Tameness and Abstract Elementary Classes

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Outline

- Give a basic overview of AECs
- Discuss tameness and its applications
- Pose some open questions

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An important note: I've included some dates to give a sense of time frame, but there's some imprecision in the the mixing of publication dates and circulation of preprints dates, the latter being more common with more recent work.

Beyond First Order Model Theory

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4. classification over a predicate
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6. omitting types, infinitary logic, and extra quantifiers

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6. omitting types, infinitary logic, and extra quantifiers

Rather than exploring each class individually, the framework of Abstract Elementary Classes allows one to analyze them in a uniform manner.

What is an Abstract Elementary Class?

$(K, \prec_K)$ is an Abstract Elementary Class (AEC) iff

0. every element of K is a $L(K)$ structure;
1. $\prec_K$ is a partial order on K;
2. if $M \prec_K N$, then $M \subseteq N$;
3. $(K, \prec_K)$ respects $L(K)$ isomorphisms;
4. if $M_0 \prec_K M_2$, $M_1 \prec_K M_2$, and $M_0 \subseteq M_1$, then $M_0 \prec_K M_1$;
5. suppose $\langle M_i \in K : i < \alpha \rangle$ is a $\prec_K$-increasing continuous chain, then
    1. $\cup_{i<\alpha} M_i \in K$ and, for all $i < \alpha$, we have $M_i \prec_K \cup_{i<\alpha} M_i$; and
    2. if there is some $N \in K$ so that, for all $i < \alpha$, we have $M_i \prec_K N$, then we also have $\cup_{i<\alpha} M_i \prec_K N$.; and
6. *(Lowenheim-Skolem number)* $LS(K)$ is the minimal infinite cardinal above $|L(K)|$ so for any $M \in K$ and $A \subset M$, there is some $N \prec_K M$ such that $A \subset N$ and $\|N\| = |A| + LS(K)$.

Why Abstract Elementary Classes?

- The AEC axioms capture the model theoretic structure that exists *without the compactness theorem*.

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What's the point of AECs?

"Goal"

Shelah's Categoricity Conjecture: For every λ, there is μ_λ such that, if K is an AEC with $LS(K) = \lambda$ that is categorical in some cardinal $\geq \mu_\lambda$, then it is categorical in every cardinal $\geq \mu_\lambda$.

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Goal

To develop classification theory for AECs.

Shelah's Presentation Theorem

Everything I've said so far about AECs is semantic, but there is a syntactic description.

Theorem (Shelah's Presentation Theorem)

If K is an AEC with $LS(K) = \lambda$, then there is some $L_1 \supset L$ of size λ, an L_1-theory T_1, and a set Γ of quantifier free types such that

$$K = PC(T_1, \Gamma, L) := \{M_1 \upharpoonright L : M_1 \models T_1 \text{ and omits } \Gamma\}$$

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Note that PC classes themselves are pretty poorly behaved: they fail

- the chain axioms;
- the existence of an LS number; and
- Shelah's Categoricity Conjecture
  Silver showed there is a PC class that is categorical exactly at $\kappa = \beth_\alpha$ for α limit.

Convention and embeddings

- Writing $f : M \to N$ means that f is a K-embedding, i.e. $f(M) \prec N$.
- We write K for $(K, \prec_K)$ and $\prec$ for $\prec_K$.
- We assume that K has a monster model $\mathfrak{C}$.
  - $\mathfrak{C}$ is μ-model homogeneous for very large cofinality μ.
  - The existence is equivalent to amalgamation, joint embedding, and no maximal models (amalgamation is the key property most of the time)
  - Gives a very simplified definition of Galois types

Galois types

Definition

$A = \langle a_i : i \in I \rangle$ and $B = \langle b_i : i \in I \rangle$ have the same Galois type over M, written as $gtp(A/M) = gtp(B/M)$, iff

$$\textit{there is } f \in Aut_M \mathfrak{C} \textit{ so that } f(a_i) = b_i \textit{ for all } i \in I$$

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Definition

$$gS^\alpha(M) = \{gtp(\langle a_i : i < \alpha \rangle / M) : a_i \in \mathfrak{C}\}$$

If $M \prec N$ and $p = gtp(a/N)$, then $p \upharpoonright M = gtp(a/M)$.

This definition is purely semantic. In first order, they agree with semantic types.

Syntactict vs. Galois

- Syntactic types are very local.
- If two syntactic types differ, you can see this difference finitely: there is a finite parameter set and finite subset of the tuples that already witness the idfference

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Key Question

Do the restrictions of Galois types determine the type?

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- Give a basic overview of AECs
- **Discuss tameness** and its applications
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Tameness

As they say, a definition can't be wrong:

Definition (Grossberg-VanDieren, 2004?)

An AEC K is $< \kappa$-tame iff for all $M \in K$ and $p, q \in gS(M)$, the two following equivalent conditions hold:

- *if $p \neq q$, then there is $M^- \prec M$ of size $< \kappa$ such that $p \upharpoonright M^- \neq q \upharpoonright M^-$.*
- *if $p \upharpoonright M^- = q \upharpoonright M^-$ for all $M^- \prec M$ of size $< \kappa$, then $p = q$.*

"κ-tameness" is "$< \kappa^+$-tameness."
"$(< \kappa, \lambda)$-tameness" restricts the size of the domain to λ.

Tameness - A little history

- The first time something like tameness shows up is in [Sh394], where Shelah deduces *weak tameness* from categoricity and amalgamation
- Grossberg and VanDieren isolated κ-tameness in the course of the latter's thesis for an argument about Galois stability and later proved a categoricity transfer result from it
- Later authors (especially Baldwin) introduced various parameterizations and tweaks (locality, compactness, type shortness)

Variations of tameness

Tameness and locality are two of several "locality" properties for Galois types:

Definition

- *K is κ-local iff for all $M = \cup_{i<\kappa} M_i$ and $p \neq q \in gS(M)$, there is $j < \kappa$ such that $p \upharpoonright M_j \neq q \upharpoonright M_j$.*

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- *K is κ-type short iff for all X, Y of size κ and M such that $gtp(X/M) \neq gtp(Y/M)$, there is $X_0 \subset X$ and $Y_0 \subset Y$ such that $gtp(X_0/M) \neq gtp(Y_0/M)$.*

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- *K is κ-compact iff for all $M = \cup_{i<\kappa} M_i$, if $p_i \in gS(M_i)$ in increasing, then there is an upper bound $p \in gS(M)$.*

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- *These can also be parameterized based on the length of types involved.*

- Note I'm being vague about some of the other parameters: the length of tameness/locality/compactness and the size of the domain of type shortness

How far from syntactic are we?

- An important/powerful class of AECs are those that are $< \kappa$-tame and -type short for some κ
- In these classes, Galois types are determined by their restriction to small pieces, where 'small' means '$< \kappa$ sized'

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- Doing this allows many first order arguments built on formulas to be redone in the AEC context (more on this later)
- This intuition has recently been made explicit

Galois Morleyizations and tameness

Definition (Vasey)

Given K and κ, the $< \kappa$-Galois Morleyization is obtained by adding predicates of lengths less than κ for all $< \kappa$-Galois types over the emptyset.

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Proposition (Vasey, 2015)

K is $< \kappa$-tame and type short iff Galois types map bijectively to syntactic types in the $< \kappa$-Galois Morleyization.

Examples

Definition

An AEC K is $< \kappa$-tame iff for all $M \in K$ and $p, q \in S(M)$, if $p \neq q$, then there is $M^- \prec M$ of size $< \kappa$ such that $p \upharpoonright M^- \neq q \upharpoonright M^-$.

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- quasiminimal classes are $\aleph_0$-tame (Zilber)
- Hrushovski fusions are $\aleph_0$-tame (Villaveces-Zambrano, 2005)
- Homogeneous model theory is $\aleph_0$-tame
- ${}^{\perp}N$ is $\aleph_0$-tame when N is an abelian group (Baldwin-Ekloff-Trlifaj. 2007)
- torsion modules over a PID are $\aleph_0$-tame (B, 2014)
- classically valued fields are $\aleph_0$-tame (B, 2015)

General ways of getting tameness

Proposition

If K has a "nonforking-like" notion satisfying Uniqueness, Local Character, Base Monotonicity, and Invariance, then the class is tame (for some parameters depending on what the equivalent of $\kappa(\perp)$ *is).*

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If K has a "nonforking-like" notion satisfying Uniqueness, Local Character, Base Monotonicity, and Invariance, then the class is tame (for some parameters depending on what the equivalent of $\kappa(\perp)$ is).

Theorem (B, 2013)

Let K be an AEC essentially below κ.

- *If κ is weakly compact, then K is $(<\kappa, \kappa)$-tame.*
- *If κ is measurable, then K is $(<\lambda, \lambda)$-tame when cf $\lambda = \kappa$.*
- *If κ is nearly θ-strongly compact, then K is $(<\kappa, \theta)$-tame.*
- *If κ is strongly compact, then K is $<\kappa$-tame.*

Non-examples

- For each $k < \omega$, there is $\psi_k \in L_{\omega_1,\omega}$ that is $(\aleph_0, \aleph_k)$-tame, but not $(\aleph_k, \aleph_{k+1})$-tame. (Hart-Shelah 1990, Baldwin-Kolesnikov 2009)
- Short exact sequences of an almost free, non-free, non-Whitehead group of size κ are not $(< \kappa, \kappa)$ tame (Baldwin-Shelah 2008)
- The large cardinals used on the previous slide are near strict (Shelah 2013?, B-Unger 2015)

Large cardinals and eventual tameness

- Global tameness principles are closely connected with large cardinal principles
- Shelah has an example showing the following:
  - If regular κ has no θ^+-complete, uniform measure on it, there is K with $LS(K) = \theta^\omega$ that is not κ-local

Proposition

Suppose $\mu^\omega < \kappa$ for every $\mu < \kappa$.

$$\begin{pmatrix} \text{Every AEC with } LS(K) < \kappa \\ \text{is } \kappa\text{-local} \end{pmatrix} \iff \begin{pmatrix} \kappa \text{ is measurable or} \\ \text{a limit of measurables} \end{pmatrix}$$

Large cardinals and eventual tameness

Say κ is almost-strongly compact iff for every $\mu < \kappa$, every κ-complete filter can be extended to a μ-complete ultrafilter iff for all $\mu < \kappa \leq \lambda$, there is a μ-complete, fine ultrafilter on $P_\kappa \lambda$.

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Proposition (B-Unger, 2015)

1. *Suppose $\mu^\omega < \kappa$ for every $\mu < \kappa$.*

$$\begin{pmatrix} \text{Every AEC with } LS(K) < \kappa \\ \text{is } < \kappa\text{-tame} \end{pmatrix} \iff \begin{pmatrix} \kappa \text{ is almost-strongly} \\ \text{compact} \end{pmatrix}$$

2. $$\begin{pmatrix} \text{Every AEC is} \\ \text{eventually tame} \end{pmatrix} \iff \begin{pmatrix} \text{There are class many} \\ \text{almost-strongly compact cardinals} \end{pmatrix}$$

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- Give a basic overview of AECs
- **Discuss** tameness and **its applications**
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Applications of tameness

There are three areas of classification theory that tameness has seen application to:

- Categoricity transfer
- Nonforking
- Stability Transfer

Categoricity Transfer

Theorem (Grossberg-VanDieren, 2006ish)

Suppose K has a monster model, is χ-tame, and categorical in some $\lambda^+ > LS(K)^+ + \chi$. Then K is categorical in all $\mu \geq \lambda^+$.

This is Shelah's Categoricity Conjecture for successors in tame AECs with a monster model.

SCC from Large Cardinals

Corollary (B, 2013)

Suppose there are class many strongly compact cardinals. If an AEC is categorical in a successor cardinal above $\mu(LS(K)) = \min\{\kappa > LS(K) : \kappa \text{ is strongly compact }\}$*, then it is categorical in all* $\lambda \geq \mu(LS(K))$*.*

This uses results of Grossberg-VanDieren, Shelah, and Boney and a little more. Note that there is *no* monster model assumption.

Nonforking notions

- A main line of research is trying to find good notions of nonforking in various classes of AECs.
- Unfortunately, there's not (yet?) a single definition that specializes to all other in each circumstance.
- Still have some good results, especially when tameness holds (and a monster model exists)

Coheir

Definition

Define $A \underset{M_0}{\overset{(ch)}{\downarrow}} N$ *iff for all* $a \in {}^{<\kappa}A$ *and* $N^- \prec N$ *of size* $< \kappa$, $gtp(a/N^-)$ *is realized in* M_0.

This is like the first-order notion of coheir, replacing "finitely satisfiable" with "$< \kappa$ satisfiable."

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Definition

Define $A \underset{M_0}{\overset{(ch)}{\downarrow}} N$ *iff for all* $a \in {}^{<\kappa}A$ *and* $N^- \prec N$ *of size* $< \kappa$, $gtp(a/N^-)$ *is realized in* M_0.

Theorem (B-Grossberg, 2013)

Suppose $\kappa > LS(K)$. *If*

1. K *is fully* $< \kappa$*-tame and -type short;*
2. K *does not have the* κ*-order property; and*
3. $\overset{(ch)}{\downarrow}$ *satisfies existence/extension*

then $\overset{(ch)}{\downarrow}$ *is a "stable-like" independence relation.*

Coheir

If κ is strongly compact, then this is simpler.

Theorem (B-Grossberg)

Suppose $\kappa > LS(K)$ is strongly compact and $\overset{(ch)}{\downarrow}$ satisfiers existence. If K does not have the κ-order property, then κ is a "superstable-like" independence relation.

Existence (in this case) follows from categoricity λ with cf $\lambda > \kappa$.

Good λ-frames

- Shelah's focus in this area (especially recently) has been on good λ-frame $\mathbf{s}$. This is a "superstable-like" notion of nonforking in a single cardinal; also comes equipped with a notion of basic types.
- Two things are done: first, prove a frame exists in some cardinal and, second, try to transfer this to larger cardinals.
- The second part uses a construction $\geq \mathbf{s}$ that always exists, but doesn't always satisfy the desired properties

Both parts of this project have used non-ZFC combinatorics to get nonstructure results.

Tameness and frame existence

Previous results about the existence of frames in general required strong model and set theoretic hypotheses:

Theorem (Shelah, 2001)

If

- $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ *and WDmId*(λ^+) *is not* λ^{++}*-saturated;*
- K *is categorical in* λ *and* λ^+*; and*
- $1 \leq I(\lambda^{++}, K) < \mu_{unif}(\lambda^{++}, 2^{\lambda^+})$*;*

then there is a good λ^+*-frame for* K*.*

Basic types are λ-rooted minimal types, nonforking is if the base contains the root.
(Note: $\mu_{unif}(\lambda^{++}, 2^{\lambda^+})$ is "basically" $2^{\lambda^{++}}$.)

Tameness and frame existence

Tameness can replace the set-theoretic hypotheses *and* simplify the model-theoretic ones.

Theorem (Vasey, 2014)

Suppose K has a monster model, is μ-tame, and is categorical in λ with (1) $\text{cf}\,\lambda > \mu$ or (2) $\lambda > \mu = \beth_\mu$. Then K has a type-full good $\geq \lambda$-frame.

In (1), p does not fork over M iff there is $M_0 \prec M$ of size μ so that p does not μ-split over M_0. In (2), nonforking is μ-coheir.

Tameness and frame transfer

Previous results about the transfer of frames required strong model and set theoretic hypotheses:

Theorem (Shelah, pre-2009)

If K has a good λ-frame and

- $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ *and* $WDmId(\lambda^+)$ *is not* λ^{++}*-saturated; and*
- $I(\lambda^{++}, K(\lambda^+ - saturated)) < \mu_{unif}(\lambda^{++}, 2^{\lambda^+})$;

Then there is a good λ^+*-frame for* $(K', \prec')$*, where* $K'_{\lambda^+} \subset K_{\lambda^+}$ *and* $\prec' \subset \prec$.

Tameness and frame transfer

Proposition (B, 2013)

If K has amalgamation and a good λ-frame $\mathbf{s}$, then

$$\geq_{\mathbf{s}} \text{ satisfies Uniqueness iff } K \text{ is } \lambda\text{-tame for basic types}$$

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Theorem (B, B-Vasey 2014)

Suppose K has amalgamation and a good λ-frame $\mathbf{s}$ and is λ-tame. Then

1. $\geq \mathbf{s}$ *is a good frame;*

Tameness and frame transfer

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Theorem (B, B-Vasey 2014)

Suppose K has amalgamation and a good λ-frame $\mathbf{s}$ and is λ-tame. Then

1. $\geq \mathbf{s}$ *is a good frame;*
2. $(\geq \mathbf{s})^{<\infty}$ *is a good frame (i.e. independent sequences satisfy the nonforking properties); and*
3. K *is $(\lambda + |\alpha|)$-tame for basic types of length α.*

More independence relations

Vasey has some recent work that looks to get global independence relations from more natural hypotheses. One version is:

Theorem (Vasey 2015)

If K has a monster model, is $< \kappa$-tame and type short for $\kappa = \beth_\kappa > LS(K)$, and is categorical in $\mu > (\kappa^{<\kappa})^{+5}$, then there is a superstable-like global independence relations on models of size $\geq \mu$ and types of length $\leq (\kappa^\kappa)^{+6}$.

Stability Transfer

Applications of these concepts give rise to stability transfer results.

Theorem (Grossberg-VanDieren 2004ish)

If K is Galois stable in some $\mu > \beth_{(2^{LS(K)})^+}$ and χ-tame for $\chi < \mu$, then K is Galois stable in every $\kappa = \kappa^\mu$.

Theorem (Baldwin-Kueker-VanDieren 2006)

Let K with amalgamation be Galois stable in κ and κ-weakly tame. Then K is Galois stable in κ^{+n} for all $n < \omega$.

Theorem (Vasey 2014)

Suppose K is $< \chi$-tame and stable in some $\mu \geq \chi$. Then there is some $\kappa < \beth_{(2^\chi)^+}$ such that K is stable in all $\lambda \geq \mu$ such that $\lambda^{<\kappa} = \lambda$.

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Questions

- Tameness is a relatively recent notion (2004?)
- Still lots of unanswered questions and open problems

Structural Property vs. Model theoretic property

Is tameness a structural property/dividing line OR is it just a model theoretic property? That is, can we find some non-structure from non-tameness or is it just something that some AECs have and some don't.

Structural Property vs. Model theoretic property

Is tameness a structural property/dividing line OR is it just a model theoretic property? That is, can we find some non-structure from non-tameness or is it just something that some AECs have and some don't.

Vasey's result on frame existence can be rephrased as a partial answer in the good direction: Suppose K is an AEC with a monster model and is categorical in a high enough cardinal, then

$$K \text{ is } \mu\text{-tame for some } \mu$$
$$\text{iff}$$
$$K \text{ has a good } \geq \chi\text{-frame for some } \chi$$

More examples and applications

- Examples of AECs is a pretty underdeveloped field.
- Can we find more examples of non-tame AECs?
- Find some concrete (and mathematically interesting) AECs that are tame and apply the above independence relations/ideas.

Less global tameness principles

- We saw that global tameness principles were large cardinals in disguise.
- Is there any hope of getting ZFC tameness principles in "nice" classes of AECs?
  - e.g. the class can be defined recursively or in a particular descriptive set-theoretic class
- Phrased another way: All the known non-examples are pathological in one way or another. Is there a natural AEC that is not tame?

Length of Tameness

We could parameterize tameness based on the length of tuples.

Definition

K is κ tame for α-types iff for every $p, q \in gS^\alpha(M)$, if $p \neq q$, there is $M^- \prec M$ of size κ such that $p \neq q$.

Obviously, $\alpha < \beta$ and tameness for β-types implies tameness for α-types.

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The work of B-Vasey on independent sequences gives a partial (but unsatisfactory) answer.

Thanks

Any questions?