THE FEFERMAN-VAUGHT THEOREM

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We give a self-contained proof of the Feferman-Vaught theorem. Our presentation follows Chang-Keisler [CK, Proposition 6.3] and the proof is due to, of course, Feferman and Vaught [FV].

Fix a signature L throughout.

Definition 0.1. *A formula $\phi(x_1, \dots, x_n)$ from $\mathcal{L}(L)$ is $\mathcal{L}'$-*determined *by $\langle \sigma; \psi_1, \dots, \psi_k \rangle$ iff*

(1) *ψ_i is a formula from $\mathcal{L}'(L)$ with free variables in $x_1, \dots, x_n$;*
(2) *$\sigma(y_1, \dots, y_k)$ is a formula in $\mathcal{L}'(L_{BA})$ which is monotonic, i. e.,*

$$T_{BA} \vdash \forall z_1, \dots, z_k, z'_1, \dots, z'_k \left(\sigma(z_1, \dots, z_k) \wedge \bigwedge_i z_i \leq z'_i \to \sigma(z'_1, \dots, z'_k) \right)$$

(3) *If F is a filter over I, M_i are L-structures for each $i \in I$, and $f_1, \dots, f_n \in \prod M_i$, then*

$$\prod M_i/F \vDash \phi\left([f_1]_F, \dots, [f_n]_F\right) \iff \mathcal{P}(I)/F \vDash \sigma\left(X^1/F, \dots, X^k/F\right)$$

where $X^j := \{i \in I : M_i \vDash \psi_j\left(f_1(i), \dots, f_n(i)\right)\}$.

Theorem 0.2 (Feferman-Vaught)**.** *Every first-order formula is FO-determined.*

Proof: We first define the determining formulas (by induction) and then show they work.

Work by induction on $\phi(x_1, \dots, x_n)$:

- **Atomic:**
  ϕ is determined by $\langle y = 1; \phi \rangle$.
- **Negation:** ϕ is determined by $\langle \sigma(y_1, \dots, y_k); \psi_1, \dots, \psi_k \rangle$.
  $\neg\phi$ is determined by $\langle \neg\sigma(-y_1, \dots, -y_k); \neg\psi_1, \dots, \neg\psi_k \rangle$.
- **Conjunction:** For $\ell = 1, 2$, ϕ^ℓ is determined by $\langle \sigma^\ell(y_1, \dots, y_{k_\ell}); \psi^\ell_1, \dots, \psi^\ell_{k_\ell} \rangle$.
  WLOG, we enlarge the free variables of ϕ^1 and ϕ^2 to be the same. $\phi^1 \wedge \phi^2$ is determined by $\langle \sigma^1(y_1, \dots, y_{k_1}) \wedge \sigma^2(y_{k_1+1}, \dots, y_{k_1+k_2}); \psi^1_1, \dots, \psi^1_{k_1}, \psi^2_1, \dots, \psi^2_{k_2} \rangle$.

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- **Existential:** $\phi(x, x_1, \dots, x_n)$ is determined by $\langle \sigma; \psi_1, \dots, \psi_k \rangle$.
  Let $s_1, \dots, s_{2^k}$ enumerate $\mathcal{P}(\{1, \dots, k\})$ with $s_i = \{i\}$. For $1 \leq i \leq 2^k$, set
  $$\theta_i(x_1, \dots, x_n) := \exists x \bigwedge_{j \in s_i} \psi_j(x, x_1, \dots, x_n)$$
  with the convention that the empty conjunction is true. Set $\tau(y_1, \dots, y_{2^k})$ to be
  $$\exists z_1, \dots, z_{2^k} \left(\left(\bigwedge_i z_i \leq y_i \right) \wedge \left(\bigwedge_{s_i \cup s_j = s_\ell} z_i \cap z_j = z_\ell \right) \wedge \sigma(z_1, \dots, z_k) \right)$$
  Then $\exists x \phi(x, x_1, \dots, x_n)$ is determined by $\langle \tau; \theta_1, \dots, \theta_{2^k} \rangle$.

Now we show these sequences determine ϕ by induction. Property (1) is always obvious.

- **Atomic:** $\phi(x_1, \dots, x_n) = R(x_1, \dots, x_n)$
  Functions are handled similarly, although we cannot necessarily code functions as relations. $y = 1$ is clearly monotonic. For the other property,

$$\begin{aligned}
\prod M_i/F \vDash R([f_1]_F, \dots, [f_n]_F) &\iff X := \{i \in I : M_i \vDash R(f_1(i), \dots, f_n(i))\} \in F \\
&\iff 1 - X \in F^{dual} \\
&\iff \mathcal{P}(I)/F \vDash X/F = 1
\end{aligned}$$

- **Negation:** $\phi = \neg \psi$
  Suppose $\mathbf{z}, \mathbf{y}$ are such that $\neg\sigma(-\mathbf{y})$ and $y_i \leq z_i$. Then $-y_i \geq -z_i$ and, by the monotonicity of σ, we get $\neg\sigma(-\mathbf{z})$, as desired. Also note that
  $$\begin{aligned}
  X^j &= \{i \in I : M_i \vDash \neg\psi_j(f_1(i), \dots, f_n(i))\} \\
  &= -\{i \in I : M_i \vDash \psi(f_1(i), \dots, f_n(i))\} = -Y^j
  \end{aligned}$$
  Then

$$\begin{aligned}
\prod M_i/F \vDash \neg\phi([f_1]_F, \dots, [f_n]_F) &\iff M_i/F \nvDash \phi([f_1]_F, \dots, [f_n]_F) \\
&\iff \mathcal{P}(I)/F \nvDash \sigma(Y_1/F, \dots, Y_k/F) \\
&\iff \mathcal{P}(I)/F \vDash \neg\sigma(-X_1/F, \dots, -X_k/F)
\end{aligned}$$

- **Conjunction:** $\phi = \phi^1 \wedge \phi^2$
  The conjunction of monotonic formulas is monotonic.

$$\begin{aligned}
\prod M_i/F \vDash \phi([f_1]_F, \dots, [f_n]_F) &\iff \\
\left(\prod M_i/F \vDash \phi^1([f_1]_F, \dots, [f_n]_F)\right) \wedge \left(\prod M_i/F \vDash \phi^2([f_1]_F, \dots, [f_n]_F)\right) &\iff \\
\left(\mathcal{P}(I)/F \vDash \phi^1(X_1^1/F, \dots, X_1^{k_1})/F\right) \wedge \left(\mathcal{P}(I)/F \vDash \phi^2(X_2^1/F, \dots, X_2^{k_2})/F\right) &\iff \\
\mathcal{P}(I)/F \vDash \phi^1(X_1^1/F, \dots, X_1^{k_1}/F) \wedge \phi^2(X_2^1/F, \dots, X_2^{k_2}/F)
\end{aligned}$$

- **Existential:** $\phi = \exists x \psi(x, \mathbf{y})$
  τ is monotonic because $\leq$ is transitive.
  First, suppose that $\prod M_i/F \vDash \exists x \phi\left(x, [f_1]_F, \ldots, [f_n]_F\right)$. By induction, there is an $[f_0]_F \in \prod M_i/F$ such that
  $$\mathcal{P}(I)/F \vDash \sigma\left(X^0/F, \ldots, X^k/F\right)$$
  where $X^j := \{i \in I : M_i \vDash \phi_j\left(f_0(i), \ldots, f_n(i)\right)\}$. Setting $Y^\ell := \{i \in I : M_i \vDash \exists x \bigwedge_{j \in s_\ell} \psi_j\left(x, f_1(i), \ldots, f_n(i)\right)\}$, we see that $\bigcap_{j \in s_\ell} X^j \subset Y^\ell$. Thus, these intersection are the witness that $\mathcal{P}(I)/F \vDash \tau\left(Y^1/F, \ldots, Y^{2^k}/F\right)$.
  Second, suppose that $\mathcal{P}(I)/F \vDash \tau\left(Y^1/F, \ldots, Y^{2^k}/F\right)$. Then there are $Z^1, \ldots, Z^{2^k}$ such that
  $$\begin{array}{rcl} \mathcal{P}(I)/F & \vDash & Z^i/F \leq Y^i/F \\ \mathcal{P}(I)/F & \vDash & Z^i/F \cap Z^j/F = Z^\ell/F \\ \mathcal{P}(I)/F & \vDash & \sigma\left(Z^1/F, \ldots, Z^k/F\right) \end{array}$$
  By the finite completeness of F, there is $X \in F$ such that
  $$Z^i \cap X \subset Y^i$$
  $$Z^i \cap Z^j \cap X = Z^\ell \cap X$$
  If $i \in X$, set $t^i := \{j \leq k : i \in Z^j\}$. Then there is $m < 2^k$ such that $s_m = t^i$. Note that $i \in Y^m$. Then there is $m_i \in M_i$ such that $M_i \vDash \bigwedge_{j \in t^i} \psi_j\left(m_i, f_1(i), \ldots, f_n(i)\right)$. If $i \notin X$, let $m_i \in M_i$ arbitrary.
  Set $W^j := \{i \in I : M_i \vDash \psi_j\left(m_i, f_1(i), \ldots, f_n(i)\right)\}$. If $i \in Z^j \cap X$, then $j \in t^i$. This gives that $i \in W^j$. Thus, $Z^j/F \subset W^j/F$. By monotonicity, we get
  $$\begin{array}{rcl} \mathcal{P}(I)/F & \vDash & \sigma\left(Z^1/F, \ldots, Z^k/F\right) \\ \prod M_i/F & \vDash & \psi\left([i \mapsto m_i]_F, [f_1]_F, \ldots, [f_n]_F\right) \\ \prod M_i/F & \vDash & \exists x \psi\left(x, [f_1]_F, \ldots, [f_n]_F\right) \end{array}$$

$\dagger$

My interest in the Feferman-Vaught Theorem came from hoping that the ZFC existence of countably complete filters (as opposed to the harder question of countably complete ultrafilters) would lead to some interesting compactness-like results in nonelementary model theory.

However, this turns out not to be the case. Indeed, as we exhibit below, the countably complete filter product of torsion groups need not be torsion. Reexamining the proof, the key property used in proving that a countably complete ultraproduct preserves disjunctions is given any partition of a U-large set into countably many pieces, one of the pieces is U-large.

It is an easy exercise to show that this precisely characterizes countably complete ultrafilters.

Example 0.3. *Let F be the club filter on ω_1 and $\langle G_i : i < \omega_1 \rangle$ be torsion groups of infinite exponent. We claim that $\prod G_i / F$ is not torsion. First find countably many disjoint, stationary sets $\langle X_n : n < \omega \rangle$. Define a function $f \in \prod G_i$ by, if $i \in X_n$, picking $f(i) \in G_i$ to have order larger than n. In particular, $n \cdot f(i) \neq 0$ for all $i \in X_n$.*

By the Feferman-Vaught Theorem, for each $n < \omega$,

$$\prod G_i / F \vDash n \cdot [f]_F = 0$$

iff $\{i < \omega_1 : G_i \vDash n \cdot f(i) = 0\} \in F$. Since that set is disjoint from a stationary set, it contains no club. Thus, $\prod G_i / F$ is not torsion.

This example was particularly easy because the type consisted of atomic formulas.

References

[CK] C. C. Chang and H. J. Keisler, **Model Theory**. Studies in Logic **73**, North-Holland Publishing Company 1973.

[FV] S. Feferman and R. Vaught, *The first order properties of algebraic systems.* Fundamenta Mathematicae **47**, p. 57-103, 1959.

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