Independence in abstract elementary classes

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Introduction

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We provide the following answer in the framework of abstract elementary classes (AECs):

Theorem

Let $K$ be a fully tame and short AEC with a monster model. Assume $K$ is categorical in unboundedly many cardinals. Then there exists $\lambda$ such that $K\geq\lambda$ admits an independence notion with all the properties of forking in a superstable first-order theory (except it may only have extension over saturated models).
Forking is one of the key notions of modern stability theory. Is there such a notion outside of first-order (e.g. for logics such as $L_{\omega_1,\omega}$)? We provide the following answer in the framework of abstract elementary classes (AECs):

**Theorem**

Let $K$ be a fully tame and short AEC with a monster model. Assume $K$ is categorical in unboundedly many cardinals. Then there exists $\lambda$ such that $K_{\geq \lambda}$ admits an independence notion with all the properties of forking in a superstable first-order theory (except it may only have extension over saturated models).
Abstract elementary classes

Definition (Shelah, 1985)

Let $K$ be a nonempty class of structures of the same similarity type $L(K)$, and let $\leq$ be a partial order on $K$. $(K, \leq)$ is an abstract elementary class (AEC) if it satisfies:

1. $K$ is closed under isomorphism, $\leq$ respects isomorphisms.
2. If $M \leq N$ are in $K$, then $M \subseteq N$.
3. Coherence: If $M_0 \subseteq M_1 \leq M_2$ are in $K$ and $M_0 \leq M_2$, then $M_0 \leq M_1$.
4. Downward Löwenheim-Skolem axiom: There is a cardinal $\text{LS}(K) \geq |L(K)| + \aleph_0$ such that for any $N \in K$ and $A \subseteq |N|$, there exists $M \leq N$ containing $A$ of size $\leq \text{LS}(K) + |A|$.
5. Chain axioms: If $\delta$ is a limit ordinal, $\langle M_i : i < \delta \rangle$ is a $\leq$-increasing chain in $K$, then $M := \bigcup_{i<\delta} M_i$ is in $K$, and:
   5.1 $M_0 \leq M$.
   5.2 If $N \in K$ is such that $M_i \leq N$ for all $i < \delta$, then $M \leq N$. 
For $\psi \in L_{\omega_1,\omega}$, $\Phi$ a countable fragment containing $\psi$, $K := (\text{Mod}(\psi), \prec_\Phi)$ is an AEC with $\text{LS}(K) = \aleph_0$. 
Two approaches to AECs

Question (The local approach to AECs)

Make simplifying assumptions in only a few cardinals. When can we transfer them up? Can we build a structure theory cardinal by cardinal?
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- Many proofs have a set-theoretic flavor and rely on GCH-like principles.
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▶ This is the approach Shelah adopts in his books on classification theory for AECs.
▶ Many proofs have a set-theoretic flavor and rely on GCH-like principles.

Question (The global approach to AECs)

Work in ZFC, but make global model-theoretic hypotheses (like a monster model or locality conditions on types). What can we say about the AEC?
Global assumptions

Throughout the talk, we fix an AEC $K$. We assume we work inside a “big” model-homogeneous universal model $\mathcal{C}$. 
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**Fact**

Such a $\mathcal{C}$ exists if and only if $\mathcal{K}$ has joint embedding, no maximal models, and amalgamation.
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**Fact**

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**Definition (Galois types)**

For $\bar{b} \in <\infty \mathcal{C}$, $A \subseteq |\mathcal{C}|$, let $\text{gtp}(\bar{b}/A)$ be the orbit of $\bar{b}$ under the automorphisms of $\mathcal{C}$ fixing $A$. 
Tameness

Let $\kappa$ be an infinite cardinal.

**Definition (Grossberg-VanDieren, 2006)**

$K$ is $(< \kappa)$-tame if for any $M$ and any distinct $p, q \in gS(M)$, there exists $A \subseteq |M|$ of size less than $\kappa$ such that $p \upharpoonright A \neq q \upharpoonright A$. 

**Definition (Boney, 2013)**

$K$ is fully $(< \kappa)$-tame and short if for any $\alpha$, any $M$, and any distinct $p, q \in gS_\alpha(M)$, there exists $A \subseteq |M|$ and $I \subseteq \alpha$ of size less than $\kappa$ such that $p I \upharpoonright A \neq q I \upharpoonright A$. 

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Fact (Makkai-Shelah, Boney)

Let $\kappa > \text{LS}(K)$ be strongly compact. Then:

1. (No need for $K$ to have a monster model) If $K$ is categorical in some $\lambda > \beth_{\kappa+1}(\kappa)$, then $K_{\geq \kappa}$ has a monster model.
Tame AECs and large cardinals

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1. (No need for $K$ to have a monster model) If $K$ is categorical in some $\lambda > \beth_{\kappa+1}(\kappa)$, then $K_{\geq \kappa}$ has a monster model.

2. $K$ is fully ($< \kappa$)-tame and short.
Axioms of superstable forking

Definition

An AEC $K$ with a monster model is *good* if:

1. $K$ is stable in all $\lambda \geq \text{LS}(K)$.
2. There is a relation "$p$ does not fork (dnf) over $M$", for $p \in gS_{<\infty}(N)$, $M \leq N$, which satisfies:
   2.1 Invariance: If $f \in \text{Aut}(C)$, $p$ dnf over $M$, then $f(p)$ dnf over $f[M]$.
   2.2 Monotonicity: if $M \leq M' \leq N' \leq N$, $I \subseteq \alpha$, and $p \in gS_{\alpha}(N)$ dnf over $M$, then $p_{I} : N' \text{dnf over } M'$.
   2.3 Existence of unique extension: If $p \in gS_{\alpha}(M)$ and $N \geq M$, there exists a unique $q \in gS_{\alpha}(N)$ extending $p$ and not forking over $M$. Moreover $q$ is algebraic if and only if $p$ is.
   2.4 Set local character: If $p \in gS_{\alpha}(M)$, there exists $M_0 \leq M$ with $\|M_0\| \leq |\alpha| + \text{LS}(K)$ such that $p$ dnf over $M_0$.
   2.5 Chain local character: If $\langle M_i : i \leq \delta \rangle$ is increasing continuous, $p \in gS_{\alpha}(M_{\delta})$ and $\text{cf}(\delta) > \alpha$, then there exists $i < \delta$ such that $p$ dnf over $M_i$. 
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For \( \alpha \) a cardinal, \( \mathcal{F} \) an interval of cardinals, we say \( K \) is \((< \alpha, \mathcal{F})\)-good if it is good when we restrict types to have length less than \( \alpha \), and models to have size in \( \mathcal{F} \).
For $\alpha$ a cardinal, $\mathcal{F}$ an interval of cardinals, we say $K$ is $(<\alpha, \mathcal{F})$-good if it is good when we restrict types to have length less than $\alpha$, and models to have size in $\mathcal{F}$.

For example, good means $(<\infty, \geq \text{LS}(K))$-good. In Shelah’s terminology, $(\leq 1, \geq \lambda)$-good means $K$ has a type-full good $(\geq \lambda)$-frame.
Challenges in proving goodness

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For types of length one, this follows from local character.

But for infinite types, this is much harder.
Some previous work on independence in AECs

Fact (Shelah)

Let $K$ be an AEC, categorical in $\lambda$, $\lambda^+$, with at least one but “few” models in $\lambda^{++}$.

If $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and the weak diamond ideal on $\lambda^+$ is not $\lambda^{++}$-saturated, then $K$ is $(\leq \lambda^+, \lambda^+)$-good.

Fact (V.)

If $K$ is $(\leq \mu)$-tame and categorical in a $\lambda$ with $\text{cf}(\lambda) > \mu$, then $K$ is $(\leq 1, \geq \lambda)$-good.

Fact (Makkai-Shelah, Boney-Grossberg)

Let $\kappa > \text{LS}(K)$ be strongly compact and let $K$ be categorical in a $\lambda = \lambda^+ < \kappa$. Then $K$ is $\geq \lambda$ is good.
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Let $\kappa > \text{LS}(K)$ be strongly compact and let $K$ be categorical in a $\lambda = \lambda^{<\kappa}$. Then $K_{\geq \lambda}$ is good.
Main theorem

Theorem

Let $\kappa = \beth_\kappa > \text{LS}(K)$. Assume $K$ is categorical in $\lambda > \kappa$. 

Corollary

If $K$ is $(<\kappa)$-tame, $\kappa = \beth_\kappa > \text{LS}(K)$, and $K$ is categorical in $\lambda > \kappa$, then $K$ is stable in all cardinals.

Remark

We can replace categoricity by a natural definition of superstability, analog to $\kappa(T) = \aleph_0$. 
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1. If $K$ is $(< \kappa)$-tame, then $K_{\geq \lambda}$ is $(\leq 1, \geq \lambda)$-good.
2. If $\lambda > (2^\kappa)^+5$ and $K$ is fully $(< \kappa)$-tame and short, then $K_{\geq \lambda}$ is $(\leq \lambda, \geq \lambda)$-good. Moreover it is good, except it may only have extension over saturated models.
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**Corollary**

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We can replace categoricity by a natural definition of superstability, analog to $\kappa(T) = \aleph_0$. 
Conjecture (Shelah)

Let $K$ be an AEC. If $K$ is categorical in unboundedly many cardinals, then $K$ is categorical on a tail of cardinals.

\[1\text{Shelah claims stronger results in Chapter IV of his book.}\]
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Let $K$ be an AEC. If $K$ is categorical in unboundedly many cardinals, then $K$ is categorical on a tail of cardinals.

Claim (Shelah, to appear in Sh:842)

If $K$ has an $\omega$-successful good $\lambda$-frame and weak GCH holds, then $K$ is categorical in some $\mu > \lambda^{+\omega}$ if and only if $K$ is categorical in all $\mu > \lambda^{+\omega}$.

\footnote{Shelah claims stronger results in Chapter IV of his book.}
Shelah’s categoricity conjecture from large cardinals?

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It turns out our construction gives an $\omega$-successful good frame. Thus modulo Shelah’s claim, we get\(^1\):

Corollary

Assume weak GCH. If there are unboundedly many strongly compact cardinals, then Shelah’s categoricity conjecture holds.

\(^1\)Shelah claims stronger results in Chapter IV of his book.
Main steps of the proof

Fix a “nice-enough” AEC $K$.

1. Using methods such as Galois-Morleyization and previous results of Boney-Grossberg, show that coheir has some (not all) of the properties of a good independence relation.
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2. Show that coheir induces a good $(\leq 1, \lambda)$-independence relation (for suitable $\lambda$).
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5. Use tameness and shortness to obtain a good $(< \infty, \geq \lambda)$-independence relation (we can only prove extension over saturated models).
Thank you!

- For further reference, see: Sebastien Vasey, *Independence in abstract elementary classes*.
- A preprint can be accessed from my webpage: http://svasey.org/
- For a direct link, you can take a picture of the QR code below: