SHELAH'S OMITTING TYPES THEOREM IN EXCRUCIATING DETAIL

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We aim to prove Shelah's Omitting Types Theorem for AECs [Sh394, Lemma 8.7] in detail. The proofs in [Bal09, Theorem 14.8] and [MaSh285, Proposition 3.3] are only special cases, with [MaSh285] being a very helpful guide. We've introduced a slight generalization (allowing the M_α's to be different models, rather than just one), but this is standard.

Sebastien Vasey initially pointed this result out to me and conversations with him were very helpful in working through the details.

We assume a familiarity with AECs, Galois types, etc.; a reference for these is [Bal09].

One nonstandard piece of notation:

Definition 1. *If $N \prec M$ and $p \in gS(N)$ with $\chi \leq \|N\|$, then we say that M omits p/E_χ iff for every $c \in M$, there is some $N^- \prec N$ of size $\prec \chi$ such that c does not realize $p \upharpoonright N^-$.*

Note that this is implied by omitting p and is the same under χ-tameness (or weak tameness if N is saturated). So we can think of this as a strong form of type omission. However, this is weaker than omitting the set of restrictions of p, $\{p \upharpoonright N^- : N^- \prec N \text{ and } \|N^-\| \leq \chi\}$. Each of the types in that set might be realized in M; however, there is no element of M that simultaneously realizes them all.

In the following all types are of length $< \omega$.

Theorem 2 (Shelah's Omitting Types Theorem)**.** *Let $\mathcal{K}$ be an AEC with $LS(\mathcal{K}) \leq \chi \leq \lambda$ with*

1. *$N_0 \prec N_1$ with $\|N_0\| \leq \chi$ and $\|N_1\| = \lambda$;*
2. *$\Gamma_0 = \{p_i^0 : i < i_0^*\}$ are Galois types over N_0; and*
3. *$\Gamma_1 = \{p_i^1 : i < i_1^*\}$ are Galois types over N_1 with $i_1^* \leq \chi$.*

Suppose that, for each $\alpha < (2^\chi)^+$, there is $M_\alpha \in \mathcal{K}$ such that

1. *$\|M_\alpha\| \geq \beth_\alpha(\lambda)$;*
2. *M_α omits Γ_0; and*
3. *M_α omits p_i^1/E_χ for each $i < i_1^*$.*

Then we can find $\Phi \in \Upsilon[\mathcal{K}]$; increasing, continuous $\langle N_n' \in \mathcal{K}_{\leq\chi} : n \leq \omega\rangle$; and increasing Galois types $p_{i,n}^1 \in gS(N_n')$ for $n < \omega, i < i_1^*$ such that

Date: January 18, 2016.

(1) *$N_0 = N'_0 = EM_\tau(\emptyset, \Phi)$;*
(2) *for each $n < \omega$, we have $N'_n \prec EM_\tau(n, \Phi)$ and $f_n : EM_\tau(n, \Phi) \to M_{\alpha_n}$ for some $\alpha_n < (2^\chi)^+$ such that $f_n(N'_n) \prec N_1$.*
(3) *$p^1_{i,n} := f_n^{-1}(p^1_i \restriction f_n(N'_n)) \in gS(N'_n)$; and*
(4) *for every infinite[1] I, $EM_\tau(I, \Phi)$ omits Γ_0 and omits any type that extends $\{p^1_{i,n} : n < \omega\}$ in the following strong sense: if $p^1_{i,*} \in gS(N'_\omega)$ extends each $p^1_{i,n}$ and $J \subset I$ is of size $n < \omega$ with $a \in EM_\tau(J, \Phi)$, then a doesn't realize $p^1_{i,*} \restriction N'_n = p^1_{i,n}$.*

Afterwards, we will state some corollaries and have a discussion, but for now: the proof!

Proof: Stage 1 will build a language τ^+; it is essentially a language from Shelah's Presentation Theorem with some extra aspects tacked on. Stage 2 builds a "tree of indiscernibles." Stages 3 uses this tree to build the template Φ and finishes the proof.

Stage 1: Set $\tau^+ := \tau \cup \{F^i_n : i < \chi\}$, as in Shelah's Presentation Theorem. Let M be a τ structure such that $N_1 \prec M$ and M omits p^1_i/E_χ for each $i < i^*_1$. We describe a procedure to expand M to a τ^+-structure M^+ with certain properties: we want to define a cover $\{M_{\mathbf{a}} \in \mathcal{K} : \mathbf{a} \in M\}$ with the following properties:

(1) If $\mathbf{a} \in N_0$, then $M_{\mathbf{a}} = N_0$ (so in particular, this is true for $\mathbf{a} = \emptyset$)
(2) If $\mathbf{a} \in N_1$, then $M_{\mathbf{a}} \prec N_1$
(3) For all $\mathbf{a}$, set $M_{\mathbf{a},1} := M_{\mathbf{a}} \cap N_1$. Then
$$N_0 \prec M_{\mathbf{a},1} \prec M_{\mathbf{a}}$$
(4) For $i < i^*_1$, we have $p^1_i \restriction (M_{\mathbf{a},1})$ is omitted in $M_{\mathbf{a}}$.

We build this cover in ω many steps, building increasing covers $\{M^n_{\mathbf{a}} : n < \omega\}$ that get closer and closer.

$\mathbf{n = 0}$: Nothing special happens here. Start with $M^0_{\mathbf{a}} = N_0$ for all $\mathbf{a} \in N_0$. Then extend this to a cover of N_1, and then to a cover of M. Note we ignore conditions (3) and (4) here. Also, if $\mathbf{a} \in N_1$, we will not change $M^0_{\mathbf{a}}$ in the rest of the construction.

$\mathbf{2n + 1}$: Suppose the increasing covers up to $2n$ are built. We take care of (3) in this step. First note that, for $\mathbf{a} \in N_1$, (3) is guaranteed, so no change should be done. This step is itself made up of ω many steps. Do the following construction by induction on the length of $\mathbf{a}$:
It might be the case that $\left(M^{2n}_{\mathbf{a}} \cup \bigcup_{\mathbf{b} \subsetneq \mathbf{a}} M^{2n+1}_{\mathbf{b}}\right) \cap N_1$ is not a τ-structure or in $\mathcal{K}$. However, we can find $N^{1,0} \prec N_1$ containing it of size χ. Then find $N^{2,0} \prec M$ containing $M^{2n}_{\mathbf{a}} \cup \bigcup_{\mathbf{b} \subsetneq \mathbf{a}} M^{2n+1}_{\mathbf{b}}$ of size χ. Iterate this process so

- $N^{1,i+1} \prec N_1$ contains $N^{2,i} \cap N_1$ and is of size χ; and

[1]Note if I is not infinite, then N'_ω does not appear as a strong substructure.

- $N^{2,i+1} \prec M$ contains $N^{1,i+1}$

In the end, set $M^{2n+1}_{\mathbf{a}} := \cup_{i<\omega} N^{2,i}$. Then we have $M^{2n+1}_{\mathbf{a}} \cap N_1 = \cup_{i<\omega} N^{1,i}$, which is a strong substructure of N_1, as desired. Also, since we included the $\bigcup_{\mathbf{b} \subsetneq \mathbf{a}} M^{2n+1}_{\mathbf{b}}$ term, this will form an increasing cover.

$\mathbf{2n+2}$: In this step, we take care of (4). Note that, by the odd step, $M^{2n+1}_{\mathbf{a},1} := M^{2n+1}_{\mathbf{a}} \cap N_1 \prec N_1$ is well defined. Again, we are going to expand our cover $\{M^{2n+1}_{\mathbf{a}} : \mathbf{a} \in M\}$ by induction on the length of $\mathbf{a}$:
Suppose that $M^{2n+2}_{\mathbf{b}}$ is defined for all proper subtuples $\mathbf{b}$ of $\mathbf{a}$. For each $i < i^*_1$, it might be the case that $m \in M^{2n+1}_{\mathbf{a}}$ realizes $p \restriction M^{2n+1}_{\mathbf{a},1}$. For each such i and m, pick $M_{i,m} \prec M$ of size χ such that m does not realize $p \restriction M_{i,m}$; such a model exists precisely because M omits p/E_χ. An important point is that $m \in N_1$ implies that $m \in M^{2n+1}_{\mathbf{a},1}$ and, therefore, already omits $p \restriction M^{2n+1}_{\mathbf{a},1}$. In particular, if $\mathbf{a} \in N_1$, then no expansion is undertaken in this step. Then let $M^{2n+2}_{\mathbf{a}} \prec M$ be of size χ such that it contains

$$\bigcup_{\mathbf{b} \subsetneq \mathbf{a}} M^{2n+2}_{\mathbf{b}} \cup \bigcup \{M_{i,m} : i < i^*_1, m \in M^{2n+1}_{\mathbf{a}} \text{ for which this is defined}\}$$

Note that the fact we can choose $M^{2n+2}_{\mathbf{a}} \in \mathcal{K}_\chi$ uses that $|i^*_1| \leq \chi$.

At **stage** ω, set $M_{\mathbf{a}} = \cup_{n<\omega} M^n_{\mathbf{a}}$. Note that $\{M_{\mathbf{a}} : \mathbf{a} \in M\}$ forms a cover of M because covers are closed under increasing unions. The first two conditions are satisfied because they were satisfied at stage 0 and no later stage changed $M^0_{\mathbf{a}}$ when $\mathbf{a} \in N_1$. For (3), notice that

$$M_{\mathbf{a}} \cap N_1 = \bigcup_{n<\omega} M^{2n+1}_{\mathbf{a}} \cap N_1 = \bigcup_{n<\omega} M^{2n+1}_{\mathbf{a},1}$$

which is an increasing union of strong substructures of N_1. For (4), let $m \in M_{\mathbf{a}}$ for some $\mathbf{a}$. Then m appears in some $M^{2n+1}_{\mathbf{a}}$. By construction, m does not realize $p \restriction M^{2n+2}_{\mathbf{a}}$. This carries upwards, so m does not realize $p \restriction M_{\mathbf{a}}$.

Now that we have this cover, we can expand M to a τ^+ structure M^+, where F^i_n is n-ary by letting $\{F^i_{\ell(\mathbf{a})} : i < \chi\}$ enumerate $M_{\mathbf{a}}$ such that the first n many functions are projections. The expansions of $M_{\mathbf{a}}$ and $M_{\mathbf{a},1}$ to τ^+ are denoted $M^*_{\mathbf{a}}$ and $M^*_{\mathbf{a},1}$, respectively.

Now, for each $\alpha < (2^\chi)^+$, set M^+_α to be this expansion of M_α. Furthermore, we will denote the parts of the cover as $M_{\alpha,\mathbf{a}}$ and $M_{\alpha,\mathbf{a},1}$ (so their expansion are $M^*_{\alpha,\mathbf{a}}$ and $M^*_{\alpha,\mathbf{a},1}$). Since they never get changed, we require the the expansions of N_0 and N_1 (denoted N^+_0 and N^+_1) are the same in each M^+_α. $\dagger_{\text{Stage 1}}$

Given a τ^+-structure M^+ and $X \subset M^+$, $\text{cl}^{\tau^+}_{M^+}(X)$ denotes the closure of X under the functions of τ^+. By construction, we will have $\text{cl}^{\tau^+}_{M^+}(X) \restriction \tau \prec M^+$.

Stage 2: We want to define some indiscernibles via Morley's Method. Rather than mucking about with nonstandard models of set theory, we use (in a sense) a tree of indiscernibles from M^+ (if that doesn't make sense, ignore it or see after the proof). Recall $\|N_1\| = \lambda$. The goal is to build, for $n < \omega$ and $\alpha < (2^\chi)^+$, injective functions f^n_α with domain $\beth_\alpha(\lambda)$ and range $M_{\beta_n(\alpha)}$ for some $\alpha \leq \beta_n(\alpha) < (2^\chi)^+$ such that

(1) for fixed $\alpha < (2^\chi)^+$ and $n < \omega$, we have that
  (a) $N^*_{(\alpha,n)} := M^*_{\beta_n(\alpha),\mathbf{a},1}$ is a constant τ^+-substructure of $M_{\beta_n(\alpha)}$ for $\mathbf{a} = f^n_\alpha(i_1), \dots, f^n_\alpha(i_n)$, where $i_1 < \dots < i_n < \beth_\alpha(\lambda)$; and
  (b) $q^\alpha_n := tp^{\tau^+}_{qf}(\mathbf{a}/N^*_{(\alpha,n)}; M^+_{\beta_n(\alpha)})$ is constant with the same notation;

(2) for each $n < \omega$, there is some $N^*_{(\cdot,n)} \subset N^+_1$ such that
  (a) $N^*_{(\cdot,0)} \upharpoonright \tau = N_0$;
  (b) for $m < n$, there is a τ^+-embedding $h_{m,n} : N^*_{(\cdot,m)} \to N^*_{(\cdot,n)}$ that form a coherent system;
  (c) for each $\alpha < (2^\chi)^+$, there is $g^n_\alpha : N^*_{(\cdot,n)} \cong N^*_{(\alpha,n)}$; and
  (d) for all $\alpha < (2^\chi)^+$ and $m < n$, there is $\alpha < \beta < (2^\chi)^+$ such that $\langle f^n_\alpha(i) : i < \beth_\alpha(\lambda)\rangle$ is an increasing[2] subset of $\langle f^m_\beta(i) : i < \beth_\beta(\lambda)\rangle$ and the following commutes

$$\begin{array}{ccc} N^*_{(\cdot,m)} & \xrightarrow{h_{m,n}} & N^*_{(\cdot,n)} \\ {\scriptstyle g^m_\beta}\downarrow & & \downarrow{\scriptstyle g^n_\alpha} \\ N^*_{(\beta,m)} & \xrightarrow{id} & N^*_{(\alpha,n)} \end{array}$$

  ; and

(3) fixing $n < \omega$, for each $\alpha < (2^\chi)^+$, we have that

$$q_n := (g^n_\alpha)^{-1}(q^\alpha_n) \in S(N^*_{(\cdot,n)})$$

is constant (as a syntactic type), as is

$$p^1_{(i,n)} := (g^n_\alpha)^{-1}\left(p^1_i \upharpoonright (N^*_{(\alpha,n)} \upharpoonright \tau)\right) \in gS(N^*_{(\cdot,n)} \upharpoonright \tau)$$

for each $i < i^1_*$ (as a Galois type in $\mathcal{K}$).

The construction of this is standard; one thing to note is the fixing of the Galois type in (3). In Stage 3, the syntactic types will correspond to Φ and the Galois types will correspond to pieces of p^1_i.

Construction: We do this by induction on $n < \omega$ and, inside that, on $\alpha < (2^\chi)^+$.

[2]According to the order inherited by the enumerations

$\mathbf{n = 0}$: For this case, there's not much to do: $N^*_{(\alpha,0)}$ always has universe N_0 and we can pick g^0_α to be the identity. Set $\beta_0(\alpha) = \alpha$ and let f^0_α enumerate M_α.

$\mathbf{n + 1}$: This is where it gets fun and, more importantly, where we see the importance of our cardinal arithmetic. Fix $\alpha < (2^\chi)^+$. First, we color $n+1$-tuples from $\{f^n_{\alpha+\omega}(i) : i < \beth_{\alpha+\omega}(i)\}$ with their qf-type over N_1^+; recall that the n-tuples all have the same type by construction. Erdős-Rado tells us that $\beth_{\alpha+\omega}(\lambda) \to (\beth_\alpha(\lambda))^{n+1}_{2^\lambda}$; well, really it says $\beth_{\alpha+n}(\lambda)^+ \to (\beth_\alpha(\lambda)^+)^{n+1}_{\beth_\alpha(\lambda)}$, but this follows. Thus, we can find $Y^{n+1}_\alpha \subset \beth_{\alpha+\omega}(\lambda)$ such that this type is constant. Note that this already gives us (1) one the construction: $\{f^n_{\alpha+\omega}(i) : i \in Y^{n+1}_\alpha\}$ are $n+1$-indiscernibles over N_1^+, so for each $M^*_{\beta_n(\alpha),\mathbf{a},1}$ and $tp^{\tau^+}_{qf}(\mathbf{a}/N^*_{(\alpha,n)}; M^+_{\beta_n(\alpha)})$ are constant for all $n+1$-tuples $\mathbf{a}$ that are increasing from $\{f^n_{\alpha+\omega}(i) : i \in Y^{n+1}_\alpha\}$. Call these $\hat{N}^*_{(\alpha,n+1)}$ and $\hat{q}^\alpha_{n+1}$ for now; not every α will make it and there's some reindexing, so it's premature to define the unhatted version yet.

From this, we have that $\hat{N}^*_{(\alpha,n+1)} \supset N^*_{(\alpha+\omega,n)}$. Now, color each $\alpha < (2^\chi)^+$ with the isomorphism type of $\hat{N}^*_{(\alpha,n+1)}$ over $N^*_{(\alpha+\omega,n)}$ through $(g^n_{\alpha+\omega})^{-1}$; this needlessly obtuse phrase means that we extend $(g^n_{\alpha+\omega})^{-1}$ to an isomorphism containing the $\hat{N}^*_{(\alpha,n+1)}$ in the domain (call this t_α in a notational respite) and we compare isomorphism types of

$$\left\{ \left(t_\alpha(\hat{N}^*_{(\alpha,n+1)}), N^*_{(\cdot,n)} \right) : \alpha < (2^\chi)^+ \right\}$$

We color $(2^\chi)^+$ many things with $\leq 2^\chi$ many colors, so we can find $X^0_{n+1} \subset (2^\chi)^+$ of size $(2^\chi)^+$ such that this isomorphism type is constant. Once we've fixed this set, we can fix a representative of this class $N^*_{(\cdot,n+1)}$–pick, for instance, $\hat{N}^*_{(\min X^0_{n+1},n+1)}$–; a τ^+-embedding $h_{n,n+1} := g^n_{\min X^0_{n+1}}$ from $N^*_{(\cdot,n)}$ to $N^*_{(\cdot,n+1)}$–from which we form the rest of the $h_{m,n+1}$–; and isomorphisms $\hat{g}^{n+1}_\alpha : N^*_{(\cdot,n+1)} \cong \hat{N}^*_{(\alpha,n+1)}$ such that the following picture commutes

$$\begin{array}{ccc} N^*_{(\cdot,n)} & \xrightarrow{h_{n,n+1}} & N^*_{(\cdot,n+1)} \\ {\scriptstyle g^n_{\alpha+\omega}}\downarrow & & \downarrow{\scriptstyle \hat{g}^{n+1}_\alpha} \\ N^*_{(\alpha+\omega,n)} & \longrightarrow & \hat{N}^*_{(\alpha,n+1)} \end{array}$$

To find $\hat{g}^{n+1}_\alpha$, use the fact that $\alpha, \min X^0_{n+1} \in X^0_{n+1}$ to find

$$s_\alpha : t_\alpha(\hat{N}^*_{(\alpha,n+1)}) \cong_{N_{(\cdot,n)}} t_{\min X^0_{n+1}}(N^*_{(\cdot,n+1)})$$

Then set $\hat{g}^{n+1}_\alpha := t_\alpha^{-1} \circ s_\alpha^{-1} \circ t_{\min X^0_{n+1}}$ and chase the following diagram

$$\begin{array}{ccccccc}
\hat{N}^*_{(\alpha,n+1)} & \xrightarrow{t_\alpha} & t_\alpha(\hat{N}^*_{(\alpha,n+1)}) & \xrightarrow{s_\alpha} & t_{\min X^0_{n+1}}(N^*_{(\cdot,n+1)}) & \xleftarrow{t_{\min X^0_{n+1}}} & N^*_{(\cdot,n+1)} \\
\uparrow & & \nwarrow & & \nearrow & & \uparrow \\
N^*_{(\alpha+\omega,n)} & \xrightarrow{(g^n_{\alpha+\omega})^{-1}} & & N_{(\cdot,n)} & & \xleftarrow{(g^n_{\min X^0_{n+1}+\omega})^{-1}} & N^*_{(\min X^0_{n+1}+\omega,n)}
\end{array}$$

This guarantees (2). We shrink again to get (3), but this part will give us (2) in any set we shrink to.

Now color each $\alpha \in X^0_{n+1}$ with the pair

- $(\hat{g}^{n+1}_\alpha)^{-1}(\hat{q}^\alpha_{n+1})$; and
- $(\hat{g}^n_\alpha)^{-1}\left(p^1_i \restriction (\hat{N}^*_{(\alpha,n+1)} \restriction \tau)\right)$

Again, there are $(2^\chi)^+$ many objects colored with 2^χ many colors, so there is $X^1_{n+1} \subset X^0_{n+1}$ such that each of these are constant.

Now we are ready to pick our final sets. We have sets that Y^{n+1}_α of order type $\beth_\alpha(\lambda)$ and X^1_{n+1} of order type $(2^\chi)^+$. For some j in the proper set, we will use $Y^{n+1}_\alpha(j)$ and $X^1_{n+1}(j)$ to denote the jth element of that set under the only possible ordering (the ordering inherited from the ordinals). Thus, we finish by setting, for each $\alpha < (2^\chi)^+$ and $i < \beth_\alpha(\lambda)$,

- $\beta_{n+1}(\alpha) := \beta_n\left(X^1_{n+1}(\alpha) + \omega\right)$
- $f^{n+1}_\alpha(i) := f^n_{X^1_{n+1}(\alpha)+\omega}\left(Y^{n+1}_{X^1_{n+1}(\alpha)}(i)\right)$
- $N^*_{(\alpha,n+1)} := \hat{N}^*_{(X^1_{n+1}(\alpha),n+1)}$
- $q^\alpha_{n+1} = \hat{q}^{X^1_{n+1}(\alpha)}_{n+1}$
- $g^{n+1}_\alpha = \hat{g}^{n+1}_{X^1_{n+1}(\alpha)}$
- $q_{n+1} = (g^{n+1}_\alpha)^{-1}(q^\alpha_{n+1})$
- $p^1_{(i,n+1)} = (g^{n+1}_\alpha)^{-1}(p^1_i(N^*_{(\alpha,n+1} \restriction \tau))$

noting that the last two items don't depend on α. This is a notational mess, but we essentially just replace every instance of α by the αth member of X^1_{n+1} and every instance of i by the ith member of Y^{n+1}_α.

Then this works. $\dagger_{\text{Construction, Stage 2}}$

Stage 3: Here, we use the objects constructed in Stage 2 to define the appropriate Φ.

First, we want to show that both the q_n's and $p^1_{(i,n)}$'s are increasing with n (after being hit with $h_{m,n}$).

Claim 3. *For every $s \subset n$ with $|s| = m$, $q_n^s \upharpoonright (N^*_{(\cdot,m)}) = h_{m,n}(q_m)$. In particular, $h_{m,n}(q_m) \subset q_n$.*

Proof of Claim 3: Set $s = \{s_1 < \cdots < s_m\} \subset n$. Fix $\alpha < (2^\chi)^+$ and $i_1 < \cdots < i_n < \beth_\alpha(\lambda)$ and write $\mathbf{a} = f^n_\alpha(i_1), \ldots, f^n_\alpha(i_n)$. By (2.b), there is $\beta > \alpha$ and $j_1 < \cdots < j_m < \beth_\beta(\lambda)$ such that $f^n_\beta(j_\ell) = f^n_\alpha(i_{s_\ell})$ for $\ell \leq n$. Then

$$\begin{aligned} q_m &= (g^m_\beta)^{-1}\left(tp^{\tau^+}_{qf}(f^m_\beta(j_1), \ldots, f^m_\beta(j_m)/N^*_{(\beta,m)}, M^+_{\beta_m(\beta)})\right) \\ &= (g^m_\beta)^{-1}\left(tp^{\tau^+}_{qf}(\mathbf{a}^s/N^*_{(\beta,m)}, M^+_{\beta_m(\beta)})\right) \\ &= h^{-1}_{m,n} \circ (g^n_\alpha)^{-1}\left(tp^{\tau^+}_{qf}(\mathbf{a}^s/N^*_{(\alpha,n)}, M^+_{\beta_m(\beta)}) \upharpoonright N^*_{(\beta,m)}\right) \\ &= h^{-1}_{m,n} \circ (g^n_\alpha)^{-1}((q^\alpha_n)^s) \upharpoonright (g^n_\alpha)^{-1}(N^*_{(\beta,m)}) \\ &= h^{-1}_{m,n}\left(q^s_n \upharpoonright N^*_{(\cdot,m)}\right) \\ h_{m,n}(q_m) &= q^s_n \upharpoonright N^*_{(\cdot,m)} \end{aligned}$$

as desired. $\dagger_{\text{Claim 1}}$

Claim 4. *Let $i < i^1_*$. For $m < n$, $p^1_{(i,n)} \upharpoonright \left(N^*_{(\cdot,m)} \upharpoonright \tau\right) = h_{m,n}(p^1_{(i,m)})$.*

Proof of Claim 4: This is similar to the above, but without mucking around with the f^n_α's. Let $\alpha < (2^\chi)^+$ and let β be as in (2.b), although we only use the commutative diagram. Then

$$\begin{aligned} p^1_{(i,m)} &= (g^m_\beta)^{-1}\left(p^1_i \upharpoonright (N^*_{(\beta,m)} \upharpoonright \tau)\right) \\ &= h^{-1}_{m,n} \circ (g^n_\alpha)^{-1}\left(\left[p^1_i \upharpoonright (N^*_{(\alpha,n)} \upharpoonright \tau)\right] \upharpoonright (N^*_{(\beta,m)} \upharpoonright \tau)\right) \\ &= h^{-1}_{m,n}((g^n_\alpha)^{-1}\left(p^1_i \upharpoonright (N^*_{(\alpha,n)} \upharpoonright \tau)\right) \upharpoonright (g^n_\alpha)^{-1}(N^*_{(\beta,m)} \upharpoonright \tau)) \\ h_{m,n}(p^1_{(i,m)}) &= p^1_{(i,n)} \upharpoonright (N^*_{(\cdot,m)} \upharpoonright \tau) \end{aligned}$$

$\dagger_{\text{Claim 2}}$

This means that the sequences $\{h^{-1}_{0,n}(q_n) : n < \omega\}$ and $\{h^{-1}_{0,n}(N^*_{(\cdot,n)}) : n < \omega\}$ are increasing. Remove this directed nonsense by setting $\bar{q}_n := h^{-1}_{0,n}(q_n)$ and $\bar{N}^*_{(\cdot,n)} := h^{-1}_{0,n}(N^*_{(\cdot,n)})$; note that the first is increasing by Claim 1 and the second is increasing by construction.

Now set $\Phi = \cup_{n<\omega}\bar{q}_n$ and $\bar{N}^*_{(\cdot,\omega)} = \cup_{n<\omega}\bar{N}^*_{(\cdot,n)}$. Moreover, $\langle \bar{N}^*_{(\cdot,n)} \upharpoonright \tau : n < \omega\rangle$ is an $\prec$-increasing sequence, so $\bar{N}^*_{(\cdot,\omega)} \upharpoonright \tau \in \mathcal{K}$; however, there's no reason to expect that $\bar{N}^*_{\cdot,\omega} \upharpoonright \tau \prec M$ or appears as some strong substructure of it. Set $\bar{N}_{(\cdot,n)} := \bar{N}^*_{(\cdot,n)} \upharpoonright \tau$ and similarly for $\bar{N}_{(\cdot,\omega)}$. Moreover, $\bar{q}_n$ is a type over $\bar{N}^*_{(\cdot,n)}$, so we can add a constant to the language of Φ for each element of $\bar{N}^*_{(\cdot,n)}$. The following claim says that this changes nothing.

Claim 5. *Φ is a template proper for linear orders in $\mathcal{K}$ such that $\tau(\Phi)$ has constants for every element in $\bar{N}_{(\cdot,\omega)}$; one could write this as $\Phi \in \Upsilon_\chi[\mathcal{K}_{\bar{N}_{(\cdot,\omega)}}]$.*

Proof of Claim 5: That Φ is a template for $\mathcal{K}$ (rather than $\mathcal{K}_{\bar{N}_{(\cdot,\omega)}}$) already follows. The only potential problem in the additional step is that, for $n < m$, q_n doesn't specify the diagram over $N^*_{(\cdot,m)}$. However, using Claim 1, we can see this is fine because any way of enlarging an n-tuple to an m-tuple gives the same q_m type, which specifies this diagram. $\dagger_{\text{Claim 3}}$

Thus, we have that, for any I, $EM_{\tau(N_{(\cdot,\omega)})}(I, \Phi) \in \mathcal{K}_{\bar{N}_{(\cdot,\omega)}}$. This gives a canonical isomorphism of $\bar{N}_{(\cdot,\omega)}$ into $EM_\tau(I, \Phi)$, so we will assume that this is just the identity.

We have now defined everything from the theorem statement: N'_n is (the canonical copy of) $\bar{N}^*_{(\cdot,n)}$ in $EM_\tau(n, \Phi)$ and $p^1_{i,n}$ is (the corresponding copy of) $h^{-1}_{0,n}(p^1_{(i,n)}) \in gS(\bar{N}^*_{(\cdot,n)} \upharpoonright \tau)$. The first three conditions are clear. The omission of Γ_0 is standard: given $\mathbf{a} \in EM_\tau(I, \Phi)$, we have that $\mathbf{a} \in EM_\tau(J, \Phi)$ for some finite $J \subset I$. Then we can find $f : EM_\tau(J, \Phi) \to_{N_0} M$ by construction. Then, $EM_\tau(J, \Phi)$ omits Γ_0 since M does. Since Γ_0 are types over N_0, this is preserved by f, so $\mathbf{a}$ doesn't realize any type in Γ_0.

The final piece of the theorem is contained in the next claim.

Claim 6. *Fix $i < i^1_*$ and let $p^1_{(i,\omega)}$ be any type over N'_ω that extends each $p^1_{i,n}$. For any infinite I, $EM_\tau(I, \Phi)$ omits each $p^1_{(i,\omega)}$. In particular, if finite $J \subset I$ and $x \in EM_\tau(J, \Phi)$, then x does not realize $p^1_{i,|J|}$.*

Proof of Claim 6: Let $J \subset I$ be finite with $n := |J|$. Then

$$J \vDash \bar{q}_n = h^{-1}_{0,n} \circ (g^n_\alpha)^{-1} \left(tp^{\tau^+}_{qf}(\mathbf{a}/N^*_{(\alpha,n)}; M^+_{\beta_n(\alpha)}) \right)$$

where $\mathbf{a} = f^n_\alpha(i_1), \ldots, f^n_\alpha(i_n)$ for some/any $\alpha < (2^\chi)+$ and $i_1 < \cdots < i_n < \beth_\alpha(\lambda)$; the some/any doesn't matter because of the construction, especially (1). This equality of quantifier free types (pushed from $(g^n_\alpha)^{-1}$) gives rise to a τ^+-isomorphism

$$h : EM_\tau(J, \Phi) \cong \mathrm{cl}^{\tau^+}_{M^+_{\beta_n(\alpha)}}(\mathbf{a})$$

that extends $h^{-1}_{0,n} \circ (g^n_\alpha)^{-1}$. At long last, reaching back to (4) from the **Stage 1**, we obtain that $\mathrm{cl}^{\tau^+}_{M^+_{\beta_n(\alpha)}}(\mathbf{a}) \upharpoonright \tau$ omits the Galois types $p^1_i \upharpoonright N_{(\alpha,n)}$ for each $i < i^1_*$ (recalling here that $N_{(\alpha,n)} = M^*_{\beta_n(\alpha),\mathbf{a},1}$). Hitting this with h (and recalling that it extends $h^{-1}_{0,n} \circ (g^n_\alpha)^{-1}$), we get that $EM_\tau(J, \Phi)$ omits

$$h^{-1}\left(p^1_i \upharpoonright N_{(\alpha,n)}\right) = h^{-1}_{0,n} \circ (g^n_\alpha)^{-1}\left(p^1_i \upharpoonright N_{(\alpha,n)}\right) = \bar{p}^1_{(i,n)} = p^1_{i,n}$$

as desired. $\dagger_{\text{Claim 4, Stage 3, Theorem}}$

Some commentary is in order. First note that the ω-compactness of AECs means that some $p^1_{i,*}$ always exists. The omission of Γ_0 is not new and follows from previous work of Morley; a good proof appears in [Bal09, Appendix A]. The proof is the origin of the (cryptic?) phrase of "by Morley's Method." There are typically two ways of presenting this proof: either with a tree of indiscernibles (as we did) or with nonstandard models of set theory. The basic approach is to use the fact that M is big compared to N_0 to find indiscernibles by successive applications of Erdős-Rado; however, well-foundedness means that we can't actually apply it ω-many times. The tree of indiscernibles because we essentially build a tree of height ω that has no branch, but the extensions contain enough of the same information to pretend we do. The nonstandard model method just builds a model with exactly the illfoundedness needed to apply Erdős-Rado theorem.

My feeling is that these methods are essentially equivalent and indeed the creation of the nonstandard model needs some version of the theorem to already be proved. The advantage of the nonstandard method is that it is slicker and requires less bookkeeping: every property you want to preserve is just coded as a first-order property of some V_χ. On the other hand, the advantage of the tree method is that it's easier to work through the details and see what implies what. Since I was (initially) skeptical of this theorem and wanted to work through the details, the tree method made more sense to use.

The strange conclusion of this theorem–the strong omission of some reflection of Γ_1–is noteworthy for a few reasons.

- The price to pay (in terms of model size) is much less: $\beth_{(2^\chi)^+}(\lambda)$ typically grows much slower than $\beth_{(2^\lambda)^+}$.
- The hypothesis requires a tameness like omission of p^1_i, while the conclusion gives a locality like omission of $\cup_{n<\omega} p^1_{i,n}$.
- Just like M doesn't actually necessarily contain any indiscernibles, the canonical copies of the models N'_n don't necessarily cohere or increase in M. Thus, N'_ω doesn't need to appear in M.
- Relatedly, we have crafted some type omission out of thin air!

The following are some basic applications. First we spell out what happens in the first order case.

Corollary 7. *Let T be some first order theory, $N \prec M$ be models of T, and $\Gamma = \{p_i \mid i < \mu \leq \|N\|\}$ a set of types over N such that $\|M\| \geq \beth_{(2^{|T|})^+}(\|N\|)$ and M omits Γ.*

For any $N_0 \prec N$ of size $|T|$, there are

- *a template Φ;*
- *increasing continuous $\langle N'_n \mid n \leq \omega\rangle$ that model T of size T; and*
- *$f_n : N'_n \to N$ with $N_0 = N'_0$;*

such that for any infinite I*,* $EM_\tau(I,\Phi)$ *omits*

$$\bigcup_{n<\omega} f_n^{-1}\left(p_i \upharpoonright f_n(N_n')\right)$$

Proof: We use Theorem 2 with Γ_0 being empty and Γ_1 being Γ. This AEC is very tame and local, so omitting $p/E_{|T|}$ is the same as omitting p and there is only one type extending all $f_n^{-1}\left(p_i \upharpoonright f_n(N_n')\right)$. †

Corollary 8. *Let* T *be some first order theory,* $N \prec M$ *be models of* T*, and* $\Gamma = \{p_i \mid i < \mu \leq \|N\|\}$ *a set of types over* N *such that* $\|M\| \geq \beth_{(2^{|T|})^+}(\|N\|)$ *and* M *omits* Γ*.*

For any $N_0 \prec N$ *of size* $|T|$*, there is some* $N' \succ N_0$ *of size* $|T|$*, template* Φ*, and* $p_i^* \in S(N')$ *extending* $p_i \upharpoonright N'$ *such that no* $EM_\tau(I,\Phi)$ *realizes* p_i^**.*

This doesn't seem like a very good increase. However, it turns out to be very useful to use not λ-saturated models to build a template that always builds unsaturated models. Here we contrast the two results available:

1. Morley: If M is not λ^+-saturated and $\|M\| \geq \beth_{(2^\lambda)^+}$, then there is a template such that for any infinite I, $EM_\tau(I,\Phi)$ is not λ-saturated.
   Proof: By Morley's Method ;). M omits some $p \in S(A)$ with $|A| = \lambda$. Add this to the language and use, e. g., [Bal09, Theorem A.3.(1)] to build a template omitting p.
2. Shelah: If $M$is not λ^+-saturated and $\|M\| \geq \beth_{(2^{|T|})^+}(\lambda)$, then there is a template such that for any infinite I, $EM_\tau(I,\Phi)$ is not $|T|^+$-saturated.
   Proof: By Shelah's Method, I guess. M omits some $p \in S(N)$ with $\|N\| = \lambda$. Since first-order logic is $< \omega$-tame, M omits $p/E_{|T|}$. Then we can use Corollary 8 to find a model N_0, a type $p' \in S(N_0)$, and a template Φ such $EM_\tau(I,\Phi)$ always omits p'. In particular, it is not $|T|^+$-saturated.

Indeed, this second point was how Shelah used it in [Sh394] and Makkai and Shelah used it to prove the second part of the main theorem in [MaSh285] (Vasey has generalized this to AECs in [Vas, Theorem 3.3]).

Looking at $\mathbb{L}_{\lambda,\omega}$ (or some other logic), we can get variations by allowing Γ_0 and Γ_1 to be syntactic types (or even a mix). The proof is exaclty the same, except that the appropriate sort of type is used. The only property of types we really use are isormophism preservation.

Here is an interesting question: As I've tried to emphasize, the connection between the types of Γ_1 and the type omitted is weak. Indeed the uses of Shelah's Omitting Type Theorem I'm aware of never really use what the type actually is, just that there is an template that generates not $LS(K)^+$-saturated models. Is there some result that requires this fine analysis? Is there some tighter connection that one can use for the types omitted?

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