# Gromov–Witten Invariants

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## Some motivation: two problems

- How many rational curves of degree $d$ are there on a quintic threefold $Q \subset \mathbf{P}^4$?
  - define the quintic threefold
- How many rational curves of degree $d$ are there in $\mathbf{P}^2$ which pass through $3d-1$ general points?
  - this is the degree of the relevant Severi variety

## The first problem could make sense

We're thinking about degree-$d$ holomorphic maps $f : \mathbf{P}^1 \to Q$. The tangent space to the space of such maps is

$$H^0(\mathbf{P}^1, f^\star TQ)$$

Riemann–Roch says that we expect the dimension of the space of such maps to be 3

But this counts *parametrized* maps; we should regard two such maps $f_1$, $f_2$ as the same if they differ by a reparametrization of the domain.

Aut($\mathbf{P}^1$) is 3-dimensional, so we expect $\{f : \mathbf{P}^1 \to Q\}/\sim$ to consist of isolated points.

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## The second problem could make sense

General approach:

$$\{\text{maps from } n\text{-pointed curves to } X\}/\sim \quad \xrightarrow{\text{ev}_i} \quad X$$

We want to compute

$$\#\left(\text{ev}_1^{-1}(p_1) \cap \ldots \cap \text{ev}_n^{-1}(p_n)\right)$$

Applying Riemann–Roch again, we expect that

$$\dim_{\mathbf{C}}\{\text{maps}\}/\sim \quad = \quad n + (1-g)(\dim_{\mathbf{C}} X - 3) + \langle c_1(TX), d\rangle$$

Take $X = \mathbf{P}^2$, degree $= d$, $n = 3d - 1$. Then the expected dimension is $6d - 2$, so...

## Compactifying our spaces of maps

Since we want to intersect cycles, we should compactify our spaces of maps.

Model example: Deligne–Mumford space $\overline{\mathcal{M}}_{g,n}$.

- compactification of the space of smooth curves of genus $g$ with $n$ distinct marked points

Definition of $\overline{\mathcal{M}}_{g,n}$: we allow nodal curves, but require *stability*.

- geometrically meaningful compactification
- these are smooth orbifolds (or varieties for $g = 0$, $n$ odd)

Examples: $\overline{\mathcal{M}}_{0,4}$, $\overline{\mathcal{M}}_{0,5}$

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## Moduli spaces of stable maps

Stability for $\overline{\mathcal{M}}_{g,n}$ says "no infinitesimal automorphisms".

We mimic this definition, but work over the base $X$.

Definition of the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, d)$:

- introduced by Kontsevich (1994)
- construct via Hilbert schemes; get a stack / orbispace

Key property : if $X$ is a projective variety, $\overline{\mathcal{M}}_{g,n}(X, d)$ is compact.

## Good examples

$\overline{\mathcal{M}}_{0,n}(X, 0) = X \times \overline{\mathcal{M}}_{0,n}$

- check: this has the expected dimension

$\overline{\mathcal{M}}_{0,0}(\mathbf{P}^2, 2)$ is the space of complete conics in $\mathbf{P}^2$

- tiny subtlety: automorphisms

The moduli spaces $\overline{\mathcal{M}}_{0,n}(\mathbf{P}^N, d)$ are smooth orbifolds and have the expected dimension.

- see *e.g.* Fulton–Pandharipande, *Notes on stable maps...*

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## Bad examples

$\overline{\mathcal{M}}_{1,1}(X, 0) = X \times \overline{\mathcal{M}}_{1,1}$

- virtual dimension = 1
- dimension = $\dim X + 1$

$\overline{\mathcal{M}}_{1,0}(\mathbf{P}^2, 3)$

- "compactifying strata" have bigger dimension than the "main stratum"

In the general (non-convex) case, spaces of stable maps are usually also non-reduced and singular.

# Properties

$\overline{\mathcal{M}}_{g,n}(X,d)$ is compact.

In the case where $X = \mathbf{P}^N$ and $g = 0$:

- $\overline{\mathcal{M}}_{0,n}(X,d)$ is a smooth orbifold of the expected dimension
- the set $\mathcal{M}_{0,n}(X,d)$ of stable maps from smooth curves is open
- the complement is a divisor with normal crossings
- the set $\overline{\mathcal{M}}^{*}_{0,n}(X,d)$ of automorphism-free stable maps is open

In general, all this remains "virtually true".

- virtual fundamental class: Li–Tian, Behrend–Fantechi

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# Gromov–Witten invariants

Definition

Example:

$$\int_{\overline{\mathcal{M}}_{0,3}(X,0)} \mathrm{ev}_1^\star\alpha \wedge \mathrm{ev}_2^\star\beta \wedge \mathrm{ev}_3^\star\gamma = \int_X \alpha\wedge\beta\wedge\gamma$$

These are the structure constants for the cup product with respect to the Poincaré pairing.

Example:

$$\int_{\overline{\mathcal{M}}_{0,3d-1}(\mathbf{P}^2,d)} \mathrm{ev}_1^\star P^2 \wedge \mathrm{ev}_2^\star P^2 \ldots \wedge \mathrm{ev}_{3d-1}^\star P^2$$

This gives the number of degree-$d$ rational curves in $\mathbf{P}^2$ through $3d-1$ general points.

# Topologically twisted non-linear sigma models

Fano or Calabi–Yau manifold $X \longrightarrow$ topologically twisted NL$\sigma$M

NL$\sigma$M: fields are maps $f : \Sigma \to X$ (bosonic) plus sections of spin bundles on the Riemann surface $\Sigma$ (fermionic).

topological twisting: modify fields $\longrightarrow$ supersymmetry

consequences:

- correlation functions of physical operators are independent of the metric on $\Sigma$, so this is a 'topological field theory''
- physical states $\longleftrightarrow$ cohomology classes on $X$
- get an associative product on the space of physical states

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# Algebra structure: what?

Pick a basis $\phi_1, \ldots, \phi_N$ for $H^\star(X)$, so that $t \in H^\star(X)$ is

$$t = t^1 \phi_1 + \ldots + t^N \phi_N$$

Define the *genus-zero GW potential* $F^0 : H^\star(X) \to \mathbf{C}[[Q]]$ by

$$\Phi(t) = \sum_{n,d} \sum_{i_1,\ldots,i_n} \frac{Q^d \, t_{i_1} \ldots t_{i_n}}{n!} \int_{\overline{\mathcal{M}}_{0,n}(X,d)} \mathrm{ev}_1^\star \phi_{i_1} \wedge \ldots \wedge \mathrm{ev}_n^\star \phi_{i_n}$$

This is a formal series in $t^1, \ldots, t^N$ and $Q$ whose Taylor coefficients are genus-zero Gromov–Witten invariants.

Let $g_{ab} = (\phi_a, \phi_b)$ — Poincaré pairing — and $\partial_a = \frac{\partial}{\partial t^a}$. Then

$$\phi_a \star \phi_b = C_{ab}{}^c(t) \phi_c$$

where $C_{ab}{}^c(t) = \partial_a \partial_b \partial_k \Phi(t) g^{kc}$.

# Algebra structure: why?

This algebra is manifestly commutative: $\partial_a\partial_b\partial_k\Phi(t) = \partial_b\partial_a\partial_k\Phi(t)$.

For associativity, we need:

$$\partial_a\partial_b\partial_k\Phi(t)g^{kl}\partial_l\partial_c\partial_d\Phi(t) = \partial_a\partial_d\partial_k\Phi(t)g^{kl}\partial_l\partial_b\partial_c\Phi(t)$$

There is a forgetful map $\mathrm{ct} : \overline{\mathcal{M}}_{0,n+4}(X,d) \to \overline{\mathcal{M}}_{0,4}$.

Now $\partial_a\partial_b\partial_c\partial_d\Phi(t)$ is

$$\sum \frac{Q^d t_{i_1}\dots t_{i_n}}{n!}\int_{\overline{\mathcal{M}}_{0,n+4}(X,d)} \mathrm{ev}_1^\star\phi_a\wedge\mathrm{ev}_2^\star\phi_b\wedge\mathrm{ev}_3^\star\phi_c\wedge\mathrm{ev}_4^\star\phi_d\wedge\mathrm{ev}_5^\star\phi_{i_1}\wedge\dots\wedge\mathrm{ev}_{n+4}^\star\phi_{i_n}$$

Consider

$$\sum \frac{Q^d t_{i_1}\dots t_{i_n}}{n!}\int_{\overline{\mathcal{M}}_{0,n+4}(X,d)} (\dots\text{as above}\dots)\cap[\mathrm{ct}^{-1}(\lambda)]$$

for $\lambda \in \overline{\mathcal{M}}_{0,4} \cong \mathbf{P}^1$ and specialize to $\lambda = 0$, $\lambda = \infty$.

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# Algebra structure: so what?

For $\mathbf{P}^2$, the potential $\Phi(x,y,z)$ is

$$\sum_{a,b,c,d\geq 0} \frac{Q^d x^a y^b z^c}{a!b!c!}\int_{\overline{\mathcal{M}}_{0,a+b+c}(\mathbf{P}^2,d)} \overbrace{\mathrm{ev}_1^\star 1\wedge\dots}^{a}\wedge\overbrace{\mathrm{ev}_{a+1}^\star P\wedge\dots}^{b}\wedge\overbrace{\mathrm{ev}_{a+b+1}^\star P^2\wedge\dots}^{c}$$

The degree-zero part is $\frac{1}{2}x^2z + \frac{1}{2}xy^2$.

There are no other $x$'s: compute

$$\int_{\overline{\mathcal{M}}_{0,a+b+c}(\mathbf{P}^2,d)} \mathrm{ev}_1^\star 1 \wedge \dots$$

via

$$\overline{\mathcal{M}}_{0,a+b+c}(\mathbf{P}^2,d) \to \overline{\mathcal{M}}_{0,a+b+c-1}(\mathbf{P}^2,d) \to \mathrm{pt}$$

# $\mathbf{P}^2$ example (continued)

Also,

$$\int_{\overline{\mathcal{M}}_{0,a+b+c}(\mathbf{P}^2,d)} \mathrm{ev}_1^\star P \wedge (\text{stuff}) = d \int_{\overline{\mathcal{M}}_{0,a+b+c-1}(\mathbf{P}^2,d)} (\text{stuff})$$

so

$$\Phi(x,y,z) = \frac{1}{2}x^2 z + \frac{1}{2}xy^2 + \sum_{d>0} Q^d e^{dy} \frac{z^{3d-1}}{(3d-1)!} N_d$$

where $N_d$ is the number of rational curves of degree $d$ in $\mathbf{P}^2$ which pass through $3d-1$ general points.

Write

$$\varphi(x,y,z) = \sum_{d>0} Q^d e^{dy} \frac{z^{3d-1}}{(3d-1)!} N_d$$

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# $\mathbf{P}^2$ example (continued)

The WDVV (associativity) equations are equivalent to

$$\varphi_{zzz} = \varphi_{yyz}^2 - \varphi_{yyy}\varphi_{yzz}$$

This gives the recursion

$$N(d) = \sum_{k+l=d} N(k)N(l)k^2 l \left[ l \binom{3d-4}{3k-2} - k \binom{3d-4}{3k-1} \right]$$

for $d \geq 2$.

Since $N(1) = 1$, we can solve:

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | |
|---|---|---|---|---|---|---|---|---|---|
| $N(d)$ | 1 | 1 | 12 | 620 | 87304 | 26312976 | 14616808192 | 13525751027392 | $\cdots$ |

# Mirror symmetry

Back to our first problem: counting curves on a quintic threefold.

Mirror symmetry (after Witten, Vafa, Hori): equivalence of the topologically twisted NL$\sigma$M with a topologically twisted Landau–Ginzburg model.

In our Calabi–Yau case (Candelas, de la Ossa, Green, Parkes, Greene, Plesser, Morrison,. . . ):

topologically twisted NL$\sigma$M with target $Q$

$\updownarrow$

*B-twisted* NL$\sigma$M with target $Q'$, the "mirror of $Q$"

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# Why this helps

Recall that the *coefficients* of the associativity equations are defined in terms of Gromov–Witten invariants of $Q$.

*Solutions* to analogous differential equations on the mirror side can be written in terms of periods of $Q'$

$$\int_{\Gamma \subset Q'} \Omega$$

where $\Omega$ is the Calabi–Yau form on $Q'$.

These satisfy Picard–Fuchs differential equations, so we can compute them.

## Open problems

Find a satisfactory mathematical formulation of mirror symmetry

Higher-genus Gromov–Witten invariants:

- how to compute them
- their connection to enumerative geometry
- Gopakumar–Vafa conjecture

Connection to integrable systems

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## Thank you for coming