Computation of Infinite Sums by Residues and Partial Fraction Expansion of Meromorphic Functions

One kind of definite integrals which we can compute by methods of residues is

$$\int_{x=-\infty}^{\infty} \frac{P(x)}{Q(x)} \, dx,$$

where $P(x)$ and $Q(x)$ are polynomials with the degree of $Q(x)$ at least 2 more than that of $P(x)$ under the additional assumption that $Q(x)$ is nowhere zero on $\mathbb{R}$.

We now discuss the application of the methods of residues to discrete infinite sums instead of definite integrals, for example, to compute explicitly the infinite sum

$$\sum_{n=-\infty}^{\infty} \frac{P(n)}{Q(n)},$$

where $P(x)$ and $Q(x)$ are polynomials with the degree of $Q(x)$ at least 2 more than that of $P(x)$ under the additional assumption that $Q(n)$ is nonzero for any $n \in \mathbb{Z}$.

The idea is to use a meromorphic function $f(z)$ on $\mathbb{C}$ whose poles include $z = n$ for $n \in \mathbb{Z}$ and to use a sequence of contours $C_n$ with the property that the domain enclosed by $C_n$ is increasing as $n$ increases and approach $\mathbb{C}$ as $n \to \infty$ and

$$\lim_{n \to \infty} \int_{C_n} f(z) \, dz = 0$$

and the residue of $f(z)$ at $z = n$ is $\frac{P(n)}{Q(n)}$, so that

$$\sum_{n=-\infty}^{\infty} \frac{P(n)}{Q(n)}$$

is equal to the negative of the sum of the residues of $f(z)$ at poles other than the points of $\mathbb{Z}$.

We have to choose a meromorphic function $f(z)$ on $\mathbb{C}$ whose residue at $z = n$ is $\frac{P(n)}{Q(n)}$. One choice is the function

$$\frac{P(z)}{Q(z)} \pi \cot \pi z$$
because
\[ \pi \cot \pi z = \cos \pi z \frac{\pi}{\sin \pi z}, \]
has a simple pole at \( z = n \) with residue 1, from \( \cos \pi n = (-1)^n \) and
\[ \lim_{z \to n} \pi (z - n) \frac{\pi}{\sin \pi z} = \lim_{z \to n} \frac{\pi (z - n)}{(-1)^n \sin \pi (z - n)} = (-1)^n. \]

If \( Q(n) \) is nonzero for \( n \in \mathbb{Z} \), we conclude that \( z = n \) is a simple pole for
\[ \frac{P(z)}{Q(z)} \pi \cot \pi z \]
with residue precisely equal to \( \frac{P(n)}{Q(n)} \).

For the contour \( C_n \) we use the square with vertices at \( (n + \frac{1}{2})(\pm 1 \pm i) \). Observe that when \( |y| > \frac{1}{2\pi} \) we have
\[ |\cot \pi z| \leq \left| \frac{e^{2i\pi y} + 1}{e^{2i\pi y} - 1} \right| \leq \frac{e^{2\pi y} + 1}{e^{2\pi y} - 1} = 1 + \frac{2}{e^{2\pi y} - 1} \leq 1 + \frac{2}{e - 1} \]
and hence uniformly bounded. Also observe that \( \cot \pi z \) is bounded on the line segment joining \( \frac{1}{2}(1 - i) \) to \( \frac{1}{2}(1 + i) \) and we can use the periodicity
\[ \cot \pi (z + 1) = \cot \pi z \]
of \( \cot \pi z \) with period 1 to conclude that \( \cot \pi z \) is uniformly bounded on \( C_n \).
From the assumption that the degree of \( Q(z) \) is at least 2 more than the degree of \( P(z) \) it now follows that
\[ \lim_{n \to \infty} \int_{C_n} \frac{P(z)}{Q(z)} \pi \cot \pi z \, dz = 0. \]
Finally from the residue theorem (which now simply says that the sum of all the residues is zero) we have the formula
\[ \sum_{n=-\infty}^{\infty} \frac{P(n)}{Q(n)} = - \sum_{j=1}^{k} \text{Res}_{z=a_j} \left( \frac{P(z)}{Q(z)} \pi \cot \pi z \right), \]
where \( a_1, \ldots, a_k \) are the distinct zeroes of the polynomial \( Q(z) \) (i.e., each zero being counted only once by ignoring its multiplicity).
As a simple example, we compute
\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2}
\]
for \(a > 0\). The two points whose residue for
\[
\frac{1}{z^2 + a^2} \pi \cot \pi z
\]
we have to compute are the two zeroes \(ai\) and \(-ai\) of \(z^2 + a^2\). Since both poles are simple, we have
\[
\text{Res}_{z=ai} \left( \frac{1}{z^2 + a^2} \pi \cot \pi z \right) = \lim_{z \to ai} \left( \frac{z - ai}{z^2 + a^2} \pi \cot \pi z \right) = \left( \frac{1}{z + ai} \pi \cot \pi z \right)_{z=ai} = \pi \cot \pi ai = \frac{2ai}{\pi i(e^{\pi ai} + e^{-\pi ai})} = \frac{\pi}{2a} e^{\pi a} - e^{-\pi a} = -\frac{\pi}{2a} \coth \pi a
\]
and, with \(a\) replaced by \(-a\) and the odd property of the function \(z \mapsto \coth \pi z\),
\[
\text{Res}_{z=-ai} \left( \frac{1}{z^2 + a^2} \pi \cot \pi z \right) = -\frac{\pi}{2a} \coth \pi a.
\]
Hence
\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = - \left( \text{Res}_{z=ai} \left( \frac{1}{z^2 + a^2} \pi \cot \pi z \right) + \text{Res}_{z=-ai} \left( \frac{1}{z^2 + a^2} \pi \cot \pi z \right) \right) = - \left( -\frac{\pi}{2a} \coth \pi a - \frac{\pi}{2a} \coth \pi a \right) = \frac{\pi \coth \pi a}{a}.
\]
Another kind of infinite sum which can be similarly computed in an explicit way is
\[
\sum_{n=-\infty}^{\infty} (-1)^n \frac{P(n)}{Q(n)}.
\]
where $P(x)$ and $Q(x)$ are polynomials with the degree of $Q(x)$ at least 2 more than that of $P(x)$ under the additional assumption that $Q(n)$ is nonzero for any $n \in \mathbb{Z}$. For this kind of infinite sum the meromorphic function $f(z)$ to be used is modified to be

$$
\frac{P(z)}{Q(z)} \pi \csc \pi z,
$$

because $\cos \pi z$ is $(-1)^n$ at $z = n$. The same contour $C_n$ of the square with vertices at $(n + \frac{1}{2})(\pm 1 \pm i)$ is used.

$$
\text{Res}_{z=n} \left( \frac{P(z)}{Q(z)} \pi \csc \pi z \right) = \lim_{z \to n} \frac{P(z)}{Q(z)} \frac{\pi(z-n)}{\sin \pi z}
= \frac{P(n)}{Q(n)} \lim_{z \to n} \frac{\pi(z-n)}{(-1)^n \sin \pi(z-n)}
= (-1)^n \frac{P(n)}{Q(n)}.
$$

When $y > \frac{1}{\pi}$ we have

$$
|\csc \pi z| = \left| \frac{2i}{e^{i\pi y} - e^{-i\pi y}} \right|
= \left| \frac{2}{e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{\pi y}} \right|
\leq \frac{2}{e^{\pi y} - e^{-\pi y}} \leq \frac{2}{e - e^{-1}}.
$$

When $y < -\frac{1}{\pi}$ we have

$$
|\csc \pi z| = \left| \frac{2i}{e^{i\pi y} - e^{-i\pi y}} \right|
= \left| \frac{2}{e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{\pi y}} \right|
\leq \frac{2}{e^{-\pi y} - e^{\pi y}} \leq \frac{2}{e - e^{-1}}.
$$

Hence cosec $\pi z$ is uniformly bounded on $|y| > \frac{1}{\pi}$. Moreover, cosec $\pi z$ is bounded on the line segment joining $\frac{1}{2}(1 - i)$ to $\frac{1}{2}(1 + i)$ and we can use the periodicity

$$
|\csc \pi (z + 1)| = |\csc \pi z|
$$
of $|\csc \pi z|$ with period 1 to conclude that $\csc \pi z$ is uniformly bounded on $C_n$. From the assumption that the degree of $Q(z)$ is at least 2 more than the degree of $P(z)$ it now follows that

$$\lim_{n \to \infty} \int_{C_n} \frac{P(z)}{Q(z)} \pi \csc \pi z \, dz = 0.$$  

Finally from the residue theorem (which now simply says that the sum of all the residues is zero) we have the formula

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{P(n)}{Q(n)} = -\sum_{j=1}^{k} \text{Res}_{z=a_j} \left( \frac{P(z)}{Q(z)} \pi \csc \pi z \right),$$

where $a_1, \cdots, a_k$ are the distinct zeroes of the polynomial $Q(z)$ (i.e., each zero being counted only once by ignoring its multiplicity).

**Infinite Product Expansion of Sine Function.** The identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi \coth \pi a}{a},$$

which was derived for $a > 0$, holds when $a$ is replaced by any complex number $z$, because the left-hand side of

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + z^2} = \frac{\pi \coth \pi z}{z}$$

defines a meromorphic function on $\mathbb{C}$ and the identity simply follows from applying the identity theorem the meromorphic function which is the difference of the two sides. Note that the coefficients of a Laurent series with an isolated singularity are computed along a circle centered at the isolated singularity so that the identity theorem applied to the complement of its poles determine a meromorphic function completely. To make it more convenient to factor the denominator of each term on the left-hand side, we replace $z$ by $iz$ to get

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 - z^2} = \frac{\pi \coth \pi iz}{iz}$$

$$= \frac{\pi(e^{\pi iz} + e^{-\pi iz})}{iz(e^{\pi iz} - e^{-\pi iz})}$$

$$= -\frac{\pi \cot \pi z}{z}$$
or 
\[ \pi \cot \pi z = \sum_{n=-\infty}^{\infty} \frac{z}{z^2 - n^2}, \]
which can be rewritten as
\[ \pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z + n} + \frac{1}{z - n} \right) \]
\[ = \frac{1}{z} + \sum_{n \in \mathbb{Z} - \{0\}} \left( \frac{1}{z - n} + \frac{1}{n} \right). \]

The identity 
\[ \pi \cot \pi z = \frac{1}{z} + \sum_{n \in \mathbb{Z} - \{0\}} \left( \frac{1}{z - n} + \frac{1}{n} \right) \]
is the partial fraction expansion of the cotangent function or the expansion of the cotangent function into a sum of its principal parts. Since
\[ \frac{d}{dz} (\log \sin \pi z \pi z) = \pi \cot \pi z, \]
we can rewrite the above partial fraction expansion of the cotangent function as
\[ \frac{d}{dz} \left( \log \frac{\sin \pi z}{\pi z} \right) = \sum_{n \in \mathbb{Z} - \{0\}} \left( \frac{1}{z - n} + \frac{1}{n} \right). \]

Using
\[ \int_{\zeta=0}^{\zeta=z} \left( \frac{1}{\zeta - n} + \frac{1}{n} \right) d\zeta = \log \left( \left( 1 - \frac{z}{n} \right) e^{\frac{z}{n}} \right), \]
we can integrate both sides of
\[ \frac{d}{dz} \left( \log \frac{\sin \pi z}{\pi z} \right) = \sum_{n \in \mathbb{Z} - \{0\}} \left( \frac{1}{z - n} + \frac{1}{n} \right) \]
from the origin to \( z \) and then exponentiate the results of both sides, we obtain
\[ \frac{\sin \pi z}{\pi z} = C \prod_{n \in \mathbb{Z} - \{0\}} \left( 1 - \frac{z}{n} \right) e^{\frac{z}{n}}, \]
where $C$ is a constant to be determined. By setting $z = 0$, we determine $C$

to be 1 and get the following factorization of $\sin \pi z$ as an infinite product
\[
\sin \pi z = \pi z \prod_{n \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}.
\]
The convergence of the infinite product is interpreted as the convergence of
the infinite sum whose terms are the logarithm of the corresponding factors
in the infinite product.

By using the cosecant function to sum up
\[
\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2},
\]
for $a > 0$ in a completely analogous manner, we obtain the following partial
fraction expansion
\[
\csc z = \frac{1}{z} + \sum_{n \in \mathbb{Z} \setminus \{0\}} (-1)^n \left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right)
\]
of the cosecant function. In general, for a certain kind of meromorphic func-
tion on $\mathbb{C}$ we can use the reside theorem to get its partial fraction expansion
as follows.

**Partial Fraction Expansion of Meromorphic Functions**

Suppose $f(z)$ is a meromorphic function on $\mathbb{C}$ whose poles \(\{a_n\}_{1 \leq n \leq n < \infty}\)
are simple with $0 < |a_1| \leq |a_2| \leq \cdots$ so that the residue of $f(z)$ at $a_n$ is $b_n$. Suppose that there is a sequence of closed contours $C_n$ such that the
enclosure of $C_n$ includes $a_1, \cdots, a_n$ but no other poles. Assume that the
distance $R_n$ from $C_n$ to the origin goes to infinity as $n \to \infty$ and the length
$L_n$ of $C_n$ is of the order $O(R_n)$. Assume that on $C_n$ we have $f(z) = o(R_n^{p+1})$.
We are going to apply the theorem of residue to the integral
\[
\frac{1}{2\pi i} \int_{C_n} f(w) \frac{dw}{w^{p+1}(w - z)}.
\]
The residue at $w = 0$ is obtained by expanding $\frac{f(w)}{w - z}$ in Laurent series in
$w$ around $w = 0$. We have
\[
\frac{f(w)}{w^{p+1}(w - z)} = \frac{-1}{w^{p+1}} \left(\frac{1}{z} + \frac{w}{z^2} + \frac{w^2}{z^3} + \cdots\right) \left(f(0) + f'(0)w + \frac{1}{2}f''(0)w^2 + \cdots\right)
\]
and the coefficient of $\frac{1}{w}$ is

\[-\frac{1}{z} \left( \frac{f(0)}{z^p} + \frac{f'(0)}{z^{p-1}} + \cdots + \frac{f^{(p)}(0)}{p!} \right) .\]

The residue at $w = z$ is given by $f(z)$. The residue at $a_n$ is $\frac{b_n}{a_n^{p+1}(a_n - z)}$. Since as $n \to \infty$ the integral becomes zero, we get

\[-\frac{1}{z} \left( \frac{f(0)}{z^p} + \frac{f'(0)}{z^{p-1}} + \cdots + \frac{f^{(p)}(0)}{p!} \right) + \frac{f(z)}{z^{p+1}} + \sum_{n=1}^{\infty} \frac{b_n}{a_n^{p+1}(a_n - z)} = 0\]

which means that

\[f(z) = f(0) + zf'(0) + \cdots + \frac{z^p}{p!}f^{(p)}(0) + \sum_{n=1}^{\infty} \frac{b_n z^{p+1}}{a_n^{p+1}(a_n - z)}\]

\[= \sum_{\nu=0}^{p} \frac{z^\nu}{\nu!} f^{(\nu)}(0) + \sum_{n=1}^{\infty} b_n \left( \frac{1}{z - a_n} + \frac{1}{a_n} + \frac{z^2}{z^2} + \cdots + \frac{z^p}{a_n^{p+1}} \right) .\]

The last expression comes from writing $z^{p+1}$ as $(z^{p+1} - a_n^{p+1}) + a_n^{p+1}$ and then factoring $z^{p+1} - a_n^{p+1} = (z - a_n) \sum_{\nu=0}^{p} z^\nu a_n^{-\nu}$.

**Infinite Product Expansion of Gamma Function.** The Gamma function $\Gamma(z)$ which is originally defined for $\text{Re } z > 0$ by

\[\Gamma(z) = \int_{t=0}^{\infty} t^{z-1} e^{-t} dt\]

can be extended to a meromorphic function to all of $\mathbb{C}$ from the functional equation

\[\Gamma(z + 1) = z\Gamma(z)\]

by defining

\[\Gamma(z) = \frac{\Gamma(z + 1)}{z} = \frac{\Gamma(z + 2)}{(z + 1)z} = \cdots = \frac{\Gamma(z + n)}{(z + n - 1)(z + n - 2) \cdots (z + 1)z}\]

for $\text{Re } z > -n$. Inductively on $n$, starting from $n = 0$, the residue of $\Gamma(z)$ at $z = 0$ is $1 = \Gamma(1)$ from

\[\text{Res}_{z=0} \Gamma(z) = \lim_{z \to 0} (z\Gamma(z)) = \lim_{z \to 0} \Gamma(z + 1) = G(1) = 1.\]
and the residue of $\Gamma(z)$ at $z = -n$ is computed by

$$\text{Res}_{z=-n} \Gamma(z) = \lim_{z \to -n} (z + n) \Gamma(z)) = \lim_{z \to -n} \frac{\Gamma(z + n + 1)}{(z + n - 1) \cdots (z + 1)z} = \frac{(-1)^n}{n!}.$$ 

**Euler’s reflection law for the Gamma function**

$$\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin \pi z}$$

holds on $\mathbb{C}$ as an equation for meromorphic functions, because

$$\Gamma(x) \Gamma(1 - x) = \frac{\pi}{\sin \pi x}$$

for $0 < x < 1$ and because of the identity theorem for meromorphic functions.

Euler’s reflection law for the Gamma function suggests an infinite product expansion for the Gamma function which consists of half of the factors in the infinite product expansion for the sine function. One way to get an infinite product expansion of the Gamma function is to get an infinite sum expansion for the logarithmic derivative $\frac{\Gamma'(z)}{\Gamma(z)}$ of the Gamma function and then, by integrating and then exponentiating, get an infinite product expansion for the Gamma function. One way to get an infinite sum expansion for $\frac{\Gamma'(z)}{\Gamma(z)}$ is to replace the factor $\Gamma(z - h)$ in the Beta function formula

$$\frac{\Gamma(z - h) \Gamma(h)}{\Gamma(z)} = \int_0^1 (1 - t)^{z-h-1} t^{h-1} dt$$

by its Taylor expansion

$$\Gamma(z - h) = \Gamma(z) - h \Gamma'(z) + \frac{(-h)^2}{2} \Gamma''(z) + \cdots$$

and then to equate the coefficients of like powers of $h$ on both sides of the equation. More precisely, for $\text{Re } z > h > 0$ with $z$ fixed and $h$ variable, we use

$$\frac{1}{h} = \int_0^1 t^{h-1} dt$$

to get

$$\frac{\Gamma(z - h) \Gamma(h)}{\Gamma(z)} = \int_0^1 (1 - t)^{z-h-1} t^{h-1} dt = \frac{1}{h} + \int_0^1 ((1 - t)^{z-h-1} - 1) t^{h-1} dt.$$
The integral on the right-hand side now is holomorphic for $h$ in a small open neighborhood of 0 in $\mathbb{C}$ and we can write it as its value at $h = 0$ plus a term of the order $o(h)$ as $h \to 0$. Thus

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} = \frac{1}{h} + \int_0^1 ((1-t)^{z-1} - 1) t^{-1} dt + o(h).$$

This is the Laurent expansion of the Beta function $B(z-h, z)$ in the variable $h$ at $h = 0$.

We compare this to the Laurent series expansion of

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}$$

in $h$ and get

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} = \frac{1}{\Gamma(z)} (\Gamma(z) - h \Gamma'(z) + \cdots) \left( \frac{1}{h} + A + \cdots \right),$$

where $A$ is a constant. Equating the constant terms of

$$\frac{1}{\Gamma(z)} (\Gamma(z) - h \Gamma'(z) + \cdots) \left( \frac{1}{h} + A + \cdots \right)$$

we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^1 (1 - (1-t)^{z-1}) t^{-1} dt - A$$

for $\text{Re} \ z > 0$. Using

$$\frac{1}{t} = \frac{1}{1 - (1-t)} = \sum_{n=0}^{\infty} (1-t)^n,$$

we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = -A + \int_0^1 (1 - (1-t)^{z-1}) \left( \sum_{n=0}^{\infty} (1-t)^n \right) dt$$
\[= -A + \int_0^1 \left( \sum_{n=0}^{\infty} \left((1 - t)^n - (1 - t)^{n+z-1}\right) \right) dt\]
\[= -A + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+z} \right).\]

We can rewrite it as
\[\frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+z} \right) - C\]
for some constant \(C\). To determine the constant \(C\), we integrate and take exponents of both sides and get
\[\frac{1}{\Gamma(z)} = e^{Cz} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}}.\]

Setting \(z = 1\), we get
\[1 = e^{C} \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right) e^{-\frac{1}{n}}.\]

Hence
\[C = -\log \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right) e^{-\frac{1}{n}} = \lim_{N \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} - \log N \right)\]
which is equal to the Euler constant \(\gamma\). We have finally the following infinite product decomposition for \(\Gamma(z)\).
\[\frac{1}{\Gamma(z)} = e^{\gamma z} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}}.\]

We can put together the infinite product decomposition of \(\frac{1}{\Gamma(z)}\) and \(\frac{1}{\Gamma(1-z)}\) and use Euler’s reflection formula for the Gamma function to get recover the
infinite product expansion of the sine function as follows. From

\[
\frac{\sin \pi z}{\pi} = \frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{1}{\Gamma(z)(-z)\Gamma(-z)}
\]

\[
= \frac{1}{(-z)} \left( e^{\gamma z} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}} \right) \left( e^{\gamma(-z)} \prod_{n=1}^{\infty} \left( 1 + \frac{-z}{-n} \right) e^{-\frac{-z}{n}} \right)
\]

\[
= z \prod_{\mathbb{Z} \setminus \{0\}} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}}.
\]

it follows that

\[
\sin \pi z = \pi z \prod_{\mathbb{Z} \setminus \{0\}} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}}.
\]