First name: ________________________________
Last name: ________________________________
Andrew ID: ________________________________

You have one hour and twenty minutes to complete the exam. The exam should contain five problems and one extra credit question. You may not use any notes or other documents, and you may not use any electronics such as a calculator or a mobile phone. You may use your brain and a pen.

Quote of the day: “There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no ignorabimus.” (David Hilbert).

Good luck!

| Problem 1 | 20 |
| Problem 2 | 20 |
| Problem 3 | 30 |
| Problem 4 | 15 |
| Problem 5 | 15 |
| Extra Credit | 20 |
| **Total** | **100** |

**Problem 1 (20 points)**

State the following results/definitions discussed in class. Do not prove anything.

1. The AGM inequality.
2. De Morgan’s laws for logical operators.
3. The definition of $x^n$ (for $n$ a natural number).
4. The definition of being a finite set.
5. The definition of the natural numbers.

**Problem 2 (20 points)**

Say whether the following statements are true or false, and briefly justify why.

1. $\mathbb{Z}$ is equipotent to $\mathbb{Q}$.

*Date: June 9, 2014.*
(2) Every real number has a square root.
(3) If \( p(x, y) \) is a propositional function, \( \exists x \exists y \ p(x, y) \) is equivalent to \( \exists y \exists x \ p(x, y) \).
(4) If \( x \) and \( y \) are rationals, then \( x + y \) is rational.
(5) \( p \to q \) is logically equivalent to \( \neg q \lor p \).

**Problem 3 (30 points)**

It was seen in class that every natural number \( n \geq 2 \) can be written as a product of primes. Give the proof (you should give enough details to convince me that you understand the main ideas).

**Problem 4 (15 points)**

For sets \( A \) and \( B \) recall that the *symmetric difference* \( A \Delta B \) of \( A \) and \( B \) is defined by \( (A \setminus B) \cup (B \setminus A) \), where \( A \setminus B \) is the set of elements of \( A \) that are not in \( B \). Show that for any sets \( A, B, C \), \( A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C) \).

**Problem 5 (15 points)**

Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that \( f(x + y) = f(x) + f(y) \) for all real numbers \( x \) and \( y \).

(1) Give three different examples of such a function.
(2) Prove that \( f(0) = 0 \). *Hint:* Write 0 as \( 0 + 0 \).
(3) Prove that for any natural number \( n \), \( f(n) = nf(1) \).

*Extra credit (20 points)*

Assume \( p(x) \), \( q(x) \), and \( r(x) \) are propositional functions. Assume you know that:

(1) \( \forall x (p(x) \lor (q(x) \to r(x))) \)
(2) \( \neg \forall x (p(x) \lor r(x)) \)

Prove that \( \neg \forall x q(x) \).