LIMIT MODELS IN STRICTLY STABLE ABSTRACT ELEMENTARY CLASSES

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Abstract. In this paper, we examine the locality condition for non-splitting and determine the level of uniqueness of limit models that can be recovered in some stable, but not superstable, abstract elementary classes. In particular we prove:

THEOREM 1. *Suppose that $\mathcal{K}$ is an abstract elementary class satisfying*
1. *the joint embedding and amalgamation properties with no maximal model of cardinality μ.*
2. *stabilty in μ.*
3. $\kappa^*_\mu(\mathcal{K}) < \mu^+$.
4. *continuity for non-μ-splitting (i.e. if $p \in \text{ga-S}(M)$ and M is a limit model witnessed by $\langle M_i \mid i < \alpha\rangle$ for some limit ordinal $\alpha < \mu^+$ and there exists $N \prec M_0$ so that $p \restriction M_i$ does not μ-split over N for all $i < \alpha$, then p does not μ-split over N).*

For θ and δ limit ordinals $< \mu^+$ both with cofinality $\geq \kappa^*_\mu(\mathcal{K})$, if $\mathcal{K}$ satisfies symmetry for non-μ-splitting (or just (μ,δ)-symmetry), then, for any M_1 and M_2 that are (μ,θ) and (μ,δ)-limit models over M_0, respectively, we have that M_1 and M_2 are isomorphic over M_0.

Note that no tameness is assumed.

§1. Introduction. Because the main test question for developing a classification theory for abstract elementary classes (AECs) is Shelah's Categoricity Conjecture [1, Problem D.1], the development of independence notions for AECs has often started with an assumption of categoricity ([10, 18, 19] and others). Consequently, the independence relations that result are superstable or stronger (see, for instance, good λ-frames and the superstable prototype [11, Example II.3.(A)]). However, little progress has been made to understand stable, but not superstable AECs. A notable exception is the work on κ-coheir of Boney and Grossberg [3], which only requires stability in the guise of 'no weak κ-order property.' In this paper, we add to the understanding of strictly stable AECs with a different approach and under different assumptions than [3]. In particular, our analysis uses towers and the standard definition of Galois-stability. Moreover, we work without assuming any of the strong locality assumptions (tameness, type shortness, etc.) of [3]. We hope that this work will lead to further exploration in this context.

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The main tool in our analysis is a tower, which was first conceived to study superstable AECs (see, for instance [12] or [14]). The 'right analogue' of superstability in AECs has been the subject of much research. Shelah has commented that this notion suffers from 'schizophrenia,' where several equivalent concepts in first-order seem to bifurcate into distinct notions in nonelementary settings; see the recent Grossberg and Vasey [7] for a discussion of the different possibilities (and a suprising proof that they are equivalent under tameness).

Common to much analysis of superstable AECs is the uniqueness of limit models. Uniqueness of limit models was first proved to follow from a categoricity assumption in [12, 14, 15]. Later, μ-superstability, which was isolated by Grossberg, VanDieren, and Villaveces [6, Assumption 2.8(4)], was shown to imply uniqueness of limit models under the additional assumption of μ-symmetry [13]. μ-superstability was modeled on the local character characterization of superstability in first-order and was already known to follow from categoricity [12]. The connection between μ-symmetry and structural properties of towers [13] inspired recent research on μ-superstable classes: [16, 17]. Moreover, years of work culminating in the series of papers [12], [14], [15], [13], [16], [17] has led to the extraction of a general scheme for proving the uniqueness of limit models. In this paper we witness the power of this new scheme by adapting the technology developed in [13] to cover μ-stable, but not μ-superstable classes. We suspect that this new technology of towers will likely be used to answer other problems in classification theory (in both first order and non-elementary settings).

This paper focuses on the question to what degree the uniqueness of limit models can be recovered if we assume the class is Galois-stable in μ, but not μ-superstable, by refocusing the question from "*Are all* (μ, α)-limit models isomorphic (over the base)?" to "*For which* $\alpha, \beta < \mu^+$ *are* (μ, α)-limit models and (μ, β)-limit models isomorphic (over the base)?" Based on first-order results (summarized in [6, Section 2]), we have the following conjecture.

CONJECTURE 2. *Suppose $\mathcal{K}$ is an AEC with μ-amalgamation and is μ-stable. The set*

$$\{\alpha < \mu^+ : \mathrm{cf}(\alpha) = \alpha \textit{ and } (\mu, \alpha)\textit{-limit models are isomorphic to } (\mu, \mu)\textit{-limit models}\}$$

is a non-trivial interval of regular cardinals. Moreover, the minimum of this set is an important measure of the complexity of $\mathcal{K}$.

Our main result (Theorem 20) proves this conjecture under Assumption 5. Here, the "measure of complexity" is $\kappa^*_\mu(\mathcal{K})$, a generalization of the first-order $\kappa(T)$. An important feature of this work is that it explores the underdeveloped field of strictly stable AECs.

We end with a short comment of the importance of limit models. The general arguments for investigating the uniqueness of limit models have appeared before (see [14, 6]). Briefly, they give a version of saturated models without dealing with smaller models and give a sense of how difficult it is to create saturated models. However, we expect they will take on a greater importance in the context of strictly stable AECs, especially those without assumption of tameness. Of the various analogues for AECs (see [7, Theorem 1.2]), most have seen extensive analysis, but only in the context of tameness. One of the remaining notions (solvability; see [11, Chapter IV]) seems to have no natural 'degeneralization' to stability. What remains are μ-superstability and the uniqueness of limit models. Thus, it is reasonable to assume that understanding strictly stable AECs will require understanding the connection between 'μ-stability' (Assumption 5 here) and limit models. Theorem 20 is a step towards this understanding.

Section 2 reviews key definitions and facts with Assumption 5 being the key hypotheses throughout the paper. Section 3 discusses the notion of relatively full towers. Section 4 discusses reduced towers and proves the key lemma, Theorem 18. Section 5 concludes with a proof of the main theorem, Theorem 20.

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§2. Background. We refer the reader to [1], [5], [6], [14], and [13] for definitions and notations of concepts such as Galois-stability, μ-splitting, etc. We reproduce a few of the more specialized definitions and results here.

μ-superstability was isolated in [6, Assumption 2.8] by examining consequences of categoricity from [9] and [12], and the key idea is "no long splitting chains." We weaken this property by only forbidding long enough splitting chains. How long is 'long enough' is measured by $\kappa^*_\mu(\mathcal{K})$, which is a relative of [5, Definition 4.3] and universal local character [3, Definition 3.5]. Following [3], we add the * to this symbol to denote that the chain is required to have the property that M_{i+1} is universal over M_i.

DEFINITION 3. *We define $\kappa^*_\mu(\mathcal{K})$ to be the minimal, regular $\kappa < \mu^+$ so that for every increasing and continuous sequence $\langle M_i \in \mathcal{K}_\mu \mid i \leq \alpha \rangle$ with $\alpha \geq \kappa$ regular which satisfies for every $i < \alpha$, M_{i+1} is universal over M_i, and for every non-algebraic $p \in \text{ga-S}(M_\alpha)$, there exists $i < \alpha$ such that p does not μ-split over M_i. If no such κ exists, we say $\kappa^*_\mu(\mathcal{K}) = \infty$.*

In [5, Theorem 4.13], Grossberg and VanDieren show that if $\mathcal{K}$ is a tame stable abstract elementary class satisfying the joint embedding and amalgamation properties with no maximal models, then there exists a single bound for $\kappa^*_\mu(\mathcal{K})$ for all sufficiently large μ in which $\mathcal{K}$ is μ-stable. This proof works by considering the χ-order property of Shelah. We can also give a direct bound assuming tameness.

PROPOSITION 4. *Let $\mathcal{K}$ be an AEC with amalgamation that is λ-stable and (λ, μ)-tame. Then $\kappa^*_\mu(\mathcal{K}) \leq \lambda$.*

Note that the proof does not require the extensions to be universal.

PROOF. Let $\langle M_i \in K_\mu : i \leq \alpha \rangle$ be an increasing, continuous chain with $\text{cf}(\alpha) \geq \lambda$ and $p \in \text{ga-S}(M_\alpha)$. By [9, Claim 3.3.(1)] and λ-stability, there is $N_0 \prec M_\alpha$ of size λ such that p does not λ-split over N_0. By tameness, p does not μ-split over N_0. By the cofinality assumption, there is $i_* < \alpha$ such that $N_0 \prec M_{i_*}$. By monotonicity, p does not μ-split over M_{i_*}. ⊣

This definition motivates our main assumption.

ASSUMPTION 5.

1. *$\mathcal{K}$ satisfies the joint embedding and amalgamation properties with no maximal model of cardinality μ.*
2. *$\mathcal{K}$ is stable in μ.*
3. *$\kappa^*_\mu(\mathcal{K}) < \mu^+$.*
4. *$\mathcal{K}$ satisfies (limit) continuity for non-μ-splitting (i.e. if $p \in \text{ga-S}(M)$ and M is a limit model witnessed by $\langle M_i \mid i < \theta\rangle$ for some limit ordinal $\theta < \mu^+$ and there exists N so that $p \upharpoonright M_i$ does not μ-split over N for all $i < \theta$, then p does not μ-split over N).*

A few comments on the assumption is in order. Note that tameness is not assumed in this paper. Stability in μ is necessary for the conclusion of Theorem 20 to make sense; otherwise, there are no limit models! We have argued (both in principle and in practice) that varying the local character cardinal is the right generalization of superstability to stability in this context. However, we have kept the "continuity cardinal" to be ω. This seems necessary for the arguments[1]. It seems reasonable to hope that some failure of continuity for non-splitting will lead to a nonstructure result, but this has not yet been achieved.

The assumptions are (trivially) satisfied in any superstable AEC and, therefore, any categorical AEC. However, in this context, the result is already known. For a new example, we look to the context of strictly stable homogeneous structures as developed in Hyttinen [8, Section 1]. In the homogeneous contexts, Galois types are determined by syntactic types. Armed with this, Hyttinen studies the normal syntactic notion of nonsplitting under a stable, unsuperstable hypothesis [8, Assumption 1.1], and shows that syntactic splitting satisfies continuity and (more than) the equivalent of κ^* is $\aleph_1$[2]. It is easy to see that the syntactic version of nonsplitting implies our nonsplitting, which already implies $\kappa^*_\mu(\mathcal{K}) = \aleph_1$. The following argument shows that, if N is limit over M, the converse holds as well, which is enough to get the limit continuity for our semantic definition of splitting. Since the context of homogeneous model theory is very tame, we don't worry about attaching a cardinal to non-splitting because they are all equivalent.

Suppose that N is a limit model over M, witnessed by $\langle M_i \mid i < \alpha\rangle$, and $p \in \text{ga-S}(M)$ syntactically splits over M. Then, since Galois types are syntactic, there are $b, c \in N$ such that $\text{ga-tp}(b/M) = \text{ga-tp}(c/M)$ and, for an appropriate ϕ, $\phi(x, b, m) \wedge \neg\phi(x, c, m) \in p$. We can find $\beta, \beta' < \alpha$ such that $b \in N_\beta$ and $c \in N_{\beta'}$. Since b and c have the same type, we can find an amalgam $N_* \succ N_\beta$ and $f : N_\alpha \to_M N_*$ such that $f(b) = c$. Since N is universal over $N_{\beta'}$, we can find $h : N_* \to_{N'_\beta} N$. This gives us an isomorphism $h \circ f : N_\beta \cong h(f(N_\beta))$ and we claim that this witnesses the semantic version splitting: $c \in N_{\beta'}$, so $c = h(c) = h(f(b)) \in h(f(N_\beta))$ and, thus, $\neg\phi(x, c, m) \in p \upharpoonright h(f(N_\beta))$. On the other hand, $\phi(x, c, m) = h \circ f(\phi(x, b, m)) \in h \circ f(p \upharpoonright N_\beta)$. Thus, we have witnessed $h \circ f(p \upharpoonright N_\beta) \neq p \upharpoonright h(f(N_\beta))$.

[1] The first author claimed in the discussion following [2, Lemma 9.1] that only long continuity was necessary. However, after discussion with Sebastien Vasey, this seems to be an error.

[2] It shows that it is at most $\aleph_1$. However, if it were $\aleph_0$, the class would be superstable, contradicting the assumption.

Note if $\kappa^*_\mu(\mathcal{K}) = \mu$, then the conclusion is uninteresting, but the results still hold. Also, we assume joint embedding, etc. only in $\mathcal{K}_\mu$; however, to simplify presentation, we work as though these properties held in all of $\mathcal{K}$ and, thus, we had a monster model. This will allow us to write ga-tp(a/M) rather than ga-tp$(a/M; N)$ and witness Galois type equality with automorphisms. However, standard techniques can translate our proofs to ones not using a monster model.

Under these assumptions, it is possible to construct towers. This is the key technical tool in this construction. Towers were introduced in Shelah and Villaveces [12] and expanded upon in [14] and subsequent works.

Recall that, if I is well-ordered, then it has a successor function which we will denote $+1$ (or $+_I 1$ if necessary).

DEFINITION 6 ([14].I.5.1). *1. A* tower indexed by I in $\mathcal{K}_\mu$ *is a triple* $\mathcal{T} = \langle \bar{M}, \bar{a}, \bar{N} \rangle$ *where*

- $\bar{M} = \langle M_i \in \mathcal{K}_\mu \mid i \in I \rangle$ *is in increasing sequence*[3] *of limit models;*
- $\bar{a} = \langle a_i \in M_{i+1} \backslash M_i \mid i+1 \in I \rangle$ *is a sequence of elements;*
- $\bar{N} = \langle N_i \in K_\mu \mid i+1 \in I \rangle$ *such that* $N_i \prec M_i$ *with* M_i *universal over* N_i*; and*
- ga-tp(a_i/M_i) *does not* μ*-split over* N_i*.*

2. $\mathcal{K}^*_{\mu,I}$ *is the collection of all towers indexed by* I *in* $\mathcal{K}_\mu$*.*

The set of all towers indexed by I and made up of limit models of cardinality μ is denoted by $\mathcal{K}^*_{\mu,I}$. We will switch back and forth between the notation $\mathcal{K}^*_{\mu,\alpha}$ where α is an ordinal and $\mathcal{K}^*_{\mu,I}$ where I is a well ordered set (of order type α) when it will make the notation clearer. When we deal with relatively full towers, we will find the notation using I to be more convenient for book-keeping purposes.

For $\beta < \alpha$ and $\mathcal{T} = (\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ we write $\mathcal{T} \upharpoonright \beta$ for the tower made up of the sequences $\bar{M} \upharpoonright \beta := \langle M_i \mid i < \beta \rangle$, $\bar{a} \upharpoonright \beta := \langle a_i \mid i+1 < \beta \rangle$, and $\bar{N} \upharpoonright \beta := \langle N_i \mid i+1 < \beta \rangle$.

We will construct increasing chains of towers. Here we define what it means for one tower to extend another:

DEFINITION 7. *For* I *a sub-ordering of* I' *and towers* $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ *and* $(\bar{M}', \bar{a}', \bar{N}') \in \mathcal{K}^*_{\mu,I'}$*, we say*

$$(\bar{M}, \bar{a}, \bar{N}) \leq (\bar{M}', \bar{a}', \bar{N}')$$

if $\bar{a} = \bar{a}' \upharpoonright I$*,* $\bar{N} = \bar{N}' \upharpoonright I$*, and for* $i \in I$*,* $M_i \preceq_{\mathcal{K}} M'_i$ *and whenever* M'_i *is a proper extension of* M_i*, then* M'_i *is universal over* M_i*. If for each* $i \in I$*,* M'_i *is universal over* M_i *we will write* $(\bar{M}, \bar{a}, \bar{N}) < (\bar{M}', \bar{a}', \bar{N}')$*.*

For γ a limit ordinal $< \mu^+$ and $\langle I_j \mid j < \gamma \rangle$ a sequence of well ordered sets with I_j a sub-ordering of I_{j+1}, if $\langle (\bar{M}^j, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I_j} \mid j < \gamma \rangle$ is a $<$-increasing sequence of towers, then the union of these towers $\mathcal{T}$ determined by the following:

- for each $\beta \in \bigcup_{j<\gamma} I_j$, $M_\beta := \bigcup_{\beta \in I_j;\, j<\gamma} M^j_\beta$
- the sequence $\langle a_\beta \mid \exists (j < \gamma)\, \beta+1, \beta \in I_j \rangle$, and
- the sequence $\langle N_\beta \mid \exists (j < \gamma)\, \beta+1, \beta \in I_j \rangle$

[3]Importantly, we don't require continuity.

is a tower in $\mathcal{K}^*_{\mu,\bigcup_{j<\gamma} I_j}$, provided that $\mathcal{K}$ satisfies the continuity property for non-μ-splitting and that $\bigcup_{j<\gamma} I_j$ is well ordered. Note that it is our desire to take increasing unions of towers that leads to the necessity of the continuity property.

We also need to recall a few facts about directed systems of partial extensions of towers from [14]. Proposition 8 will get us through the successor step of inductive constructions of directed systems, and Proposition 9 describes how to pass through the limit stages. Note that in neither of these propositions do we require that the towers be continuous.

PROPOSITION 8. *Suppose $\mathcal{T}$ is a tower in $\mathcal{K}^*_{\mu,\alpha}$ and $\mathcal{T}'$ is a tower of length $\beta < \alpha$ with $\mathcal{T} \restriction \beta < \mathcal{T}'$, if $f \in \mathrm{Aut}_{M_\beta}(\mathfrak{C})$ and M''_β is a limit model universal over M_β such that* ga-tp(a_β/M''_β) *does not μ-split over N_β and $f(\bigcup_{i<\beta} M'_i) \prec_{\mathcal{K}} M''$, then the tower $\mathcal{T}'' \in \mathcal{K}^*_{\mu,\beta+1}$ defined by $f(\mathcal{T}')$ concatenated with the model M''_β, element a_β and submodel N_β is an extension of $\mathcal{T} \restriction (\beta+1)$.*

PROPOSITION 9. *Fix $\mathcal{T} \in \mathcal{K}^*_{\mu,\alpha}$ for α a limit ordinal. Suppose $\langle \mathcal{T}^i \in \mathcal{K}^*_{\mu,i} \mid i < \alpha\rangle$ and $\langle f_{i,j} \mid i \leq j < \alpha\rangle$ form a directed system of towers. Suppose*

- *each $\mathcal{T}^i$ extends $\mathcal{T} \restriction i$*
- *$f_{i,j} \restriction M_i = id_{M_i}$*
- *M^{i+1}_{i+1} is universal over $f_{i,i+1}(M^i_i)$.*

Then there exists a direct limit $\mathcal{T}^\alpha$ and mappings $\langle f_{i,\alpha} \mid i < \alpha\rangle$ to this system so that $\mathcal{T}^\alpha \in \mathcal{K}^*_{\mu,\alpha}$, $\mathcal{T}^\alpha$ extends $\mathcal{T}$, and $f_{i,\alpha} \restriction M_i = id_{M_i}$.

Finally, to prove results about the uniqueness of limit models we will additionally need to assume that non-μ-splitting satisfies a symmetry property over limit models. We refine the definition of symmetry Definition 3 of [13] for non-μ-splitting; this localization only requires symmetry to hold when M_0 is (μ,δ)-limit over N.

DEFINITION 10. *Fix $\mu \geq \mathrm{LS}(\mathcal{K})$ and δ a limit ordinal $< \mu^+$. We say that an abstract elementary class exhibits* (μ,δ)-symmetry for non-μ-splitting *if whenever models $M, M_0, N \in \mathcal{K}_\mu$ and elements a and b satisfy the conditions 1-4 below, then there exists M^b a limit model over M_0, containing b, so that* ga-tp(a/M^b) *does not μ-split over N. See Figure 1.*

1. *M is universal over M_0 and M_0 is a (μ,δ)-limit model over N.*
2. *$a \in M \backslash M_0$.*
3. ga-tp(a/M_0) *is non-algebraic and does not μ-split over N.*
4. ga-tp(b/M) *is non-algebraic and does not μ-split over M_0.*

§3. Relatively Full Towers. One approach to proving the uniqueness of limit models is to construct a continuous relatively full tower of length θ, and then conclude that the union of the models in this tower is a (μ,θ)-limit model. In this section we confirm that this approach can be carried out in this context, even if we remove continuity along the relatively full tower.

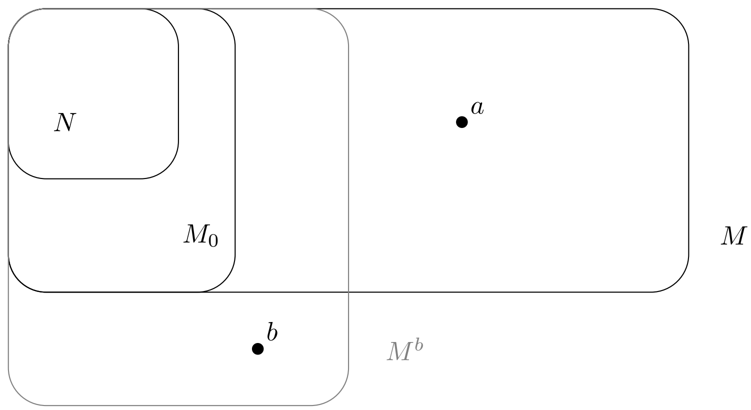

FIGURE 1. A diagram of the models and elements in the definition of (μ, δ)-symmetry. We assume the type ga-tp(b/M) does not μ-split over M_0 and ga-tp(a/M_0) does not μ-split over N. Symmetry implies the existence of M^b a limit model over M_0 so that ga-tp(a/M^b) does not μ-split over N.

DEFINITION 11 (Definition 3.2.1 of [12]). *For M a (μ, θ)-limit model, let*

$$\mathfrak{St}(M) := \left\{ (p, N) \;\middle|\; \begin{array}{l} N \prec_{\mathcal{K}} M; \\ N \textit{ is a } (\mu,\theta)\textit{-limit model}; \\ M \textit{ is universal over } N; \\ p \in \text{ga-S}(M) \textit{ is non-algebraic} \\ \textit{and } p \textit{ does not } \mu\textit{-split over } N. \end{array} \right\}$$

Elements of $\mathfrak{St}(M)$ *are called* strong types. *Two strong types* $(p_1, N_1) \in \mathfrak{St}(M_1)$ *and* $(p_2, N_2) \in \mathfrak{St}(M_2)$ *are* parallel *iff for every M' of cardinality μ extending M_1 and M_2 there exists $q \in$ ga-S(M') such that q extends both p_1 and p_2 and q does not μ-split over N_1 nor over N_2.*

DEFINITION 12 (Relatively Full Towers). *Suppose that I is a well-ordered set. Let $(\bar{M}, \bar{a}, \bar{N})$ be a tower indexed by I such that each M_i is a (μ, σ)-limit model. For each i, let $\langle M_i^\gamma \mid \gamma < \sigma \rangle$ witness that M_i is a (μ, σ)-limit model.*
The tower $(\bar{M}, \bar{a}, \bar{N})$ is full relative to $(M_i^\gamma)_{\gamma<\sigma, i\in I}$ *iff*

1. *there exists a cofinal sequence $\langle i_\alpha \mid \alpha < \theta \rangle$ of I of order type θ such that there are $\mu \cdot \omega$ many elements between i_α and $i_{\alpha+1}$ and*
2. *for every $\gamma < \sigma$ and every $(p, M_i^\gamma) \in \mathfrak{St}(M_i)$ with $i_\alpha \leq i < i_{\alpha+1}$, there exists $j \in I$ with $i \leq j < i_{\alpha+1}$ such that* (ga-tp$(a_j/M_j), N_j$) *and* (p, M_i^γ) *are parallel.*

The following proposition will allow us to use relatively full towers to produce limit models. The fact that relatively full towers yield limit models was first proved in [14] and in [6] and later improved in [4, Proposition 4.1.5]. We notice here that the proof of [4, Proposition 4.1.5] does not require that the tower be continuous and does not require that $\kappa^*_\mu(\mathcal{K}) = \omega$. We provide the proof for completeness.

Proposition 13 (Relatively full towers provide limit models). *Let θ be a limit ordinal $< \mu^+$ satisfying $\theta = \mu \cdot \theta$. Suppose that I is a well-ordered set as in Definition 12.(1).*

Let $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ be a tower made up of (μ,σ)-limit models, for some fixed σ with $\kappa^*_\mu(\mathcal{K}) \leq \mathrm{cf}(\sigma) < \mu^+$. If $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ is full relative to $(M^\gamma_i)_{i\in I,\gamma<\sigma}$, then $M := \bigcup_{i\in I} M_i$ is a (μ,θ)-limit model over M_{i_0}.

Proof. Because the sequence $\langle i_\alpha \mid \alpha < \theta\rangle$ is cofinal in I and $\theta = \mu\cdot\theta$, we can rewrite $M := \bigcup_{i\in I} M_i = \bigcup_{\beta<\theta} M_{i_\beta} = \bigcup_{\alpha<\theta}\bigcup_{\delta<\mu} M_{i_{\mu\alpha+\delta}}$.

For $\alpha < \theta$ and $\delta < \mu$, notice

$$M_{i_{\mu\alpha+\delta+1}} \text{ realizes every type over } M_{i_{\mu\alpha+\delta}}. \tag{1}$$

To see this take $p \in \text{ga-S}(M_{i_{\mu\alpha+\delta}})$. By our assumption that $\mathrm{cf}(\sigma) \geq \kappa^*_\mu(\mathcal{K})$, p does not μ-split over $M^\gamma_{i_{\mu\alpha+\delta}}$ for some $\gamma < \sigma$. Therefore $(p, M^\gamma_{i_{\mu\alpha+\delta}}) \in \mathfrak{St}(M_{i_{\mu\alpha+\delta}})$. By definition of relatively full towers, there is an a_k with $i_{\mu\alpha+\delta} \leq k < i_{\mu\alpha+\delta+1}$ so that $(\text{ga-tp}(a_k/M_k), N_k)$ and $(p, M^\gamma_{i_{\mu\alpha+\delta}})$ are parallel. Because $M_{i_{\mu\alpha+\delta}} \prec_{\mathcal{K}} M_k$ and by the definition of parallel strong types, it must be the case that $a_k \models p$.

By a back and forth argument we can conclude from (1) that $M_{i_{\mu\alpha+\mu}}$ is universal over $M_{i_{\mu\alpha}}$. Thus M is a (μ,θ)-limit model.

To see the details of the back-and-forth argument mentioned in the previous paragraph, first translate (1) to the terminology of [1]: (1) witnesses that $\bigcup_{\beta<\mu} M_{i_{\mu\alpha+\beta}}$ is 1-special over $M_{i_{\mu\alpha}}$. Then, refer to the proof of Lemma 10.5 of [1].

$\dashv$

§4. Reduced Towers. The proof of the uniqueness of limit models from [6], [14], and [15] is two dimensional. The relatively full towers are used to produce a (μ,θ)-limit model, but to conclude that this model is also a (μ,ω)-limit model, a $<$-increasing chain of ω-many continuous towers of length $\theta+1$ is constructed. We adapt this construction to prove Theorem 20. Instead of creating a chain of ω-many towers, we produce a chain of δ-many towers, and instead of each tower in this chain being continuous, we only require that these towers are continuous at limit ordinals of cofinality at least $\kappa^*_\mu(\mathcal{K})$.

The use of towers should be compared with the proof uniqueness of limit models in [11, Section II.4] (details are given in [2, Section 9]). Both proofs create a 'square' of models, but do so in a different way. The proof here will proceed by starting with a 1-dimensional tower of models and then, in the induction step, extend this tower to fill out the square. In contrast, the induction step of [11, Lemma II.4.8] adds single models at a time. This seems like a minor distinction (or even just a difference in how the induction step is carried out), but there is a real distinction in the resulting squares. In [11], the construction is 'symmetric' in the sense that θ and δ are treated the same. However, in the proof presented here, this symmetry is broken and one could 'detect' which side of the square was laid out initially as the tower by observing where continuity fails.

In [6], [14], [15], and [13] continuity of the towers is achieved by restricting the construction to reduced towers, which under the stronger assumptions of [6],

[14], [15], and [13] are shown to be continuous. We take this approach and notice that continuity of reduced towers at certain limit ordinals can be obtained with the weaker assumptions of Theorem 20, in particular $\kappa^*_\mu(\mathcal{K}) < \mu^+$.

Definition 14. *A tower $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ is said to be* reduced *provided that for every $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ with $(\bar{M}, \bar{a}, \bar{N}) \leq (\bar{M}', \bar{a}, \bar{N})$ we have that for every $i < \alpha$,*

$$(*)_i \quad M'_i \cap \bigcup_{j<\alpha} M_j = M_i.$$

The proofs of the following three results about reduced towers only require that the class $\mathcal{K}$ be stable in μ and that μ-splitting satisfies the continuity property. None of these results uses the assumption that $\kappa^*_\mu(\mathcal{K}) = \omega$.

Theorem 15 (Theorem 3.1.13 of [12]). *There exists a reduced $<$-extension of every tower in $\mathcal{K}^*_{\mu,\alpha}$.*

Theorem 16 (Theorem 3.1.14 of [12]). *Let $\langle(\bar{M}, \bar{a}, \bar{N})^\gamma \in \mathcal{K}^*_{\mu,\alpha} \mid \gamma < \beta\rangle$ be a $<$-increasing and continuous sequence of reduced towers such that the sequence is continuous in the sense that for a limit $\gamma < \beta$, the tower $(\bar{M}, \bar{a}, \bar{N})^\gamma$ is the union of the towers $(\bar{M}, \bar{a}, \bar{N})^\zeta$ for $\zeta < \gamma$. Then the union of the sequence of towers $\langle(\bar{M}, \bar{a}, \bar{N})^\gamma \in \mathcal{K}^*_{\mu,\alpha} \mid \gamma < \beta\rangle$ is itself a reduced tower.*

In fact the proof of Theorem 16 gives a slightly stronger result which allows us to take the union of an increasing chain of reduced towers of increasing index sets and conclude that the union is still reduced.

Lemma 17 (Lemma 5.7 of [6]). *Suppose that $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ is reduced. If $\beta < \alpha$, then $(\bar{M}, \bar{a}, \bar{N}) \restriction \beta$ is reduced.*

The following theorem is related to [13, Theorem 5]. Here we weaken the assumption that $\kappa^*_\mu(\mathcal{K}) = \omega$ and we get a sightly weaker conclusion. The proof is similar to the proof of [13, Theorem 5] only here we allow for our towers to be discontinuous at γ where $\mathrm{cf}(\gamma) < \kappa^*_\mu(\mathcal{K})$. We provide the details where the proof differs.

Theorem 18. *Suppose $\mathcal{K}$ satisfies Assumption 5. Let α be an ordinal and δ a limit ordinal so that $\kappa^*_\mu(\mathcal{K}) \leq \mathrm{cf}(\delta) < \alpha$. If $\mathcal{K}$ satisfies (μ, δ)-symmetry for non-μ-splitting and $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ is reduced, then the tower $(\bar{M}, \bar{a}, \bar{N})$ is continuous at δ (i.e., $M_\delta = \bigcup_{\beta<\delta} M_\beta$).*

Proof. Suppose for that sake of contradiction that reduced towers are not necessarily continuous at a δ with $\mathrm{cf}(\delta) \geq \kappa^*_\mu(\mathcal{K})$. Let δ be the minimal ordinal such that a counterexample discontinuous at δ exists. Let α be the minimal ordinal $\geq \delta$ so that there exists $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ a reduced tower of minimal length, which is discontinuous at δ. Notice that by Lemma 17, we can conclude that $\alpha = \delta + 1$. Let $b \in M_\delta \setminus \bigcup_{i<\delta} M_i$ witness the discontinuity of the tower at δ.

By Theorem 15 and Theorem 16, we can construct a $<$-increasing and continuous chain of (not necessarily continuous) reduced towers $\langle\mathcal{T}^i = (\bar{M}, \bar{a}, \bar{N})^i \in \mathcal{K}^*_{\mu,\delta} \mid i < \delta\rangle$ with $(\bar{M}, \bar{a}, \bar{N})^0 := (\bar{M}, \bar{a}, \bar{N}) \restriction \delta$. By δ-applications of Theorem 15 in between successor stages of the construction we can require that for $\beta < \delta$

$$M^{i+1}_\beta \text{ is a } (\mu,\delta)\text{-limit over } M^i_\beta \quad (2)$$
$$\text{and consequently } M^{i+1}_\beta \text{ is a } (\mu,\delta)\text{-limit over } N_\beta.$$

Let $M^\delta_\delta := \bigcup_{i<\delta,\ \beta<\delta} M^i_\beta$. See Figure 2.

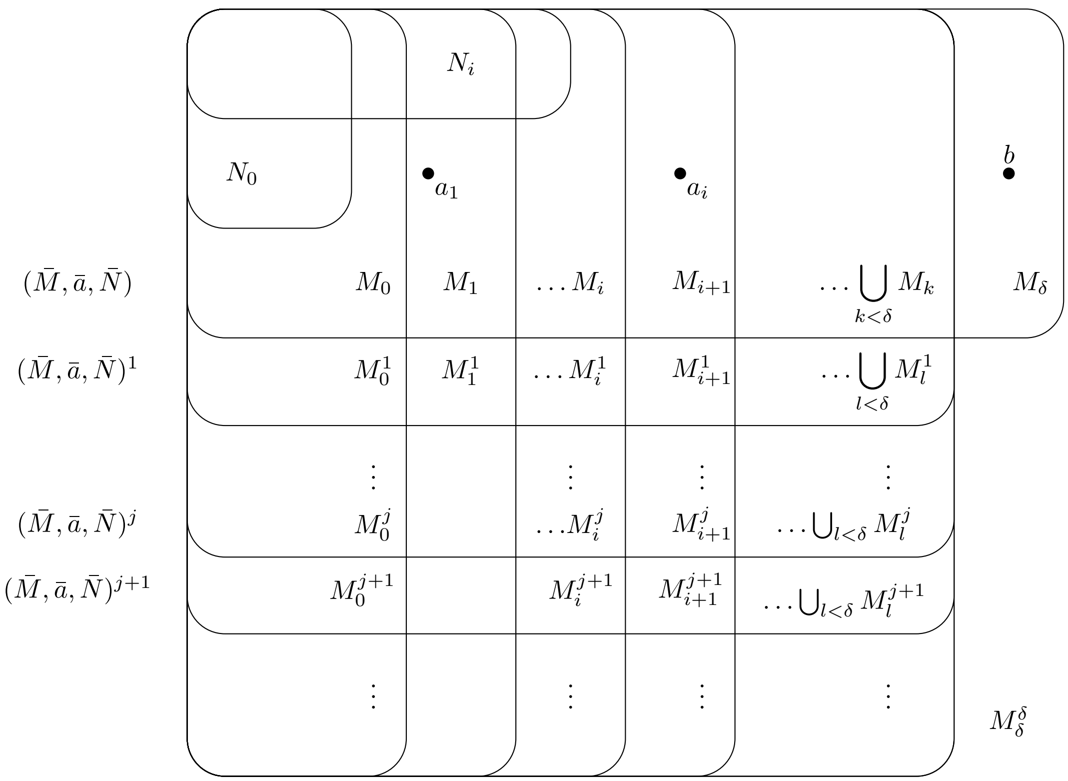

FIGURE 2. $(\bar{M},\bar{a},\bar{N})$ and the towers $(\bar{M},\bar{a},\bar{N})^j$ extending $(\bar{M},\bar{a},\bar{N}) \restriction \delta$.

There are two cases: 1) we have $b \in M^\delta_\delta$ and 2) we have $b \notin M^\delta_\delta$. If $b \in M^\delta_\delta$, then we will have found an extension of $(\bar{M},\bar{a},\bar{N}) \restriction \delta$ containing b (namely $(\bar{M},\bar{a},\bar{N})^\delta$)) which can easily be lengthened to a discontinuous extension of the entire $(\bar{M},\bar{a},\bar{N})$ tower by taking the δ^{th} model to be some extension of M^δ_δ which is also universal over M_δ. This discontinuous extension of $(\bar{M},\bar{a},\bar{N})$ along with b witness that $(\bar{M},\bar{a},\bar{N})$ cannot be reduced.

So suppose that $b \notin M^\delta_\delta$. Then ga-tp(b/M^δ_δ) is non-algebraic. Consider the sequence $\langle \check{M}_i \mid i < \delta\rangle$ defined by $\check{M}_i := M^i_i$ if i is a successor and $\check{M}_i := \bigcup_{j<i} M^j_j$ for i a limit ordinal. Notice that (2) implies that this sequence witnesses that M^δ_δ is a (μ,δ)-limit model. Because M^δ_δ is a (μ,δ)-limit model, by our assumption that $\mathrm{cf}(\delta) \geq \kappa^*_\mu(\mathcal{K})$ and monotonicity of non-splitting, there exists a successor

ordinal $i^* < \delta$ so that

$$\text{ga-tp}(b/M^\delta_\delta) \text{ does not } \mu\text{-split over } M^{i^*}_{i^*}. \tag{3}$$

Our next step is to consider the tower formed by the diagonal elements in Figure 2. In particular let $\mathcal{T}^{diag}$ be the tower in $\mathcal{K}^*_{\mu,\delta}$ extending $\mathcal{T} \restriction \delta$ whose models are M^i_i for each $i < \delta$.

By (2), $M^{i^*}_{i^*}$ is a (μ, δ)-limit over N_{i^*}. Now, referring to the Figure 1, apply (μ, δ)-symmetry to a_{i^*} standing in for a, $M^{i^*}_{i^*}$ representing M_0, N_{i^*} as N, M^δ_δ as M, and b as itself. We can conclude that there exists M^b containing b, a limit model over $M^{i^*}_{i^*}$, for which ga-tp(a_{i^*}/M^b) does not μ-split over N_{i^*}.

Define the tower $\mathcal{T}^b \in \mathcal{K}^*_{\mu,i^*+2}$ by the sequences $\bar{a} \restriction (i^*+1)$, $\bar{N} \restriction (i^*+1)$ and $\bar{M}'$ with $M'_j := M^j_j$ for $j \leq i^*$ and $M'_{i^*+1} := M^b$. Notice that $\mathcal{T}^b$ is an extension of $\mathcal{T}^{diag} \restriction (i^*+2)$ containing b. We will explain how we can use this tower to find a tower $\mathring{\mathcal{T}}^\delta \in \mathcal{K}^*_{\mu,\delta}$ extending $\mathcal{T}^{diag}$ with $b \in \bigcup_{j<\delta} \mathring{M}^\delta_j$. This will be enough to contradict our assumption that $\mathcal{T}$ was reduced.

We define $\langle \mathring{\mathcal{T}}^j, f_{j,k} \mid i^*+2 \leq j \leq k \leq \delta \rangle$ a directed system of towers so that for $j \geq i^*+2$

1. $\mathring{\mathcal{T}}^{i^*+2} = \mathcal{T}^b$
2. $\mathring{\mathcal{T}}^j \in \mathcal{K}^*_{\mu,j}$ for $j \leq \delta$
3. $\mathcal{T}^{diag} \restriction j \leq \mathring{\mathcal{T}}^j$ for $j \leq \delta$
4. $f_{j,k}(\mathring{\mathcal{T}}^j) \leq \mathring{\mathcal{T}}^k \restriction j$ for $j \leq k < \delta$
5. $f_{j,k} \restriction M^j_j = id_{M^j_j}$ $j \leq k < \delta$
6. $\mathring{M}^{j+1}_{j+1}$ is universal over $f_{j,j+1}(\mathring{M}^j_j)$ for $j < \delta$
7. $b \in \mathring{M}^j_{i^*+1}$ for $j \leq \delta$
8. ga-tp$(f_{j,k}(b)/M^k_k)$ does not μ-split over $M^{i^*}_{i^*}$ for $j < k < \delta$.

We will define this directed system by induction on k, with $i^*+2 \leq k \leq \alpha$. The base and successor case are exactly as in the proof of Theorem 5 of [13]. The only difference in the construction here is at limit stages in which $\mathcal{T}^{diag}$ is not continuous. Therefore we will concentrate on the details of the construction for stage k and $k+1$ where $k < \delta$ is a limit ordinal for which $\mathcal{T}^{diag}$ is discontinuous at k.

Case 1: k is limit where $\mathcal{T}^{diag}$ is discontinuous.
First, let $\grave{\mathcal{T}}^k$ and $\langle \grave{f}_{j,k} \mid i^*+2 \leq j < k \rangle$ be a direct limit of the system defined so far. We use the ` notation since these are only approximations to the tower and mappings that we are looking for. We will have to take some care to find a direct limit that contains b in order to satisfy Condition 7 of the construction. By Proposition 9 and our induction hypothesis, we may choose this direct limit so that for all $j < k$

$$\grave{f}_{j,k} \restriction M^j_j = id_{M^j_j}.$$

Consequently $\grave{M}^\alpha_j := \grave{f}_{j,k}(\mathring{M}^j_j)$ is universal over M^j_j, and $\bigcup_{j<k} \grave{M}^k_j$ is a limit model witnessed by Condition 6 of the construction. Additionally, the tower $\grave{\mathcal{T}}^k$ composed of the models $\grave{M}^k_j$, extends $\mathcal{T}^{diag} \restriction k$.

We will next show that for every $j < k$,

$$\text{ga-tp}(\grave{f}_{i^*+2,k}(b)/M^j_j) \text{ does not } \mu\text{-split over } M^{i^*}_{i^*}. \tag{4}$$

To see this, recall that for every $j < k$, by the definition of a direct limit, $\grave{f}_{i^*+2,k}(b) = \grave{f}_{j,k}(f_{i^*+2,j}(b))$. By Condition 8 of the construction, we know

$$\text{ga-tp}(f_{i^*+2,j}(b)/M^j_j) \text{ does not } \mu\text{-split over } M^{i^*}_{i^*}.$$

Applying $\grave{f}_{j,k}$ to this implies ga-tp$(\grave{f}_{i^*+2,k}(b)/M^j_j)$ does not μ-split over $M^{i^*}_{i^*}$, establishing (4).

Because M^{j+1}_{j+1} is universal over M^j_j by construction, we can apply the continuity of non-splitting to (4), yielding

$$\text{ga-tp}(\grave{f}_{i^*+2,k}(b)/\bigcup_{j<k} M^j_j) \text{ does not } \mu\text{-split over } M^{i^*}_{i^*}. \tag{5}$$

Because $\grave{f}_{i^*+2,k}$ fixes $M^{i^*+1}_{i^*+1}$, ga-tp$(b/M^{i^*+1}_{i^*+1}) =$ ga-tp$(\grave{f}_{i^*+2,k}(b)/M^{i^*+1}_{i^*+1})$. We can then apply the uniqueness of non-splitting extensions (see [14, Theorem I.4.12]) to (5) to see that ga-tp$(\grave{f}_{i^*+2,k}(b)/\bigcup_{j<k} M^j_j) =$ ga-tp$(b/\bigcup_{j<k} M^j_j)$. Thus we can fix g an automorphism of the monster model fixing $\bigcup_{j<k} M^j_j$ so that $g(\grave{f}_{i^*+2,k}(b)) = b$.

We will then define $\mathring{\mathcal{T}}^k$ to be the tower $g(\grave{\mathcal{T}}^k)$, and the mappings for our directed system will be $f_{j,k} := g \circ \grave{f}_{j,k}$ for all $i^* + 2 \leq j < k$. Notice that by our induction hypothesis we have that $b \in \mathring{M}^{i^*+2}_{i^*+1}$. Then, by definition of a direct limit we have $\grave{f}_{i^*+2,k}(b) \in \grave{M}^k_{i^*+1}$. Therefore $g(\grave{f}_{i^*+2,k}(b)) = b \in \mathring{M}^k_{i^*+1}$, satisfying condition 7 of the construction. Furthermore for all $j < k$, we have that $f_{j,k}(b) = b$. Therefore by (3) and monotonicity of non-splitting, condition 8 of the construction holds.

Notice that $\mathcal{T}^{diag}$ being discontinuous at k does not impact this stage of the construction since we only require that $\mathring{\mathcal{T}}^k$ be a tower of length k and therefore $\mathring{\mathcal{T}}^k$ need not contain models extending M^k_k. The discontinuity plays a role at the next stage of the construction.

Case 2: $k+1$ is successor of limit where $\mathcal{T}^{diag}$ is discontinuous.
Suppose that $\mathcal{T}^{diag}$ is discontinuous at k and that $\mathring{\mathcal{T}}^k \in \mathcal{K}^*_{\mu,k}$ has been defined.

By our choice of i^*, we have ga-tp$(b/\bigcup_{l<\alpha} M^l_l)$ does not μ-split over $M^{i^*}_{i^*}$. So in particular by monotonicity of non-splitting, we notice:

$$\text{ga-tp}(b/M^{k+1}_k) \text{ does not } \mu\text{-split over } M^{i^*}_{i^*}. \tag{6}$$

Using the definition of towers (i.e. M^{k+1}_k is a (μ, δ)-limit over N_k and ga-tp(a_k/M^{k+1}_k) does not μ-split over N_k) and the choice of i^*, we can apply (μ, δ)-symmetry to a_k, M^{k+1}_k, $\bigcup_{l<\delta} M^l_l$, b and N_k which will yield M^b_k a limit model over M^{k+1}_k containing b so that ga-tp(a_k/M^b_k) does not μ-split over N_k (see Figure 3).

Notice that M^b_k has no relationship to $\mathring{\mathcal{T}}^k$. In particular, it does not contain $\bigcup_{l<k} \mathring{M}^l_l$. Fix M' to be a model of cardinality μ extending both $\bigcup_{l<k} \mathring{M}^l_l$ and M^{k+1}_k. Since M^b_k is a limit model over M^{k+1}_k which is a limit model over M^k_k, there exits $f : M' \to M^{k+1}_k$ with $f = id_{M^k_k}$ so that M^b_k is also universal over $f(\bigcup_{l<k} \mathring{M}^l_l)$. Because ga-tp$(b/M^k_k)$ does not μ-split over $M^{i^*}_{i^*}$ and f fixes M^k_k,

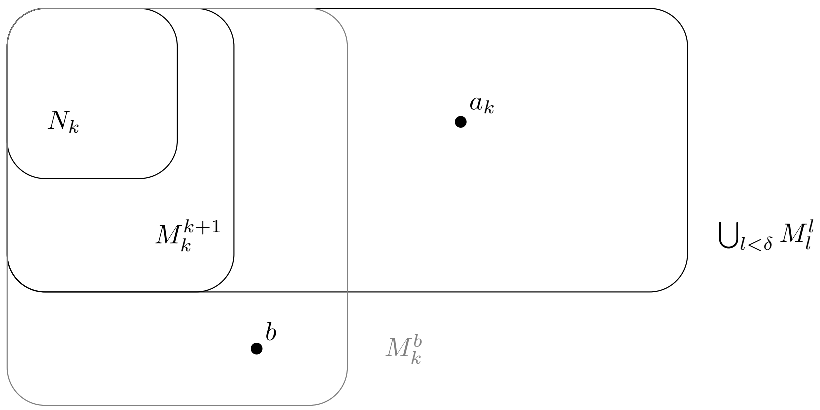

FIGURE 3. A diagram of the application of (μ, δ)-symmetry in the successor stage of the directed system construction in the proof of Theorem 18. We have ga-tp$(b/\bigcup_{l<\delta} M_l^l)$ does not μ-split over M_k^{k+1} and ga-tp(a_k/M_k^{k+1}) does not μ-split over N_k. Symmetry implies the existence of M_k^b a limit model over M_k^{k+1}. so that ga-tp(a_k/M^b) does not μ-split over N_k.

we know that ga-tp$(f(b)/M_k^k)$ does not μ-split over $M_{i^*}^{i^*}$. But because $f(b)$ and b both realize the same types over $M_{i^*+1}^{i^*+1}$, we can conclude by the uniqueness of non-splitting extensions that ga-tp$(f(b)/M_k^k) =$ ga-tp(b/M_k^k); so there is $g \in \mathrm{Aut}_{M_k^k}(\mathfrak{C})$ with $g(f(b)) = b$. Since M_k^b is universal over M_k^k and $b \in M_k^b$, we can choose g so that $g(f(M')) \prec_{\mathcal{K}} M_k^b$.

Take $\mathring{M}_k^{k+1}$ to be an extension of M_k^b which is also universal over M_{k+1}^{k+1}, and set $f_{k,k+1} := g \circ f$. To see that Condition 8 of the construction holds, just apply monotonicity and the fact that $f_{k,k+1}(b) = b$ to (3). See figure 4.

It is easy to check by invariance and the induction hypothesis that $\mathring{\mathcal{T}}^{k+1}$ defined by the models $\mathring{M}_l^{k+1} := f_{k,k+1}(\mathring{M}_l^k)$ for $l < k$ satisfies the remaining requirements on $\mathring{\mathcal{T}}^{k+1}$. Then the rest of the directed system can be defined by the induction hypothesis and the mappings $f_{l,k+1} := f_{l,k} \circ f_{k,k+1}$ for $i^* + 2 \leq l < k$.

This completes the construction.

Now that we have $\mathring{\mathcal{T}}^\delta$ a tower extending $\mathcal{T} \restriction \delta$ which contains b, we are in a situation similar to the proof in case 1). To contradict that $\mathcal{T}$ is reduced, we need only lengthen $\mathring{\mathcal{T}}^\delta$ to a discontinuous extension of the entire $(\bar{M}, \bar{a}, \bar{N})$ tower by taking the δ^{th} model to be some extension of $\bigcup_{i<\delta} \mathring{M}_i^i$ which is also universal over M_δ. This discontinuous extension of $(\bar{M}, \bar{a}, \bar{N})$ along with $b \in \mathring{M}_{i^*+1}^\delta$ witness that $(\bar{M}, \bar{a}, \bar{N})$ cannot be reduced.

$\dashv$

Although not used here, the converse of this theorem is also true, as in [13]. Note that the following does not use local character at all.

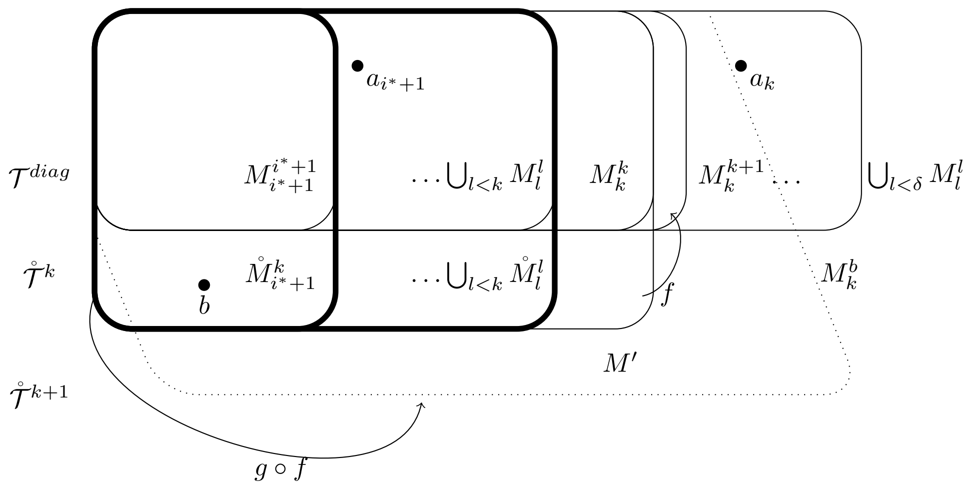

FIGURE 4. The construction of $\mathring{\mathcal{T}}^{k+1}$(dotted) from $\mathring{\mathcal{T}}^k$ (bold) with $g \circ f$ fixing M^k_k and b.

PROPOSITION 19. *Suppose $\mathcal{K}$ satisfies Assumption 5.(1), (2), and (4). Suppose further that that, for every reduced tower $(\bar{M}, \bar{a}, \bar{M}) \in \mathcal{K}^*_{\mu,\alpha}$, $\bar{M}$ is continuous at limit ordinals of cofinality δ. Then $\mathcal{K}$ satisfies (μ,δ)-symmetry for non μ-splitting.*

PROOF. The proof is an easy adaptation of [13, Theorem 5.$(b) \to (a)$]. The same argument works; the only adaptation is to require every limit model to in fact be a (μ,δ) limit model that the tower $\mathcal{T}$ be of length $\delta + 1$[4]. ⊣

§5. Uniqueness of Long Limit Models. We now begin the proof Theorem 20, which we restate here.

THEOREM 20. *Suppose that $\mathcal{K}$ is an abstract elementary class satisfying*

1. *the joint embedding and amalgamation properties with no maximal model of cardinality μ.*
2. *stabilty in μ.*
3. *$\kappa^*_\mu(\mathcal{K}) < \mu^+$.*
4. *continuity for non-μ-splitting (i.e. if $p \in \text{ga-S}(M)$ and M is a limit model witnessed by $\langle M_i \mid i < \alpha\rangle$ for some limit ordinal $\alpha < \mu^+$ and there exists N so that $p \restriction M_i$ does not μ-split over N for all $i < \alpha$, then p does not μ-split over N).*

For θ and δ limit ordinals $< \mu^+$ both with cofinality $\geq \kappa^*_\mu(\mathcal{K})$, if $\mathcal{K}$ satisfies symmetry for non-μ-splitting (or just (μ,δ)-symmetry), then, for any M_1 and M_2 that are (μ,θ) and (μ,δ)-limit models over M_0, respectively, we have that M_1 and M_2 are isomorphic over M_0.

[4]In a happy coincidence, the notation in that proof already agrees with this change.

The structure of the proof of Theorem 20 from this point on is similar to the proof in [6]. For completeness we include the details here, and emphasize the points of departure from [6].

We construct an array of models which will produce a model that is both a (μ,θ)- and a (μ,δ)-limit model. Let θ be an ordinal as in the definition of relatively full tower so that $\mathrm{cf}(\theta) \geq \kappa^*_\mu(\mathcal{K})$ and let $\delta = \kappa^*_\mu(\mathcal{K})$. The goal is to build an array of models with $\delta+1$ rows so that the bottom row of the array is a relatively full tower indexed by a set of cofinality $\theta+1$ continuous at θ. To do this, we will be adding elements to the index set of towers row by row so that at stage n of our construction the tower that we build is indexed by I_n described here.

The index sets I_β will be defined inductively so that $\langle I_\beta \mid \beta < \delta+1\rangle$ is an increasing and continuous chain of well-ordered sets. We fix I_0 to be an index set of order type $\theta+1$ and will denote it by $\langle i_\alpha \mid \alpha \leq \theta\rangle$. We will refer to the members of I_0 by name in many stages of the construction. These indices serve as anchors for the members of the remaining index sets in the array. Next we demand that for each $\beta < \delta$, $\{j \in I_\beta \mid i_\alpha < j < i_{\alpha+1}\}$ has order type $\mu\cdot\beta$ such that each I_β has supremum i_θ. An example of such $\langle I_\beta \mid \beta \leq \delta\rangle$ is $I_\beta = \theta \times (\mu\cdot\beta)\bigcup\{i_\theta\}$ ordered lexicographically, where i_θ is an element $\geq$ each $i \in \bigcup_{\beta<\delta} I_\beta$. Also, let $I = \bigcup_{\beta<\delta} I_\beta$.

To prove the main theorem of the paper, we need to prove that, for a fixed $M \in \mathcal{K}$ of cardinality μ, any (μ,θ)-limit and (μ,δ)-limit model over M are isomorphic over M. Since all (μ,θ)-limits over M are isomorphic over M (and the same holds for (μ,δ)-limits), it is enough to construct a single model that is simultaneously (μ,θ)-limit and (μ,δ)-limit over M. Let us begin by fixing a limit model $M \in \mathcal{K}_\mu$. We define, by induction on $\beta \leq \delta$, a $<$-increasing and continuous sequence of towers $(\bar{M},\bar{a},\bar{N})^\beta$ such that

1. $\mathcal{T}^0 := (\bar{M},\bar{a},\bar{N})^0$ is a tower with $M^0_0 = M$.
2. $\mathcal{T}^\beta := (\bar{M},\bar{a},\bar{N})^\beta \in \mathcal{K}^*_{\mu,I_\beta}$.
3. For every $(p,N) \in \mathfrak{St}(M^\beta_i)$ with $i_\alpha \leq i < i_{\alpha+1}$ there is $j \in I_{\beta+1}$ with $i_\alpha < j < i_{\alpha+1}$ so that $(\text{ga-tp}(a_j/M^{\beta+1}_j), N^{\beta+1}_j)$ and (p,N) are parallel.

See Figure 5.

Given M, we can find a tower $(\bar{M},\bar{a},\bar{N})^0 \in \mathcal{K}^*_{\mu,I_0}$ with $M \preceq_{\mathcal{K}} M^0_0$ because of the existence of universal extensions and because $\kappa^*_\mu(\mathcal{K}) < \mu^+$. At successor stages we first take an extension of $(\bar{M},\bar{a},\bar{N})^\beta$ indexed by $I_{\beta+1}$ and realizing all the strong types over the models in $(\bar{M},\bar{a},\bar{N})^\beta$. This tower may not be reduced, but by Theorem 15, it has a reduced extension. At limit stages take unions of the chain of towers defined so far.

Notice that by Theorem 16, the tower $\mathcal{T}^\delta$ formed by the union of all the $(\bar{M},\bar{a},\bar{N})^\beta$ is reduced. Furthermore, by Theorem 18 every one of the reduced towers $\mathcal{T}^j$ is continuous at θ because $\mathrm{cf}(\theta) \geq \kappa^*_\mu(\mathcal{K})$. Therefore $M^\delta_{i_\theta} = \bigcup_{k<\theta} M^\delta_{i_k}$, and by the definition of the ordering $<$ on towers, the last model in this tower $(M^\delta_{i_\theta})$ is a (μ,δ)-limit model witnessed by $\langle M^j_{i_\theta} \mid j < \delta\rangle$. Since $M^1_{i_\theta}$ is universal over M, we have that $M^\delta_{i_\theta}$ is (μ,δ)-limit over M.

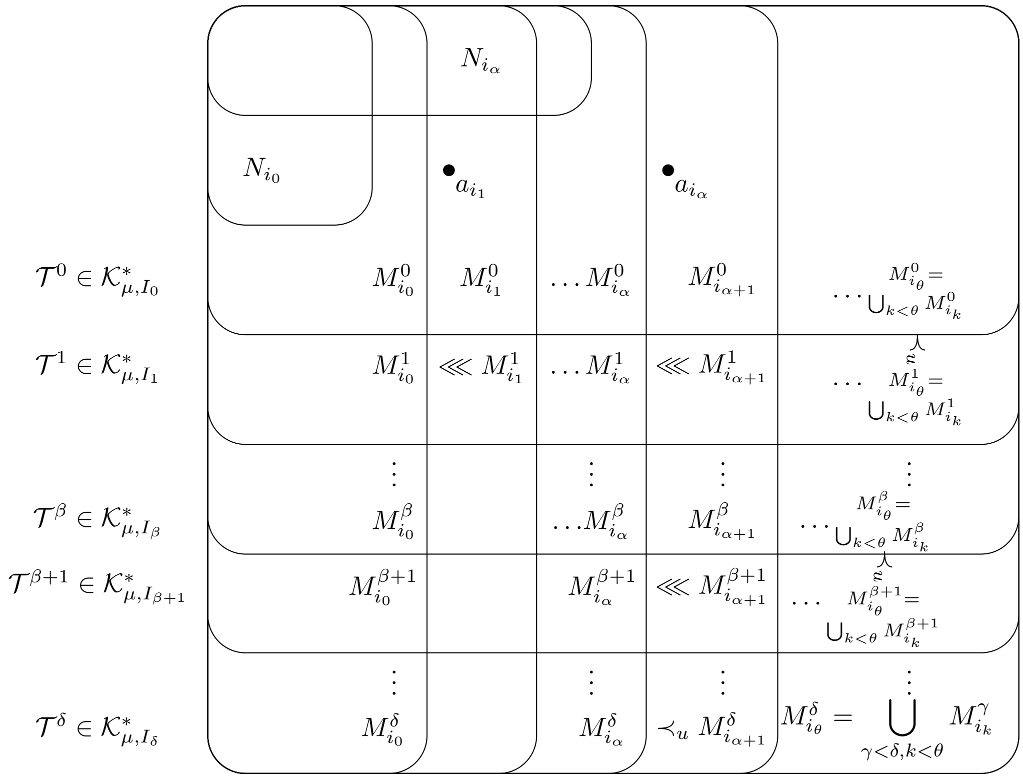

FIGURE 5. The chain of length δ of towers of increasing index sets I_j of cofinality $\theta + 1$. The symbol $\lll$ indicates that there are μ many new indices between i_β and $i_{\beta+1}$ in $I_{j+1} \backslash I_j$. The elements indexed by these indices realize all the strong types over the model $M^j_{i_\alpha}$. The notation $\prec_u$ is an abbreviation for a universal extension.

Next to see that $M^\delta_{i_\theta}$ is also a (μ,θ)-limit model, notice that $\mathcal{T}^\delta$ is relatively full by condition 3 of the construction and the same argument as Claim 5.11 of [6]. Therefore by Theorem 18 and our choice of δ with $\mathrm{cf}(\delta) \geq \kappa^*_\mu(\mathcal{K})$, the last model $M^\delta_{i_\theta}$ in this relatively full tower is a (μ,θ)-limit model over M.

This completes the proof of Theorem 20.

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