ZILBER'S PSEUDO-EXPONENTIAL FIELDS

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The goal of this talk is to prove the following facts above Zilber's pseudoexponential fields

(1) they are axiomatized in $L_{\omega_1,\omega}(Q)$ and this logic is essential.
(2) they form a quasiminimal class
(3) $\mathbb{C} \equiv \mathbb{B}$ implies $\mathbb{C} \cong \mathbb{B}$

The results and presentation are drawn from Kirby "A note on the axioms for Zilber's Pseudo-Exponential Fields" [Kir] and Bays and Kirby "Excellence and uncountable categoricity of Zilber's Exponential Fields" [BaKi].

The class of pseudo-exponential fields is denoted $\mathcal{EC}^*_{st,ccp}$ and is the collection of structures $\langle F; +, \cdot, \exp\rangle$ satisfying the following axioms:

(1) **ELA-field**: F models ACF_0 and $\exp : F \to F^{\cdot}$ is a surjective homomorphism
(2) **Standard kernel**: $\ker \exp$ is an infinite cyclic group generated by a transcendental element $2\pi i$.
(3) **Schanuel property**: For all $\mathbf{x} \in F$,
$$\delta(\mathbf{x}) := td(\mathbf{x}, \exp(\mathbf{x})) - ldim_{\mathbb{Q}}(\mathbf{x}) \geq 0$$
(4) **Strong exponential-algebraic closedness**: Every system of exponential polynomials has a solution, unless this would violate Schanuel.
(5) **Countable Closure Property**: ecl^F is a pregeometry such that, if $C \subset F$ is finite, then $ecl^F(C)$ is countable.

The fourth axiom is imprecisely stated, but we will state it more precisely. We will change the language (essentially but conservatively) to show this is a quasiminimal class.

First, we define some useful notions.

$$\delta(\mathbf{x}/A) := td(\mathbf{x}, \exp \mathbf{x}, A, \exp A/A, \exp A) - ldim_{\mathbb{Q}}(\mathbf{x}/A)$$

Second, we discuss the axioms.

(1) **ELA-field**: Clear.

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(2) **Standard kernel**: This is $L_{\omega_1,\omega}$ expressible. The important bit is that this defines the integers:

$$Z(F) := \{r \in F : \forall x[x \in \ker \exp \to rx \in \ker \exp]\}$$

An alternate approach would to just say that $Z(F)$ is standard (using $L_{\omega_1,\omega}$); then the standard kernel is first-order.

(3) **Schanuel property**: This is expressible as a first order scheme, modulo the integers being standard: for each variety $V \subset \mathbb{G}_a^n \times \mathbb{G}_m^n$ defined over $\mathbb{Q}$ of dimension $n-1$,

$$\forall \mathbf{x} \exists \mathbf{m} \in Z(F) - \{0\} \left((\mathbf{x}, \exp(\mathbf{x})) \in V \to \sum_{i=1}^{n} m_i x_i = 0 \right)$$

Essentially, this says that any tuple with $td(\mathbf{x}, \exp(\mathbf{x})) < n$ has some linear dependence.

This property is a big part of the interest in Zilber's work. Schanuel's Conjecture says that $\mathbb{C}$ has satisfies this and, not to understate it's importance, Wikipedia[1] says something like proving Schanuel's Conjecture "would generalize most known results in transcendental number theory."

(4) **Strong exponential-algebraic closedness**: Before we state this formally, we need some definitions. Write G for $\mathbb{G}_a \times \mathbb{G}_m$.

Given a matrix $M \in Mat_{n\times n}(\mathbb{Z})$, this acts on G^n by being an additive map on $\mathbb{G}_a^n$ and a multiplicative map on $\mathbb{G}_m^n$. Write $G \cdot V$ for the image of $V \subset G^n$ under M. An irreducible variety $V \subset G^n$ is *rotund* iff for every $M \in Mat_{n\times n}(\mathbb{Z})$, $dim M \cdot V \geq rk M$.

Let $(\mathbf{x}, \mathbf{y})$ be a generic point of V over F. V is *additively free* iff the x_i don't satisfy

$$\sum_{i<n} m_i x_i \in F$$

for any $m_i \in \mathbb{Z}$, not all 0.

Similarly, V is *multiplicatively free* iff the y_i don't satisfy

$$\Pi_{i<n} y_i^{m_i} \in F$$

for any $m_i \in \mathbb{Z}$, not all 0.

The property is:

If V is a rotund, additively and multiplicatively free subvariety of $\mathbb{G}_a^n \times \mathbb{G}_m^n$ defined over F and of dimension n and $\mathbf{a}$ is a finite tuple from F, then there is $\mathbf{x} \in F$ such that $(\mathbf{x}, \exp \mathbf{x}) \in V$ is generic in V over $\mathbf{a}$.

So if there are some exponential polynomials that are nice enough, then there is a root.

[1]http://en.wikipedia.org/wiki/Schanuel's_conjecture#Consequences

Proposition 0.1 ([Kir]). *This is first-order expressible modulo the previous axioms.*

Proof: The key is that we have $\mathbb{Z}$ already definable. Suppose we have a parametric family of varieties $(V_p)_{p\in P}$ from G^n, where P is the parameterizing variety.
It is known (fibre dimension theory) that $\{p \in P : V_p$ is irreducible of dimension $n\}$ is first order definable.
With $\mathbb{Z}^2$, we can define additive freeness:

$$\forall \mathbf{m} \in \mathbb{Z} - \{0\} \forall z \exists \mathbf{x} \left(\mathbf{x} \in V_p \wedge \sum_{i<n} m_i x_i \neq z \right)$$

This gives additive freeness: the generic over F satisfies all equations over F that every member of V_p does, and this says that, for each element of F, there is some tuple that doesn't linearly combine to it.
Similarly, one can get rotundness and multiplicative freeness[3]. Alternatively, we can appeal to [Zil05, Theorem 3.2] says that these are first-order definable in the field language.

Given a parametric family of varieties $(V_p)_{p\in P}$ from G^n, where P is the parameterizing variety, let $P' \subset P$ be the subset so V_p is irreducible, of dimension n, rotund, and additively and multiplicatively free. Then we add the following scheme

$$\forall p \in P' \forall \mathbf{a} \in F^r \exists \mathbf{x} \in F^n \forall \mathbf{m} \in \mathbb{Q}$$
$$\left[(\mathbf{x}, \exp \mathbf{x}) \in V_p \wedge \left(\sum_{i<n} m_i x_i + \sum_{i<r} m_{n+i} a_i = 0 \to \wedge_{i<n} m_i = 0 \right) \right]$$

In a sense, this says that $span\{\mathbf{x}\} \cap span\{\mathbf{a}\} = \{0\}$ and $\mathbf{x}$ is linearly independent.
First, note that this follows from axiom 4: Given such a V_p, there is $(\mathbf{x}, \exp \mathbf{x}) \in V_p$ which is generic over $\mathbf{a}$. By additive freeness, this means that $\mathbf{x}$ does not $\mathbb{Q}$-linearly combine to an element of F, which this gives.
Second, suppose this holds. Let V be a rotund, additively and multiplicatively free subvariety of $\mathbb{G}_a^n \times \mathbb{G}_m^n$ defined over F and of dimension n. Then it is V_p in some parametric family[4]. Let $\mathbf{a} \in F$. Adding things to $\mathbf{a}$, we may assume that $\delta(\mathbf{y}/\mathbf{a}) \geq 0$ for all $\mathbf{y} \in F^5$ and that V is defined over $\mathbf{a}$.

[2]Necessary; $x_1 + px_2 = 0$ is additively free iff $p \notin \mathbb{Q}$
[3]Note that x^y is definable as $\exp(y \log x)$, which is well-defined for integer y.
[4]Parametric varieties are the equivalent of fixing the degree and quantifying over all coefficients in algebraically closed fields.
[5]To do this, let $\mathbf{a}'$ be a minimizer of $\delta(\mathbf{y}/\mathbf{a})$ as $\mathbf{y}$ ranges over F. Then we have $\delta(\mathbf{y}/\mathbf{aa}') = \delta(\mathbf{yaa}') - \delta(\mathbf{aa}') \geq 0$ (by choice of $\mathbf{a}'$), so use $\mathbf{aa}'$ in place of $\mathbf{a}$.

The scheme then gives a $\mathbf{x}$ such that $ldim_{\mathbb{Q}}(\mathbf{x}/\mathbf{a}) = n$. By Schanuel, then $td(\mathbf{x}, \exp \mathbf{x}/\mathbf{a}, \exp \mathbf{a}) \geq n$. Since V is of dimension n, we have $(\mathbf{x}, \exp \mathbf{x})$ is generic in V over $(\mathbf{a}, \exp \mathbf{a})$, which is more than we need. †

[Kir] says that this was originally described by [Zil05] as a "slight saturatedness" of the structure, so it being first order was a bit of a surprise. Having it first order allows for the some of the details to be simplified. On the other hand, the analogy to algebraic closure makes it being first order more expected.

(5) **Countable Closure Property**: There are two definitions of exponential closure that are the same with axiom 2 (we will not prove this, see [Kir10, Theorem 1.3]).

We define the one used in [Kir]. An exponential polynomial $f(\mathbf{X}) = p(\mathbf{X}, \exp(\mathbf{X}))$, where $p \in F[\mathbf{X}, \mathbf{Y}]$. A *Khovanskii system of width* n is exponential polynomials $f_0, \ldots, f_{n-1}$ of arity n such that

$$f_i(\mathbf{x}) = 0 \forall i < n$$
$$\begin{vmatrix} \frac{\partial f_0}{\partial x_0} & \cdots & \frac{\partial f_0}{\partial x_{n-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{n-1}}{\partial x_0} & \cdots & \frac{\partial f_{n-1}}{\partial x_{n-1}} \end{vmatrix} (\mathbf{x}) \neq 0$$

Then

$$ecl^F(C) := \{a_0 \in F : \mathbf{a} \in F \text{ is the solution to the Khovanski system given by } f_0, \ldots, f_{n-1} \text{ defined over } \mathbb{Q}(C)\}$$

By [Kir10, Theorem 1.1], this is a pregeometry in any exponential field. We also have an easy proof that this is $L(Q)$.

Proposition 0.2 ([Kir]). *Axiom 5 is an $L(Q)$ scheme.*

Proof: Given exponential polynomials , let $\chi(\mathbf{x}, \mathbf{z})$ state that $\mathbf{x}$ is a solution to the Khovanskii system with coefficients $\mathbf{z}$.

$$\forall \mathbf{z} \neg Q x_0 \exists x_1, \ldots, x_{n-1} \chi_f(\mathbf{x}, \mathbf{z})$$

†

We're going to discuss two interesting things coming from [Kir] and then look at the language used for quasiminimality.

Q **cannot be eliminated**: We're going a little out of order here, because we will use some of the quasiminimality in this proof.

Proposition 0.3 ([Kir]). *Countable closure is not $L_{\infty,\omega}$ expressible.*

Proof: Let F_0 be the zero dimensional Zilber field. Then we can adjoin some a so $\exp a = a$ and form the strong exponential algebraic closure to get F_1[6]
. Note that F_1 is also a zero dimensional Zilber field, so $F_0 \cong F_1$. Our goal is to show that $F_0 \prec_{L_{\infty,\omega}} F_1$.

Let $\mathbf{b} \in F_0$ and let B be the smallest ELA-subfield containg $\mathbf{b}$. By general Fraisse construction or [Kir11, Proposition 6.9], both F_0 and F_1 are isomorphic to the strong exponential algebraic closure of $\mathbf{b}$. That is $F_0 \cong_B F_1$. Thus, given any $L_{\infty,\omega}(\mathbf{b})$ formula, F_0 and F_1 agree on it.

We iterate this process to form $\{F_\alpha : \alpha < \omega_1\}$. Note at limit stages, $L_{\infty,\omega}$ substructure is well-behaved ($L(Q)$ is not) and taking unions cannot grow the exponential transcendence degree. Thus F_{ω_1} is $L_{\infty,\omega}$ equivalent to all of these, but does not satisfy the countable closure property. †

Our restriction to $L_{\infty,\omega}$ formulas comparing F_0 and F_1 was not crucial; this is true of any logic. However, the key step is that $\prec_{L_{\infty,\omega}}$ is stable under unions of chains of any (esp. uncountable) length. Note that this gives the first[7] explicit example of a non-finitary AEC.

Elementary equivalence implies categoricity: Again, we go out of order. This is [Kir, §2.5]. The key is that all of the axioms except 2 and 5 are first order (modulo 2). Obviously, $\mathbb{C}$ has standard $\mathbb{Z}$, so we only need to see that countable closure property holds. [Zil05, Lemma 5.12] proves this, but the Khovanskii-system definition of *ecl* gives a shorter proof.

Suppose $f_0, \ldots, f_{n-1}$ is a Khovanskii system with coefficients from $\mathbb{Q}(C)$. The inequation in the system says that the Jacobian of $f_0, \ldots, f_{n-1}$ is nonzero. Then we can use the implicit function theorem to find an open set around each solution to the Kohvanskii system such that it is the *only* solution in that set: Let $f(y, \mathbf{x}) = (f_0(\mathbf{x}), \ldots, f_{n-1}(\mathbf{x}))$ and note $f(0, \mathbf{a}) = 0$. By the implicit function theorem, there is

- open $U \ni 0$
- open $V_{\mathbf{a}} \ni \mathbf{a}$
- $g : U \to V$ such that

$$\{(x, \mathbf{y}) \in U \times V_{\mathbf{a}} : f(x, \mathbf{y}) = 0\} = \{(x, g(x)) : x \in U\}$$

The key point is that the only solution in $V_{\mathbf{a}}$ is $\mathbf{a}$. Thus, the solutions are isolated in the complex topology, and there are only countably many of them.

[6]In doing this, we need to ensure that F_1 has no nonstandard kernel elements. First, in the $\mathbb{Q}$-linear span of F_0 and a, we have that $\exp(f + qa) = 1$ iff $\exp(f)a^q = 1$ iff $\exp(f) = 1$ and $q = 0$. So no new kernel elements are in this span. To get F_1, we form the ELA-closure and then the strong exponential algebraic closure of this span, each of which does not add new kernel elements.
More details are in Kirby [Kir11].

[7]Potentially not the first chronologically, but at least the first ones that people got excited about.

Thus, $\mathbb{C}$ is the Zilber field of dimension continuum. So, $\mathbb{C} \cong \mathbb{B}$. †

The language of quasiminimality: One facet of quasiminimal classes is that quantifier free types must be the right one, i. e., Galois type. This is not true in the current language, so we expand it. For each algebraic variety $V \subset \mathbb{G}_a^{r+n} \times \mathbb{G}_m^{r+n}$ that is defined and irreducible over $\mathbb{Q}$, we set the r-ary relation

$$R_{V,n}(\mathbf{x}) := \exists \mathbf{y} \forall \mathbf{m} \in \mathbb{Q}^{r+n}((\mathbf{x}, \mathbf{y}, \exp(\mathbf{x}), \exp(\mathbf{y})) \in V \wedge \left(\sum_{i<r} m_i x_i + \sum_{i<n} m_{i+r} y_i = 0 \to \wedge_{r \leq i < r+n} m_i = 0 \right))$$

Set $L = \langle +, 0, \lambda\cdot, R_{V,n} \rangle_{\lambda \in \mathbb{Q}}$. Note the similarity between these relations and the statement of Axiom 4

$$\forall p \in P' \forall \mathbf{a} \in F^r \exists \mathbf{x} \in F^n \forall \mathbf{m} \in \mathbb{Q} \left[(\mathbf{x}, \exp \mathbf{x}) \in V_p \wedge \left(\sum_{i<n} m_i x_i + \sum_{i<r} m_{n+i} a_i = 0 \to \wedge_{i<n} m_i = 0 \right) \right]$$

If $(V_p)_{p \in P}$ is a parameterizing variety in $(\mathbf{x}, \exp \mathbf{x})$ and $(W_p)_{p \in P}$ is the parameterizing variety in $(\mathbf{x}, \mathbf{y}, \exp \mathbf{x}, \exp \mathbf{y})$ requiring only the same relationship between $\mathbf{y}$ and $\exp \mathbf{y}$, then an instance of Axiom 4 is equivalent to

$$\forall p \in P' \forall \mathbf{a} \in F^r R_{W_p,n}(\mathbf{a})$$

The reason we change the language is the following lemma of Zilber:

Fact 0.4 ([Zil05].5.7)**.** *Suppose $A \prec B$ and $\mathbf{a}, \mathbf{b}$ are finite tuples so $A\mathbf{a} \prec B$ and $A\mathbf{b} \prec B$. If there is an isomorphism so $F_{A\mathbf{a}} \cong_{F_A} F_{A\mathbf{b}}$*[8] *that sends $\mathbf{a}$ to $\mathbf{b}$, then $tp_{qf,L}(\mathbf{a}/A) = tp_{qf,L}(\mathbf{b}/A)$.*

Additionally, making the language more relational means that structures can be smaller.

We call this change conservative since L has the same definable sets as the original language (in some Zilber field):

First, note that $V = V(x_1 x_2 = x_3)$ and $R_{V,0}$ define the graph of multiplication:

$$R_{V,0}(a_1, a_2, a_3) \iff a_1 a_2 = a_3$$

and $W = V(x_2 = x_1')$ and $R_{W,0}$ define the graph of exponentiation:

$$R_{V,0}(a_1, a_2) \iff a_2 = \exp(a_1)$$

Second, $\mathbb{Q}$ and, thus, each $R_{V,n}$ is definable in each member of $\mathcal{EC}^*_{st,ccp}$.

[8] F_X is the field generated by $X \cup \exp(X)$ with exp regarded as a partial function defined only on X. Note that this is not a structure in the original language, but *is* a structure in the new one.

Zilber initially proves that this is a quasiminimal class in [Zil05], but Bays and Kirby fill some gaps/make some corrections. Note, following [BHH+12], proving excellence is not necessary.

References

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