Forking and superstability in tame abstract elementary classes

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Introduction

- Forking is one of the key notions of modern stability theory.
- Is there such a notion outside of first-order (e.g. for logics such as $L_{\omega_1,\omega}$)?
- One way to formalize this question is to use Shelah’s definition of a *good* $\lambda$-frame.
- Roughly, a good $\lambda$-frame is a nice class of models of size $\lambda$, together with a forking-like notion for types of singletons.
- Shelah showed how to build a good frame using GCH-like set-theoretic assumptions and local model-theoretic hypotheses.
Forking is one of the key notions of modern stability theory. Is there such a notion outside of first-order (e.g. for logics such as $L_{\omega_1,\omega}$)?

One way to formalize this question is to use Shelah’s definition of a *good* $\lambda$-frame.

Roughly, a good $\lambda$-frame is a nice class of models of size $\lambda$, together with a forking-like notion for types of singletons.

Shelah showed how to build a good frame using GCH-like set-theoretic assumptions and local model-theoretic hypotheses.

We show how to build one in ZFC, paying with more global (but very natural) model-theoretic hypotheses.
Abstract elementary classes

Definition (Shelah)

Let $K$ be a nonempty class of structures of the same similarity type $L(K)$, and let $\leq$ be a partial order on $K$. $(K, \leq)$ is an abstract elementary class (AEC) if it satisfies:

1. $K$ is closed under isomorphism, $\leq$ respects isomorphisms.
2. If $M \leq N$ are in $K$, then $M \subseteq N$.
3. Coherence: If $M_0 \subseteq M_1 \leq M_2$ are in $K$ and $M_0 \leq M_2$, then $M_0 \leq M_1$.
4. Downward Löwenheim-Skolem axiom: There is a cardinal $\text{LS}(K) \geq |L(K)| + \aleph_0$ such that for any $N \in K$ and $A \subseteq |N|$, there exists $M \leq N$ containing $A$ of size $\leq \text{LS}(K) + |A|$.
5. Chain axioms: If $\delta$ is a limit ordinal, $\langle M_i : i < \delta \rangle$ is a $\leq$-increasing chain in $K$, then $M := \bigcup_{i<\delta} M_i$ is in $K$, and:
   5.1 $M_i \leq M$ for all $i < \delta$.
   5.2 If $N \in K$ is such that $M_i \leq N$ for all $i < \delta$, then $M \leq N$. 
Example: For $\psi \in L_{\omega_1, \omega}$, $\Phi$ a countable fragment containing $\psi$, $K \coloneqq \left( \text{Mod}(\psi), \prec_\Phi \right)$ is an AEC with $\text{LS}(K) = \aleph_0$. 

The main test question in the study of AECs is the categoricity conjecture: 

**Conjecture (Shelah)**

For every $\kappa$, there exists a cardinal $\mu = \mu(\kappa)$ such that whenever $K$ is an AEC with $\text{LS}(K) = \kappa$ and $K$ is categorical in some cardinal above $\mu$, then $K$ is categorical in all cardinals above $\mu$. 
Example: For $\psi \in L_{\omega_1,\omega}$, $\Phi$ a countable fragment containing $\psi$, $K := (\text{Mod}(\psi), \prec_\Phi)$ is an AEC with $\text{LS}(K) = \aleph_0$.

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Simplifying assumptions

Let \( K \) be an AEC.

**Definition**

- \( f : M \to N \) is a \((K-)embedding\) if \( f : M \cong f[M] \) and \( f[M] \leq N \). (From now on, every mapping will be assumed to be an embedding).

- \( K \) has **no maximal models** if for any \( M \in K \) there exists \( N \in K \) so that \( M < N \) (i.e. \( M \leq N \) and \( M \neq N \)).

- \( K \) has **joint embedding** if for any \( M_1, M_2 \in K \), there exists \( N \in K \) and \( f_\ell : M_\ell \to N, \ell = 1, 2 \).

- \( K \) has **amalgamation** if for any \( M_0, M_1, M_2 \in K \) with \( M_0 \leq M_\ell, \ell = 1, 2 \), there exists \( N \in K \) and \( f_\ell : M_\ell \to N \) so that \( f_1 \upharpoonright M_0 = f_2 \upharpoonright M_0 \).
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- If we assume they hold globally, we can build a homogeneous monster model $\mathcal{C}$ in which every model of the AEC embeds.
- Another simplifying assumption is tameness, a weak compactness property that was first isolated by Grossberg and VanDieren to prove an approximation to Shelah’s categoricity conjecture.

**Definition (Tameness)**

Let $K$ be an AEC with amalgamation. $K$ is $\mu$-tame if for any $M \in K$ and distinct $p, q \in S(M)$ there exists $M_0 \leq M$ of size $\leq \mu$ such that $p \upharpoonright M_0 \neq q \upharpoonright M_0$. 

Two approaches to AECs

Question (The global approach to AECs)
Work in ZFC, but assume a monster model and maybe some tameness. What can we say about the AEC?
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Make simplifying assumptions in only a few cardinals. When can we transfer them up? Can we build a structure theory cardinal by cardinal?
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Question (The local approach to AECs)
Make simplifying assumptions in only a few cardinals. When can we transfer them up? Can we build a structure theory cardinal by cardinal?

- This is the approach Shelah adopts in his books on classification theory for AECs.
- Many proofs have a set-theoretic flavor and rely on GCH-like principles.
- The key notion there is that of a good $\lambda$-frame, a local AEC version of superstability.
**Pre-frame**

**Definition (Shelah)**

Let $\lambda$ be a cardinal. A *pre-$\lambda$-frame* is a triple $s = (K, \sqsubset, S^{bs})$, where $K$ is an AEC with $\lambda \geq \text{LS}(K)$, $K_\lambda \neq \emptyset$, for all $M \in K_\lambda$, $S^{bs}(M)$ is a set of non-algebraic types, and:

1. $\sqsubset$ is a relation on quadruples of the form $(M_0, M_1, a, N)$, where $a \in N$ and $M_0 \leq M_1 \leq N$ are all in $K_\lambda$.
2. The following properties hold:
   2.1 **Invariance**: Both $\sqsubset$ and $S^{bs}$ are invariant under isomorphisms.
   2.2 **Monotonicity**: If $a \not\sqsubset M_1$,
   $$M_0 \leq M'_0 \leq M'_1 \leq M_1 \leq N' \leq N \leq N'' \text{ with } a \in N' \text{ and } N'' \in K_\lambda,$$
   then $a \not\sqsubset M'_1$ and $a \not\sqsubset M'_1$.  
     
   2.3 **Nonforking types are basic**: If $a \not\sqsubset M$, then 
   $$\text{tp}(a/M; N) \in S^{bs}(M).$$
Definition (Shelah)

$s = (K, \perp, S^{bs})$ is a good $\lambda$-frame if it is a pre-$\lambda$-frame and:

- $K_\lambda$ has amalgamation, joint embedding, and no maximal models.
- Stability: $|S^{bs}(M)| \leq \|M\|$ for all $M \in K_\lambda$.
- Density of basic types: If $M < N$ are both in $K_\lambda$, then there is $a \in N$ such that $tp(a/M; N) \in S^{bs}(M)$.
- Full existence: If $p \in S^{bs}(M)$ and $N \geq M$, then there exists $q \in S^{bs}(N)$ extending $p$ that does not fork over $M$.
- Uniqueness: If $p, q \in S^{bs}(N)$ do not fork over $M$ and $p \upharpoonright M = q \upharpoonright M$, then $p = q$. 
Symmetry: If \( a_1 \downarrow_{M_0} M_2, \ a_2 \in M_2, \) and
\[
\text{tp}(a_2/M_0; N) \in S^{bs}(M_0),
\]
then there is \( M_1 \) containing \( a_1 \) and \( N' \geq N \) such that \( a_2 \downarrow_{M_0} M_1. \)

Local character: If \( \delta \) is a limit ordinal, \( \langle M_i : i \leq \delta \rangle \) is an increasing chain in \( K_{\lambda} \) with \( M_\delta = \bigcup_{i<\delta} M_i \), and \( p \in S^{bs}(M_\delta) \), then there exists \( i < \delta \) such that \( p \) does not fork over \( M_i. \)

Continuity: If \( \delta \) is a limit ordinal, \( \langle M_i : i \leq \delta \rangle \) is an increasing chain in \( K_{\lambda} \) with \( M_\delta = \bigcup_{i<\delta} M_i \), \( p \in S(M_\delta) \) is so that \( p \upharpoonright M_i \) does not fork over \( M_0 \) for all \( i < \delta \), then \( p \) does not fork over \( M_0. \)

We say a good frame is **type-full** if the basic types are all the nonalgebraic types (in that case the density of basic types becomes trivial).
Existence of good frames

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Fact (Shelah)

Assume $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and the weak diamond ideal in $\lambda^+$ is not $\lambda^{++}$-saturated.

Let $K$ be an AEC and let $\lambda \geq \text{LS}(K)$ be a cardinal. Assume:

1. $K$ is categorical in $\lambda$ and $\lambda^+$.
2. $0 < I(\lambda^{++}, K) < \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$

Then $K$ has a good $\lambda^+$-frame.
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Existence of good frames from global assumptions

In ZFC, we can show:

1. Let $\mathcal{K}$ be an AEC with a monster model. If $\mathcal{K}$ is categorical in a "high-enough" successor $\lambda^+$, then $\mathcal{K}$ has a type-full good $\lambda$-frame.

2. Let $\mathcal{K}$ be an AEC with a monster model. If $\mathcal{K}$ is $\mu$-tame and categorical in a cardinal $\lambda$ with $\text{cf}(\lambda) > \mu$, then $\mathcal{K}$ has a type-full good ($\geq \lambda$)-frame.

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The idea is to start with $\mu$-nonsplitting as a candidate for our nonforking relation, and to refine it until we obtain all the desired properties. Instead of categoricity, we will assume only:

1. $K$ is stable in $\mu$ and $\mu$-nonsplitting has local character for $\langle M_i \subseteq K : i \leq \omega \rangle$-increasing continuous with $M_0 = M$ and $M_\omega = N$.
2. All models in $K_\lambda$ are $\mu^+$-saturated.

Both follow from the categoricity hypothesis. Even if we do not assume the second, our nonforking notion will still be well-behaved for $\mu^+$-saturated bases.
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So we assume $K$ is an AEC with a monster model, $\mu$-tameness, and:

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4. Symmetry holds as well: If not, we get the order property, and thus unstability.
An explicit description of forking

Unpacking our definition of forking, we get:

Proposition

For $M \leq N$ both in $K \geq \lambda$, $p \in S(N)$ does not fork over $M$ if and only if there is $M_0 \leq M$ in $K$ such that $p$ does not $\mu$-split over $M_0$.

Definition (Shelah)

For $M \leq N$ in $K$, $M \in K \mu$, $p \in S(N) \mu$-splits over $M$ if there exists $N_1, N_2 \in K \mu$ with $M \leq N_\ell \leq N$, $\ell = 1, 2, h$ such that $h(p \upharpoonright N_1) \neq p \upharpoonright N_2$.
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Let $K$ be an AEC with a monster model. Assume $K$ is $\mu$-tame and categorical in a cardinal $\lambda$ with $\text{cf}(\lambda) > \mu$.

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1. $K$ has a unique limit model in all $\lambda' \geq \lambda$. More precisely, if $\delta$ and $\rho$ are limit ordinals, $\langle M_i \in \mathcal{K}_{\lambda'} : i \leq \delta \rangle, \langle N_i \in \mathcal{K}_{\lambda'} : i \leq \rho \rangle$ are $\text{<}_{\text{univ}}$-increasing continuous, then $M_\delta \cong M_\rho$, and if in addition $M_0 = N_0$, then $M_\delta \cong_{M_0} N_\rho$. 

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2. $K$ is stable in all cardinals.
The stability spectrum in tame AECs

We can be more precise and establish a partial stability spectrum theorem:

- Theorem
  - Let $K$ be an AEC with a monster model. Assume $K$ is $\mu$-tame and stable in $\mu$. Let $\kappa$ be the least regular cardinal such that $\mu$-nonsplitting has local character for $<\mu,\omega$-increasing chains of cofinality $\geq \kappa$. The following hold:
    1. If $\kappa = \aleph_0$, then $K$ is stable in all $\lambda \geq \mu$.
    2. If GCH holds, then $K$ is stable in all $\lambda \geq \mu$ such that $\lambda = \lambda^\mu$.

Remark
- The following were already known:
  1. (Shelah) $\kappa \leq \mu^+$.  
  2. (Grossberg-VanDieren) $K$ is stable in all $\lambda$ such that $\lambda = \lambda^\mu$.  
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Remark

The following were already known:

1. (Shelah) $\kappa \leq \mu +$.
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**Theorem**
Let $K$ be an AEC with a monster model. Assume $K$ is $\mu$-tame and stable in $\mu$. Let $\kappa$ be the least regular cardinal such that $\mu$-nonsplitting has local character for $<\mu, \omega$-increasing chains of cofinality $\geq \kappa$. The following hold:

1. If $\kappa = \aleph_0$, then $K$ is stable in all $\lambda \geq \mu$.
2. If GCH holds, then $K$ is stable in all $\lambda \geq \mu$ such that $\lambda^{<\kappa} = \lambda$.

**Remark**
The following were already known:

1. (Shelah) $\kappa \leq \mu^+$.
2. (Grossberg-VanDieren) $K$ is stable in all $\lambda$ such that $\lambda = \lambda^\mu$.
3. (Baldwin-Kueker-VanDieren) $K$ is stable in $\mu^+$. 
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Future work: A nonforking notion for models

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Question

Let \( K \) be an AEC with a monster model that is tame and totally categorical. Does \( K \) have a nonforking notion for models?
For further reference, see: Sebastien Vasey, *Forking and superstability in tame AECs*. A preprint can be accessed from my webpage: http://math.cmu.edu/~svasey/ For a direct link, you can take a picture of the QR code below: