \beth_\lambda)$-extender E such that for all $\alpha, \beta < \lambda$, we have*
$$h(\alpha) \in h(\beta) \iff \{s \in {}^{\{\alpha,\beta\}}\kappa \mid h(s(\alpha)) \in h(s(\beta))\} \in E_{\{\alpha,\beta\}}$$
3. *For* every *rank-layering bijection $h : \beth_\lambda \to V_\lambda$, there is a $(\kappa, \beth_\lambda)$-extender E such that for all $\alpha, \beta < \lambda$, we have*
$$h(\alpha) \in h(\beta) \iff \{s \in {}^{\{\alpha,\beta\}}\kappa \mid h(s(\alpha)) \in h(s(\beta))\} \in E_{\{\alpha,\beta\}}$$

In each of the cases, we have that $x \in V_\lambda$ is the image of $[\{h^{-1}(x)\}, s \mapsto h \circ s \circ h^{-1}(x)]_E$ after the transitive collapse.

Proof of Proposition 5.2: (3) implies (2) is obvious, and (2) implies (1) is proven by working through the standard arguments. (1) implies (3) is also proven by the standard arguments, except one seeds the extender by

$$X \in E_{\mathbf{a}} \iff j(h)^{-1} \circ h \circ j^{-1} \restriction j(\mathbf{a}) \in j(X)$$

†

Now we turn to the issue of how necessary logics beyond $\mathbb{L}_{\infty,\infty}$ are to characterize large cardinals and focus on strong cardinals. Theorem 4.7 characterizes strong cardinals in terms of the logic $\mathbb{L}^2$; is there a characterization of strong cardinals solely in terms of $\mathbb{L}_{\infty,\infty}$?

For $\kappa \leq \lambda$, consider the following (definable-class) $\mathbb{L}_{\kappa,\omega}(Q^{WF})$-theory and types for $y \in V_\lambda$:

$$\begin{aligned}
\tau &= \{E, c_a, c, d_{a'}\}_{a \in V, a' \in V_\lambda} \\
T &= ED_{\mathbb{L}_{\kappa,\omega}(Q^{WF})}(V, \in, x)_{x \in V} \cup \{d_i = c_i < c < c_\kappa \mid i < \kappa\} \cup \{d_b \in d_a \mid b \in a \in V_\lambda\} \\
p_y(x) &= \{xEd_y \wedge \neg(x = d_z) \mid z \in y\}
\end{aligned}$$

It follows from the methods of the previous sections that T has a model omitting each p_y iff κ is λ-strong. However, what we lack from Proposition 3.9 is an appropriate type omitting-compactness scheme and a decomposition of T along that scheme that is satisfiable. The difficulty in constructing this becomes more clear if we look at the extender product construction.

Given a (κ, λ)-extender E, for each $\mathbf{a} \in [\lambda]^{<\omega}$, we form the ultraproduct $\prod V/E_{\mathbf{a}}$. The coherence axiom and Łoś' Theorem insures that the restriction function induces a coherent system of $\mathbb{L}_{\kappa,\kappa}$-elementary embedding from $\prod V/E_{\mathbf{a}}$ to $\prod V/E_{\mathbf{b}}$ when $\mathbf{a} \subset \mathbf{b}$. Since $[\lambda]^{<\omega}$ is directed, we can get a colimit $\prod V/E$ and $\mathbb{L}_{\kappa,\omega}$-elementary embeddings. Since E is well-founded, so is $\prod V/E$, so we can form its transitive collapse $\mathcal{M}_E$.

We can generalize this to a more general model-theoretic context. Fix some $\mathbf{a} \in [\lambda]^{<\omega}$ and suppose that we have a collection of τ-structures $\{M_s \mid s \in {}^{\mathbf{a}}\kappa\}$ and form the ultraproducts $\prod_{s \in {}^{\mathbf{b}}\kappa} M_{s \restriction \mathbf{a}}/E_{\mathbf{a}}$ for $\mathbf{b} \supset \mathbf{a}$. Again, the coherence gives a coherent system of embeddings, so we can form the extender product $\prod M_s/E$ as the colimit of this system. This structure has universe $\{[\mathbf{b}, f]_E \mid \mathbf{a} \subset \mathbf{b} \in [\lambda]^{<\omega}, f \in {}^{\mathbf{b}}\kappa\}$ where $[\mathbf{b}, f]_E = [\mathbf{c}, g]_E$ iff $\{s \in {}^{\mathbf{bc}}\kappa \mid f(s \restriction \mathbf{b}) = g(s \restriction \mathbf{c})\} \in E_{\mathbf{bc}}$. Then we have following result for Łoś' Theorem.

Proposition 5.3. *Let E be a κ-complete (κ, λ)-coherent ultrafilter. TFAE*

(1) *E is well-founded*

(2) *Łoś' Theorem holds for $\mathbb{L}_{\kappa,\omega}(Q^{WF})$ formulas. That is, given τ-structures $\{M_s \mid s \in {}^{\mathbf{a}}\kappa\}$, $\phi(x_1, \ldots, x_n) \in \mathbb{L}_{\kappa,\omega}(Q^{WF})(\tau)$, and $[\mathbf{b}_1, f_1]_E, \ldots, [\mathbf{b}_n, f_n]_E \in \prod M_s/E$, we have*

$$\prod M_s/E \vDash \phi\left([\mathbf{b}_1, f_1]_E, \ldots, [\mathbf{b}_n, f_n]_E\right)$$
$$\Longleftrightarrow$$
$$\left\{s \in {}^{\cup \mathbf{b}_i}\kappa \mid M_{s \restriction \mathbf{a}} \vDash \phi\left(f_1(s \restriction \mathbf{b}_1), \ldots, f_n(s \restriction \mathbf{b}_n)\right)\right\} \in E_{\cup \mathbf{b}_i}$$

Proof: For one direction, it is known that E is well-founded iff $\prod V/E$ is well-founded, which follows from Łoś' Theorem applied to $Q^{WF}xy(x \in y)$.

For the other direction, fix $\mathbf{b} \in [\lambda]^{<\omega}$ and τ-structures $\{M_s \mid s \in {}^{\mathbf{b}}\kappa\}$. We show Łoś' Theorem for $\mathbb{L}_{\kappa,\omega}(Q^{WF})$ by induction. Standard arguments take care of everything but the Q^{WF} quantifier. So suppose Łoś' Theorem holds for $\phi(x, y, \mathbf{z})$ and $[\mathbf{a}, f]_E \in M_E := \prod M_s/E$.

First, suppose that $\{s \in {}^{\mathbf{a}}\kappa \mid M_{s \restriction \mathbf{b}} \vDash Q^{WF}xy\phi(x, y, f(s))\} \notin E_{\mathbf{a}}$. Set X to be the complement of this set. For $s \in X$, there is $c^s_r \in M_{s \restriction \mathbf{b}}$ for $r < \omega$ such that $M_{s \restriction \mathbf{b}} \vDash \phi(c^s_{r+1}, c^2_r, f(s)$. Then $c_r := [\mathbf{a}, c^s_r]_E \in M_E$ witnesses the illfoundedness of ϕ.

Second, suppose that $X_0 = \{s \in {}^{\mathbf{a}}\kappa \mid M_{s \restriction \mathbf{b}} \vDash Q^{WF}xy\phi(x, y, f(s))\} \in E_{\mathbf{a}}$ and $M_E \vDash \neg Q^{WF}xy\phi(x, y, [\mathbf{a}, f]_E)$. Then there is $[\mathbf{a}_r, f_r]_E \in M_E$ such that

$$M_E \vDash \phi\left([\mathbf{a}_{r+1}, f_{r+1}]_E, [\mathbf{a}_r, f_r]_E, [\mathbf{a}, f]_E\right)$$

for all $r < \omega$ and, WLOG, $\mathbf{a} \subset \mathbf{a}_r \subset \mathbf{a}_{r+1}$. Then

$$X_{r+1} := \{s \in {}^{\mathbf{a}_{r+1}}\kappa \mid M_{s \restriction \mathbf{b}} \vDash \phi(f_{r+1}(s), f_r(s \restriction \mathbf{a}_r), f(s \restriction \mathbf{a})\} \in E_{\mathbf{a}_{r+1}}$$

The well-foundedness of E gives $h : \cup \mathbf{a} \to \kappa$ such that $h \restriction \mathbf{a}_{r+1} \in X_{r+1}$ and $h \restriction \mathbf{a} \in X_0$. Then, for $r < \omega$,

$$M_{h \restriction \mathbf{b}} \vDash \phi(f_{r+1}(h \restriction \mathbf{a}_{r+1}), f_r(h \restriction \mathbf{a}_r), f(h \restriction \mathbf{a}))$$

Thus, $\langle f_r(h \restriction \mathbf{a}_r) \in M_{h \restriction \mathbf{b}} \mid r < \omega\rangle$ witnesses the illfoundedness of $\phi(x, y, f(h \restriction \mathbf{a}))$ in $M_{h \restriction \mathbf{b}}$, contradicting $h \restriction \mathbf{a} \in X_0$. †

So, to decompose T above, one could try to construe $\prod V/E$ as an extender product $\prod M_s/E$ in the appropriate language and for the appropriate $\mathbf{a}$, and then see what fragment of T M_s satisfies. However, the problem is that the factors that make up $\prod V/E$ don't have expansions to τ-structures. Rather, the analysis of $\prod V/E$ crucially uses that it can be seen as the extender power $\prod_{s \in {}^{\mathbf{a}}\kappa} V/E$ for any choice of $\mathbf{a}$. Thus, there is no way to analyze which parts of the types each factor omits.

However, there is a nice criterion for when an extender product (or just a coherent ultraproduct by a κ-complete, well-founded coherent ultrafilter) omits a $\mathbb{L}_{\kappa,\omega}(Q^{WF})$-type based on the behavior of its *provided* that its over a subset that appears as an element of $\mathcal{M}_E$.

Proposition 5.4. *Let E be a $(\kappa, \beth_\lambda)$-extender witnessing that κ is λ-strong. Suppose that $\mathbf{a} \in [\beth_\lambda]^{<\omega}$, $\{M_s \mid s \in {}^{\mathbf{a}}\kappa\}$ is a collection of τ-structures, $A_\ell \subset \prod M_s/E$ of rank $\leq \lambda$ for $\ell = 0, 1$, and $\phi(x, y) \in \mathbb{L}_{\kappa,\omega}(Q^{WF})$. Set $p(x) = \{\phi(x, a) \mid a \in A_0\} \cup \{\neg\phi(x, a) \mid a \in A_1\}$ and $p_s(x) = \{\phi(x, a) \mid a \in f_0(s)\} \cup \{\neg\phi(x, a) \mid a \in f_1(s)\}$, where $[\mathbf{b}, f_\ell]_E$ represents A_ℓ in $\mathcal{M}_E$. Then, the following are equivalent:*

1. *$\prod M_s/E$ omits p.*
2. *$\{s \in {}^{\mathbf{b}}\kappa \mid M_{s \restriction \mathbf{a}}$ omits $p_s\} \in E_{\mathbf{b}}$.*

Note that we have restricted both to the case of ϕ-types and the case where E is an extender witnessing strength for simplicity and it because it suffices for our assumption. We could remove these assumptions, instead requiring that A_ℓ is an element of the subset sort of $\prod(M_s, \mathcal{P}(M_s), \in)/E$.

Proof: The structure $\prod M_s/E$ is (isomorphic to) $j_E(g)(\mathbf{a})$, where g is the function taking $s \in {}^{\mathbf{a}}\kappa$ to M_s and $\mathbf{a} = [\mathbf{a}, s \mapsto s(\mathbf{a})]_E$. Then, since $A_\ell \in V_\lambda \in \mathcal{M}_E$, Łoś' Theorem for extenders tells us that

$$\begin{aligned}
\mathcal{M}_E \vDash \text{“}j_E(g)(\mathbf{a}) \text{ omits } p\text{''} &\iff \mathcal{M}_E \vDash \text{“}j_E(g)(\mathbf{a}) \text{ omits the } \phi\text{-type generated by } A_0, A_1\text{''} \\
&\iff \{s \in {}^{\mathbf{b}}\kappa \mid g(s \restriction \mathbf{a}) \text{ omits the } \phi\text{-type generated by } f_0(s), f_1(s)\} \in E_{\mathbf{b}} \\
&\iff \{s \in {}^{\mathbf{b}}\kappa \mid M_{s\restriction \mathbf{a}} \text{ omits } p_s\} \in E_{\mathbf{b}}
\end{aligned}$$

†

Still, this doesn't give a syntactic characterization of strength because it deals with type omission for types over sets, whereas Theorem 3.4 deals just with type omission over the empty set (in the appropriate language). Thus, we are still left with the following question.

Question 5.5. *Given $\kappa \leq \lambda$, is there a syntactic property of logics such that $\mathbb{L}_{\kappa,\omega}(Q^{WF})$ (or some other sub-logic of $\mathbb{L}_{\kappa,\kappa}$) satisfies this property iff κ is λ-strong?*

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