THE BIRMAN EXACT SEQUENCE DOES NOT VIRTUALLY SPLIT

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Abstract. This paper answers a basic question about the Birman exact sequence in the theory of mapping class groups. We prove that the Birman exact sequence does not admit a section over any subgroup \( \Gamma \) contained in the Torelli group with finite index. \textit{A fortiori} this proves that there is no section of the Birman exact sequence for any finite-index subgroup of the full mapping class group. This theorem was announced in a 1990 preprint of G. Mess, but an error was uncovered and described in a recent paper of the first author.

1. Introduction

Let \( S \) be a Riemann surface of finite type. A fundamental tool in the study of the mapping class group \( \text{Mod}(S) \) of \( S \) is the \textit{Birman exact sequence} which describes the relationship between \( \text{Mod}(S) \) and the mapping class group \( \text{Mod}(S') \) of a surface \( S' \) obtained from \( S \) by filling in boundary components and/or punctures on \( S \). In its most basic form, \( S = \Sigma_{g,*} \) is a surface of genus \( g \geq 2 \) with a single puncture \( * \in \Sigma_g \), and \( S' = \Sigma_g \) is the closed surface obtained by filling in \( * \). In this case, the Birman exact sequence takes the form

\[
1 \to \pi_1(\Sigma_g,*) \to \text{Mod}(\Sigma_{g,*}) \to \text{Mod}(\Sigma_g) \to 1.
\]

Given any such short exact sequence of groups \( 1 \to A \to B \to C \to 1 \) determined by a surjective homomorphism \( f : B \to C \), it is a basic question to determine whether the sequence \textit{splits}: that is, whether there is a (necessarily injective) homomorphism \( g : C \to B \) such that \( f \circ g = \text{id}_C \). In the context of the Birman exact sequence, this question has a topological interpretation: \([1]\) can be viewed as the short exact sequence on (orbifold) fundamental groups induced by the fibration \( \mathcal{M}_{g,*} \to \mathcal{M}_g \) of the “universal curve” \( \mathcal{M}_{g,*} \) over the moduli space of Riemann surfaces \( \mathcal{M}_g \). The question of whether \([1]\) splits is equivalent\(^{1}\) to asking whether the universal curve \( \mathcal{M}_{g,*} \) admits a continuous section: that is, whether it is possible to continuously choose a marked point on every Riemann surface of genus \( g \).

The Birman exact sequence \([1]\) does not split for any \( g \geq 2 \). This is an easy consequence of two observations. For one, it is easy to construct non-cyclic torsion subgroups of \( \text{Mod}(\Sigma_g) \), while it is also simple to show that no such subgroups exist in \( \text{Mod}(\Sigma_{g,*}) \). However, this argument is somewhat unsatisfactory in that it does not address the more fundamental issue of \textit{virtual splitting}. A short exact sequence \( 1 \to A \to B \to C \to 1 \) is said to \textit{virtually split} if there exists some finite-index subgroup

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\(^{1}\)For brevity’s sake, we are ignoring the complications induced by the orbifold structure; of course, these issues disappear when passing to suitably-chosen finite covers of \( \mathcal{M}_g \).
$C' \leq C$ and a homomorphism $g : C' \to B$ such that $f \circ g = \text{id}_{C'}$. The mapping class group $\text{Mod}(\Sigma_g)$ is virtually torsion-free, i.e. there exist finite-index subgroups $\Gamma \leq \text{Mod}(\Sigma_g)$ that are torsion-free. Thus for any such $\Gamma$, the argument above breaks down. Formulated in terms of moduli spaces, this leaves a very basic question unanswered: does there exist some finite-sheeted cover $\tilde{M}_g$ of $M_g$, over which it is possible to find a continuous section of the (pullback of the) universal curve?

For $g = 2$ the Birman exact sequence does virtually split. This follows from the fact that every Riemann surface of genus 2 is hyperelliptic, and hence equipped with 6 necessarily distinct Weierstrass points. The monodromy of these Weierstrass points is the full symmetric group, but by passing to the 6-sheeted cover associated with the subgroup $S_5 \leq S_6$, one of these points becomes globally distinguishable and hence the universal curve virtually has a section.

The purpose of this note is to show that a similar phenomenon cannot occur for higher genus Riemann surfaces. For the definition of the Torelli group $I(\Sigma_g)$, see Section 2.2.

**Theorem A.** For $g \geq 4$, the Birman exact sequence does not virtually split. Moreover, for any subgroup $\Gamma \leq I(\Sigma_g)$ of finite index in the Torelli group, there is no splitting $\sigma : \Gamma \to I(\Sigma_g, \ast)$ of the Birman exact sequence restricted to $\Gamma$.

From the topological point of view, it is natural to consider the more general notion of a multisection of a fiber bundle. A multisection is a continuous choice of $n$ distinct points on each fiber. For instance, the Weierstrass points form a multisection of cardinality 6 of the universal curve in genus 2. As in that particular example, a multisection of a fiber bundle $E \to B$ always induces a genuine section of the pullback bundle over some finite-sheeted cover $B' \to B$. We thus obtain Theorem B below as an immediate corollary of Theorem A. The Torelli space is the cover $I_g \to M_g$ of $M_g$ corresponding to the subgroup $I_g \leq \text{Mod}_g$; the universal curve $M_{g, \ast}$ pulls back to give the universal family of “homologically framed curves” $I_{g, \ast} \to I_g$.

**Theorem B.** For $g \geq 4$, the universal family $I_{g, \ast} \to I_g$ does not admit any continuous multisection. A fortiori, for $g \geq 4$, the universal curve $M_{g, \ast} \to M_g$ does not admit any continuous multisection.

There is an important bibliographical comment to be made. Theorem A is claimed in the 1990 preprint [Mes90] of G. Mess. Unfortunately, as detailed in the paper [Che17] of the first author, Mess’ argument contains a fatal error. In [Che17], the first author proves Theorem A in the special case of the full Torelli group $\Gamma = I(\Sigma_g)$. The methods therein make essential use of some special relations in $I(\Sigma_g)$ which disappear upon passing to finite-index subgroups.

In the present note, we return to the outline of the argument as proposed by Mess. In brief, Mess uses the hypothesis of a splitting to construct a particular homomorphism (the homomorphism $s$ constructed and analyzed in Section 3). Mess incorrectly assumes $s$ to be valued in a certain subgroup of the codomain, and derives a contradiction predicated on this assumption. Our argument proceeds by studying $s$ and deriving a contradiction as Mess does, but a more exhaustive analysis must be carried out.
The paper is organized as follows. Section 2 collects the necessary facts from the theory of mapping class groups, and establishes some preliminary results. The proof of Theorem A is then carried out in Section 3.

2. MAPPING CLASS GROUPS

2.1. Canonical reduction systems. The central tool for the proof of Theorem A is the notion of a canonical reduction system, which can be viewed as an enhancement of the Nielsen-Thurston classification. We remind the reader that a curve $c \subset S$ is said to be peripheral if $c$ is isotopic to a boundary component or puncture of $S$. The Nielsen-Thurston classification asserts that each nontrivial element $f \in \text{Mod}(S)$ is of exactly one of the following types: periodic, reducible, or pseudo-Anosov. A mapping class $f$ is periodic if $f^n = \text{id}$ for some $n \geq 1$, and is reducible if for some $n \geq 1$, there is some nonperipheral simple closed curve $c \subset S$ such that $f^n(c)$ is isotopic to $c$. If neither of these conditions are satisfied, $f$ is said to be pseudo-Anosov. In this case, $f$ is isotopic to a homeomorphism $f'$ of a very special form. We will not need to delve into the theory of pseudo-Anosov mappings, and refer the interested reader to [FM12, Chapter 13] and [FLP12] for more details.

Definition 2.1 (Reduction systems). A reduction system of a reducible mapping class $h$ in $\text{Mod}(S)$ is a set of disjoint nonperipheral curves that $h$ fixes as a set up to isotopy. A reduction system is maximal if it is maximal with respect to inclusion of reduction systems for $h$. The canonical reduction system $\text{CRS}(h)$ is the intersection of all maximal reduction systems of $h$.

Canonical reduction systems allow for a refined version of the Nielsen-Thurston classification. For a reducible element $f$, there exists $n$ such that $f^n$ fixes each element in $\text{CRS}(f)$ and after cutting out $\text{CRS}(f)$, the restriction of $f^n$ on each component is either identity or pseudo-Anosov. See [FM12] Corollary 13.3]. In Propositions 2.2–2.6 we list some properties of the canonical reduction systems that will be used later.

Proposition 2.2. $\text{CRS}(h^n) = \text{CRS}(h)$ for any $n$.

Proof. This is classical; see [FM12] Chapter 13].

For two curves $a, b$ on a surface $S$, let $i(a, b)$ be the geometric intersection number of $a$ and $b$. For two sets of curves $P$ and $Q$, we say that $P$ and $Q$ intersect if there exist $a \in P$ and $b \in Q$ such that $i(a, b) \neq 0$. We emphasize that “intersection” here refers to the intersection of curves on $S$, and not the abstract set-theoretic intersection of $P$ and $Q$ as sets.

Proposition 2.3. Let $h$ be a reducible mapping class in $\text{Mod}(S)$. If $\{\gamma\}$ and $\text{CRS}(h)$ intersect, then no power of $h$ fixes $\gamma$.

Proof. Suppose that $h^n$ fixes $\gamma$. Therefore $\gamma$ belongs to a maximal reduction system $M$. By definition, $\text{CRS}(h) \subset M$. However $\gamma$ intersects some curve in $\text{CRS}(f)$; this contradicts the fact that $M$ is a set of disjoint curves.
Proposition 2.4. Suppose that $h, f \in \text{Mod}(S)$ and $fh = hf$. Then $\text{CRS}(h)$ and $\text{CRS}(f)$ do not intersect.

Proof. Conjugating, $\text{CRS}(hfh^{-1}) = h(\text{CRS}(f))$. Since $hfh^{-1} = f$, it follows that $\text{CRS}(f) = h(\text{CRS}(f))$. Therefore $h$ fixes the whole set $\text{CRS}(f)$. There is some $n \geq 1$ such that $h^n$ fixes all curves element-wise in $\text{CRS}(f)$. By Proposition 2.3, curves in $\text{CRS}(h)$ do not intersect curves in $\text{CRS}(f)$. □

For a curve $a$ on a surface $S$, denote by $T_a$ the Dehn twist about $a$. More generally, a Dehn multitwist is any mapping class of the form $T := \prod T_{a_i}^{k_i}$ for a collection of pairwise-disjoint simple closed curves $\{a_i\}$ and arbitrary integers $k_i$.

Proposition 2.5. Let

$$T := \prod T_{a_i}^{k_i}$$

be a Dehn multitwist. Then $\text{CRS}(T) = \{a_i\}$.

Proof. Firstly $T$ cannot contain any simple closed curves $b$ for which $i(b, a_i) \neq 0$, since no power of $T$ preserves $b$. This can be seen from the equation

$$i(\prod T_{a_i}^{k_i} (b), b) = \sum |k_i|i(a_i, b) \neq 0 = i(b, b);$$

see [FM12 Proposition 3.2]. It follows that if $S$ is any reduction system for $T$, then $S \cup \{a_i\}$ is also a reduction system, and hence that $\{a_i\} \subset \text{CRS}(T)$. If $\gamma$ is disjoint from each element of $\{a_i\}$ but not equal to any $a_i$, then there exists some curve $\delta$, also disjoint and distinct from each $a_i$, such that $i(\gamma, \delta) \neq 0$. As both $\{a_i\} \cup \{\gamma\}$ and $\{a_i\} \cup \{\delta\}$ are reduction systems for $T$, this shows that no such $\gamma$ can be contained in $\text{CRS}(T)$ and hence that $\text{CRS}(T) = \{a_i\}$ as claimed. □

The final result we will require follows from the theory of canonical reduction systems. It appears as [McC82 Theorem 1].

Proposition 2.6 (McCarthy). Let $S$ be a Riemann surface of finite type, and let $f \in \text{Mod}(S)$ be a pseudo-Anosov element. Then the centralizer subgroup of $f$ in $\text{Mod}(S)$ is virtually cyclic.

2.2. The Torelli group, separating twists, and bounding pair maps. The action of a diffeomorphism $f$ on the homology of a surface $S$ is well-defined on the level of isotopy, giving rise to the symplectic representation

$$\Psi : \text{Mod}(S) \to \text{Aut}(H_1(S; \mathbb{Z})).$$

The Torelli group is the kernel subgroup $\mathcal{I}(S) := \ker(\Psi)$. There are several classes of elements of the Torelli group that will feature in the proof of Theorem □. For context, background, and proofs of the following assertions, see [FM12 Chapter 6]. A separating twist is a Dehn twist $T_c$, where $c$ is a separating curve on $S$. Separating twists $T_c \in \mathcal{I}(S)$ are elements of the Torelli group. A pair of curves $\{a, b\} \subset S$ is said to be a bounding pair if $a, b$ are individually nonseparating, but $a \cup b$ bounds
a subsurface of $S$ of positive genus on both sides. A bounding pair map is the Dehn multitwist $T_a T_b^{-1}$; necessarily $T_a T_b^{-1} \in \mathcal{I}(S)$ for any bounding pair $\{a, b\}$.

2.3. Point- and disk-pushing subgroups. Recall the Birman exact sequence (1). The kernel $\pi_1(\Sigma_g, \ast)$ is referred to as the point-pushing subgroup of $\text{Mod}(\Sigma_g, \ast)$. Henceforth we will tidy our notation and omit reference to the basepoint. An element $\alpha \in \pi_1(\Sigma_g)$ determines a mapping class $\alpha \in \text{Mod}(\Sigma_g, \ast)$ as follows: one “pushes” the marked point $\ast$ along the loop determined by $\alpha$.

There is an analogous notion of a “disk-pushing subgroup”. Let $S = \Sigma_{g,1}$ denote a surface of genus $g$ with one boundary component. In this setting, the Birman exact sequence takes the form

$$1 \to \pi_1(UT\Sigma_g) \to \text{Mod}(\Sigma_{g,1}) \to \text{Mod}(\Sigma_g) \to 1.$$ \hfill (2)

Here, $UT\Sigma_g$ denotes the unit tangent bundle of $\Sigma_g$; i.e. the $S^1$-subbundle of the tangent bundle $T\Sigma_g$ consisting of unit-length tangent vectors (relative to an arbitrarily-chosen Riemannian metric). In this context, the kernel $\pi_1(UT\Sigma_g)$ is known as the disk-pushing subgroup. An element $\tilde{\alpha} \in \pi_1(UT\Sigma_g)$ determines a “disk-pushing” diffeomorphism of $\Sigma_{g,1}$ as follows: one treats the boundary component $\Delta$ as the boundary of a disk $D$, and “pushes” $D$ along the path determined by the image $\alpha \in \pi_1(\Sigma_g)$. The extra information of the tangent vector encoded in $\tilde{\alpha}$ is used to give a consistent framing of $\partial D$ along its path.

The proposition below records some basic facts about point- and disk-pushing subgroups. In item 5 below, the support of a (not necessarily simple) element $\alpha \in \pi_1(\Sigma_g)$ is defined to be the minimal subsurface $S_\alpha \subset \Sigma_{g,\ast}$ that contains $\alpha$ for which every component of $\partial S_\alpha$ is essential, i.e. non-nullhomotopic and nonperipheral.

**Proposition 2.7.**

1. There are containments $\pi_1(\Sigma_g) \leq \mathcal{I}(\Sigma_{g,\ast})$ and $\pi_1(UT\Sigma_g) \leq \mathcal{I}(\Sigma_{g,1})$.
2. Let $\alpha \in \pi_1(\Sigma_g)$ be a simple element. Viewed as a point-push map, $\alpha$ has an expression as a bounding pair map

$$\alpha = T_{\alpha_L} T_{\alpha_R}^{-1},$$

where $\alpha_L, \alpha_R$ are the simple closed curves on $\Sigma_{g,\ast}$ lying to the left (resp. right) of $\alpha$.
3. Let $\zeta \in \pi_1(UT\Sigma_g)$ be a generator of the kernel of the map $\pi_1(UT\Sigma_g) \to \pi_1(\Sigma_g)$. Viewed as a push map, $\zeta = T_\Delta$, the twist about the boundary component of $\Sigma_{g,1}$.
4. Let $\tilde{\alpha} \in \pi_1(UT\Sigma_g)$ be simple (in the sense that $\alpha \in \pi_1(\Sigma_g)$ is simple). Viewed as a disk-pushing map, there is an expression

$$\tilde{\alpha} = T_{\alpha_L} T_{\alpha_R}^{-1} T_k$$

for some $k \in \mathbb{Z}$.
5. Let $\alpha \in \pi_1(\Sigma_g)$ be an arbitrary (not necessarily simple) element. Then

$$\text{CRS}(\alpha) = \partial(S_\alpha),$$

the (possibly empty) boundary of the support $S_\alpha$. Moreover, $\alpha$ is pseudo-Anosov on the subsurface $S_\alpha$. 


Proof. Items 1-4 are standard; see [FM12, Chapters 4,6] for details. Item 5 is a reformulation of a theorem of Kra, adapted to the language of canonical reduction systems. See [FM12, Theorem 14.6]. □

In Section 3 we will make use of the following lemma concerning the action of separating twist maps on the underlying fundamental group.

Lemma 2.8. Let $T_c \in \mathcal{I}(\Sigma_{g,*})$ be a Dehn twist about a separating simple closed curve $c$. Let $\alpha \in \pi_1(\Sigma_g)$ be an arbitrary element, represented as a (not necessarily simple) curve based at $* \in \Sigma_{g,*}$. If

$$T_c^k(\alpha) = \alpha$$

for any $k \neq 0$, then there exists a representative of $\alpha$ that is disjoint from $c$.

Proof. The hypothesis implies that $T_c^k$ and $\alpha$ commute as elements of $\mathcal{I}(\Sigma_{g,*})$. By Propositions 2.4, 2.5 and 2.7, CRS($\alpha$) $= \partial(S_\alpha)$ and CRS($T_c^k$) $= \{c\}$ must be disjoint, and moreover

$$\{c\} \subset \Sigma_{g,*} \setminus S_\alpha.$$  

The result follows. □

2.4. Lifts of some special mapping classes. The foundation of the proof of Theorem A is an analysis of possible images of bounding pair maps and separating twists under a hypothetical section. Let $\Gamma \leq \mathcal{I}(\Sigma_g)$ be a finite-index subgroup, and suppose that $\sigma : \Gamma \to \text{Mod}(\Sigma_{g,*})$ is a section. A first observation is that in fact, $\sigma(\Gamma) \leq \mathcal{I}(\Sigma_{g,*})$. This follows readily from the fact that $\pi_1(\Sigma_g) \leq \mathcal{I}(\Sigma_{g,*})$ as observed in Proposition 2.7.

Since $\Gamma$ is a finite-index subgroup of $\mathcal{I}(\Sigma_g)$, there is no assumption that a given separating twist $T_c$ or bounding pair map $T_aT_b^{-1}$ is an element of $\Gamma$. However, the assumption that $\Gamma$ is of finite index in $\mathcal{I}(\Sigma_g)$ does imply that for each separating twist $T_c$ and each bounding pair map $T_aT_b^{-1}$, there is some $k > 0$ (depending on the individual element) such that $T_c^k \in \Gamma$, and likewise $T_c^kT_b^{-k} \in \Gamma$.

In the following lemma and throughout, for a curve $\tilde{c}$ on $\Sigma_{g,1}$ (resp. $\Sigma_{g,*}$), when we say $\tilde{c}$ is isotopic to a curve $c$ on $\Sigma_g$, we mean that $\tilde{c}$ is isotopic to $c$ after forgetting the puncture (resp. boundary component).

Lemma 2.9.

1) Let $\{a, b\}$ be a bounding pair, and fix $k > 0$ such that $(T_aT_b^{-1})^k \in \Gamma$. Up to a swap of $a$ and $b$, we have that $\sigma((T_aT_b^{-1})^k) = (T_aT_b^{-1})^k(T_a^{-1}T_a^n)^n$, where $n$ is an integer and $a’, a’, b’$ are three disjoint curves on $\Sigma_{g,1}$ such that $a’, a’, b’$ are isotopic to $a$ and $b$ is isotopic to $b$. Notice that $n$ can be zero.

2) Let $c$ be a separating curve on $\Sigma_g$ that divides $\Sigma_g$ into two subsurfaces each of genus at least two. For any $k > 0$ such that $(T_c)^k \in \Gamma$, we have that $\sigma((T_c)^k) = (T_c)^k(T_c^{-1}T_c^n)^n$ where $n$ is an integer and $c’$ and $c’$ are a pair of curves on $\Sigma_{g,1}$ that are both isotopic to $c$.

Proof. Let $(T_aT_b^{-1})^k \in \Gamma$ be a power of a bounding pair map. Since the centralizer of $(T_aT_b^{-1})^k$ contains a copy of $\mathbb{Z}^{2g-3}$ as a subgroup of $\mathcal{I}(\Sigma_g)$, the centralizer of $(T_aT_b^{-1})^k$ as a subgroup of $\Gamma$...
contains a copy of \( \mathbb{Z}^{2g-3} \) as well. By the injectivity of \( \sigma \), the centralizer of \( \sigma(T_aT_b^{-1}) \in I(\Sigma_{g,a}) \) contains a copy of \( \mathbb{Z}^{2g-3} \). When \( g > 3 \), we have that \( 2g - 3 > 3 \). Therefore \( \sigma((T_aT_b^{-1})^k) \in I(\Sigma_{g,a}) \) cannot be pseudo-Anosov because the centralizer of a pseudo-Anosov element is a virtually cyclic group by Proposition 2.6. For any curve \( \gamma' \) on \( \Sigma_{g,a} \), denote by \( \gamma \) the same curve on \( \Sigma_g \). We decompose the proof into the following three steps.

**Claim 2.10 (Step 1).** \( CRS(\sigma((T_aT_b^{-1})^k)) \) only contains curves that are isotopic to \( a \) or \( b \).

**Proof.** Suppose that there exists \( \gamma' \in CRS(\sigma((T_aT_b^{-1})^k)) \) such that \( \gamma \) is not isotopic to \( a \) or \( b \). There are two cases.

**Case 1:** \( \gamma \) intersects \( a \) or \( b \). Since a power of \( \sigma((T_aT_b^{-1})^k) \) fixes \( \gamma' \), a power of \( (T_aT_b^{-1})^k \) fixes \( \gamma \). On the other hand, \( CRS((T_aT_b^{-1})^k) = \{a, b\} \). Combined with Lemma 2.3, this shows that \( (T_aT_b^{-1})^k \) does not fix \( \gamma \). This is a contradiction.

**Case 2:** \( \gamma \) does not intersect \( a \) and \( b \). In this case by the change-of-coordinates principle, there exists a separating curve \( c \) on \( \Sigma_g \) such that \( i(a, c) = 0, i(b, c) = 0 \) and \( i(c, \gamma) \neq 0 \). Assume that \( T_c^m \in \Gamma \). Since \( (T_aT_b^{-1})^k \) and \( T_c^m \) commute in \( \Gamma \), the two mapping classes \( \sigma((T_aT_b^{-1})^k) \) and \( \sigma(T_c^m) \) commute in \( I(\Sigma_g) \). Therefore a power of \( \sigma(T_c^m) \) fixes \( CRS(\sigma(T_aT_b^{-1})) \); more specifically a power of \( T_c^m \) fixes \( \gamma \). However by Lemma 2.3, no power of \( T_c \) fixes \( \gamma \). This is a contradiction. \( \square \)

**Claim 2.11 (Step 2).** \( CRS(\sigma((T_aT_b^{-1})^k)) \) must contain curves \( a' \) and \( b' \) that are isotopic to \( a \) and \( b \), respectively.

**Proof.** Suppose that \( CRS(\sigma((T_aT_b^{-1})^k)) \) does not contain a curve \( a' \) isotopic to \( a \). Then by Step 1, \( CRS(\sigma((T_aT_b^{-1})^k)) \) either contains one curve \( b' \) isotopic to \( b \) or two curves \( b' \) and \( b'' \) both isotopic to \( b \). After cutting \( \Sigma_{g,a} \) along \( CRS(\phi((T_aT_b^{-1})^k)) \), there is some component \( C \) that is not a punctured annulus. \( C \) is homeomorphic to the complement of \( b \) in \( \Sigma_g \).

By the Nielsen-Thurston classification, a power of \( \sigma((T_aT_b^{-1})^k) \) is either pseudo-Anosov on \( C \) or else is the identity on \( C \). If a power of \( \sigma((T_aT_b^{-1})^k) \) is pseudo-Anosov on \( C \), then the centralizer of \( \sigma((T_aT_b^{-1})^k)|_C \) is virtually cyclic by Proposition 2.6. Combining with \( T_{b'} \) and \( T_{b''} \), the centralizer of \( \sigma((T_aT_b^{-1})^k) \) in \( I(\Sigma_g) \) is virtually an abelian group of rank at most 3. This contradicts the fact that the centralizer of \( \sigma((T_aT_b^{-1})^k) \) contains a subgroup \( \mathbb{Z}^{2g-3} \), since \( g \geq 4 \) and hence \( 2g - 3 > 3 \). Therefore \( \sigma((T_aT_b^{-1})^k) \) is the identity on \( C \). However, viewing \( C = \Sigma_g \setminus \{b\} \) as a subsurface of \( \Sigma_g \) that contains \( a \), we see that \( (T_aT_b^{-1})^k \) is actually not the identity on \( C \); this is a contradiction. \( \square \)

**Claim 2.12 (Step 3).** \( \sigma((T_aT_b^{-1})^k) = (T_aT_{b'}^{-1})^k(T_{a''}^{-1}T_{b''})^n \), where \( n \) is an integer and \( a', a'', b' \) are three disjoint curves on \( \Sigma_{g,a} \) such that \( a', a'' \) are isotopic to \( a \) and \( b' \) is isotopic to \( b \).

**Proof.** Suppose that \( \sigma((T_aT_b^{-1})^k) \) is pseudo-Anosov on some component \( C \) of \( \Sigma_g \setminus CRS(\sigma((T_aT_b^{-1})^k)) \).

Since the genus \( g(C) \geq 1 \), there exists a separating curve \( s \) on \( C \) such that \( \sigma(T_s) \) commutes with \( \sigma((T_aT_b^{-1})^k) \) in \( \sigma(\Gamma) \). Therefore, some power of \( \sigma((T_aT_b^{-1})^k) \) fixes \( CRS(\sigma(T_s)) \), which is either one curve or two curves isotopic to \( s \). Thus a power of \( \sigma((T_aT_b^{-1})^k) \) fixes some curve on \( C \), which means
that \( \sigma((T_aT_b^{-1})^k) \) is not pseudo-Anosov on \( C \). It follows that a power of \( \sigma((T_aT_b^{-1})^k) \) must be a product of Dehn twists about the curves in \( \text{CRS}(\sigma((T_aT_b^{-1})^k)) \). Since \( \sigma((T_aT_b^{-1})^k) \) is a lift of \( (T_aT_b^{-1})^k \), the lemma holds. \( \square \)

The same argument works for \( T_c^m \in \Gamma \) the Dehn twist about a separating curve \( c \) as long as both components of \( \Sigma_g \setminus \{ c \} \) have genus two or greater. \( \square \)

2.5. The handle-pushing subgroup. As in Mess’s approach, we will prove Theorem A by showing that certain “handle-pushing” subgroups (contained in any finite-index subgroup of \( \mathcal{I}(\Sigma_g) \)) do not admit sections to \( \mathcal{I}(\Sigma_g,*) \). To define these, let \( c \) be a separating curve. The complement \( \Sigma_g \setminus \{ c \} = \Sigma_{p,1} \cup \Sigma_{q,1} \) is a disconnected surface with two components. Let \( \mathcal{I}(c) \leq \mathcal{I}(\Sigma_g) \) be the subgroup consisting of Torelli mapping classes that are a product of mapping classes with supports on either \( \Sigma_{p,1} \) or \( \Sigma_{q,1} \). The subgroup \( \mathcal{I}(c) \) satisfies the following exact sequence:

\[
1 \rightarrow \mathbb{Z} \rightarrow \mathcal{I}(\Sigma_{p,1}) \times \mathcal{I}(\Sigma_{q,1}) \rightarrow \mathcal{I}(c) \rightarrow 1,
\]

where \( \mathbb{Z} \) is generated by \( T_c \).

**Definition 2.13** (Handle-pushing subgroup). Let \( c \) be a separating curve as in Figure 1 dividing \( \Sigma_g \setminus \{ c \} = \Sigma_{p,1} \cup \Sigma_{q,1} \). The handle-pushing subgroup on \( \Sigma_{p,1} \), written \( \mathcal{H}(\Sigma_{p,1}) \), is defined as

\[
\mathcal{H}(S) := \pi_1(UT \Sigma_p) \leq \mathcal{I}(c).
\]

More broadly, any finite-index subgroup of \( \mathcal{H}(\Sigma_{p,1}) \) will also be called a handle-pushing subgroup.

**Remark 2.14.** Every finite-index subgroup of \( \mathcal{H}(\Sigma_{p,1}) \), being isomorphic to a finite-index subgroup of \( \pi_1(UT \Sigma_p) \), is isomorphic to a non-split extension of a surface group of genus \( p' \geq p \) by \( \mathbb{Z} \).

Denote by \( A \leq \mathcal{I}(c) \) the group generated by the disk-pushing subgroup on both subsurfaces \( \Sigma_{p,1} \) and \( \Sigma_{q,1} \). Then \( A \) satisfies the following exact sequence:

\[
1 \rightarrow \mathbb{Z} \rightarrow \pi_1(UT \Sigma_p) \times \pi_1(UT \Sigma_q) \rightarrow A \rightarrow 1
\]

**Lemma 2.15.** The exact sequence (3) does not virtually split.
Proof. This will be proved via group cohomology. For a \( \mathbb{Z} \)-central extension of a group \( T \)
\[
1 \rightarrow \mathbb{Z} \rightarrow \widetilde{T} \xrightarrow{\alpha} T \rightarrow 1,
\]
there is an associated Euler class \( Eu(\alpha) \in H^2(T; \mathbb{Z}) \). The extension \( \alpha \) splits if and only if \( Eu(\alpha) \) vanishes; see [Bro94, Chapter 4.3]. \( Eu(\alpha) \) can be constructed using the Lyndon-Hochschild-Serre spectral sequence of \([4, \text{Chapter } 4.3]\), by taking \( Eu(\alpha) = d_2(1) \). Here \( d_2 \) is the differential \( d_2 : \mathbb{Z} \rightarrow H^2(T; \mathbb{Z}) \) on the \( E_2 \) page. The (rational) Betti number \( b_1(\widetilde{T}) \) can be computed from the spectral sequence as
\[
b_1(\widetilde{T}) = b_1(T) + \dim(\ker(d_2)).
\]
Therefore \( Eu(\alpha) \neq 0 \) is nonvanishing if and only if \( b_1(\widetilde{T}) = b_1(T) \).

Let \( A' \xrightarrow{i} A \) be a finite-index subgroup of \( A \). Let \( \overline{A'} = p^{-1}(A') \). The goal is to prove that the top row of the diagram
\[
\begin{array}{c}
1 \\ \downarrow \\ 1
\end{array}
\quad
\begin{array}{cccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \overline{A'} & \longrightarrow & A' & \longrightarrow & 1 \\
\downarrow & & \downarrow \beta & & \downarrow \alpha & & \downarrow \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \overline{A} & \longrightarrow & A & \longrightarrow & 1
\end{array}
\]
does not split. It suffices to show that \( Eu(\beta) \neq 0 \in H^2(A'; \mathbb{Q}) \). By the theory of the transfer homomorphism,
\[
i^*: H^2(A; \mathbb{Q}) \rightarrow H^2(A'; \mathbb{Q})
\]
is injective. By construction, \( Eu(\beta) = i^*(Eu(\alpha)) \). Therefore it suffices to establish that
\[
Eu(\alpha) \neq 0 \in H^2(A; \mathbb{Q}).
\]
By the above discussion, we only need to show that \( b_1(A) = b_1(\pi_1(UT\Sigma_p) \times \pi_1(UT\Sigma_q)) \). However, since \( p \geq 2 \) and \( q \geq 2 \) by assumption,
\[
b_1(\pi_1(UT\Sigma_p) \times \pi_1(UT\Sigma_q)) = b_1(\pi_1(\Sigma_p) \times \pi_1(\Sigma_q)).
\]
Since \( A \rightarrow \pi_1(\Sigma_p) \times \pi_1(\Sigma_q) \) is surjective, it follows that \( b_1(A) \geq b_1(\pi_1(\Sigma_p) \times \pi_1(\Sigma_q)) \), and so \( b_1(A) = b_1(\pi_1(UT\Sigma_p) \times \pi_1(UT\Sigma_q)) \) as desired. \( \square \)

As a corollary, we can refine the analysis of \( \sigma(T^k_c) \) for \( T_c \) a separating twist, as begun in Lemma 2.15.

**Lemma 2.16.** Let \( c \subset \Sigma_g \) be a separating curve such that each component of \( \Sigma_g \setminus \{c\} \) has genus at least 2, and let \( k > 0 \) be such that \( T^k_c \in \Gamma \). Then there exists a curve \( \overline{c} \subset \Sigma_g \) isotopic to \( c \) such that \( \sigma(T^k_c) = T^k_{\overline{c}} \).

**Proof.** If this is not the case, then \( \sigma(T^k_c) = T^l_c T^m_{\overline{c}} \) where \( c', c'' \) bound an annulus and \( l \neq 0, m \neq 0 \). Let \( A \) be the subgroup constructed above, relative to the separating curve \( c \). The image \( \sigma(A \cap \Gamma) \) must be contained in the centralizer of \( T^l_c T^m_{\overline{c}} \). In particular, \( \sigma(A \cap \Gamma) \) must be contained in the disk-pushing subgroups on the sides of \( c' \) and \( c'' \) not bounding the annulus. This gives a virtual splitting of exact sequence \([3]\), contradicting Lemma 2.15. \( \square \)
3. Proof of Theorem A

Beginning the proof. Let $\Gamma \leq I(\Sigma_g)$ be a subgroup of finite index, and suppose that $\sigma : \Gamma \to I(\Sigma_g,*)$ is a section. By the hypothesis that $g \geq 4$, there exists a separating simple closed curve $c \subset \Sigma_g$ that divides $\Sigma_g$ into subsurfaces $\Sigma_{p,1}$ and $\Sigma_{q,1}$ with $p, q \geq 2$. Let $T_c$ denote the corresponding Dehn twist. Choosing $k$ such that $T_c^k \in \Gamma$, Lemma 2.16 asserts that $\sigma(T_c^k) = T_{\tilde{c}}$ for some separating curve $\tilde{c} \subset \Sigma_g,*$. Without loss of generality, we assume that the marked point $* \in \Sigma_{p,1}$.

A standard argument using canonical reduction systems shows that $\sigma(\text{Mod}(\Sigma_{p,1} \cap \Gamma))$ is supported on the subsurface $\tilde{\Sigma}_{p,1} \sim \Sigma_{p,1},*$ bounded by $\tilde{c}$. The Birman exact sequences for $\Sigma_{p,1}$ and $\Sigma_{p,1,*}$ (restricted to the Torelli group) fit together in the following commutative diagram, where the group $\text{PB}_{1,1}(\Sigma_p)$ and the homomorphism $p_* \pi$ will be described below.

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \text{PB}_{1,1}(\Sigma_p) & \longrightarrow & I(\Sigma_{p,1,*}) & \longrightarrow & I(\Sigma_p) & \longrightarrow & 1 \\
& & \downarrow p_* & \downarrow \pi & \downarrow \sigma & \downarrow & \downarrow & \downarrow & \\
1 & \longrightarrow & \pi_1(UT\Sigma_p) & \longrightarrow & I(\Sigma_{p,1}) & \longrightarrow & I(\Sigma_p) & \longrightarrow & 1
\end{array}
\]

The group $\text{PB}_{1,1}(\Sigma_p)$ is defined as the fundamental group of the configuration space $P\text{Conf}_{1,1}(\Sigma_p)$, where

\[
P\text{Conf}_{1,1}(\Sigma_p) := \{(x, v) \mid x \in \Sigma_p, v \in T_y^1(\Sigma_p), x \neq y\}.
\]

Here, $T_y^1(\Sigma_p)$ denotes the space of unit-length tangent vectors in the tangent space $T_y(\Sigma_p)$, relative to an arbitrarily-chosen Riemannian metric. Projection onto either factor realizes $P\text{Conf}_{1,1}(\Sigma_p)$ as a fibration in two ways:

\[
\begin{array}{cccc}
\Sigma_{p,*} & \longrightarrow & UT(\Sigma_{p,*}) & \longrightarrow & \text{PConf}_{1,1}(\Sigma_p) & \longrightarrow & \Sigma_p \\
& & \downarrow p_1 & \downarrow & \downarrow p_* & \downarrow & \downarrow & \downarrow &
\end{array}
\]

3.1. The map $s$. We now come to the central object of study in the argument. Let $H = H(\Sigma_{p,1}) \cap \Gamma$ denote the handle-pushing subgroup. Combining diagrams (6) and (7), we obtain a homomorphism $\tilde{s} := p_{1,*} \circ \sigma : H \to \pi_1(\Sigma_p)$.

We will see that $\tilde{s}$ has paradoxical properties, leading to a contradiction that establishes the non-existence of the section $\sigma$. A first observation is that we can replace $\tilde{s}$ by a map between surface groups. Let $\varpi : \pi_1(UT\Sigma_p) \to \pi_1(\Sigma_p)$ denote the projection, and define $\overline{H} := \varpi(H)$. By construction $\overline{H}$ is a finite-index subgroup of $\pi_1(\Sigma_p)$.

\[\text{The Torelli group is not unambiguously defined for a surface } \Sigma_{p,1,*}. \text{ The meaning here of } I(\Sigma_{p,1,*}) \text{ is simply the full preimage } \pi^{-1}(I(\Sigma_{p,1})).\]
Lemma 3.1. There is a homomorphism
\[ s : \mathcal{H} \to \pi_1(\Sigma_p) \] (9)
such that \( s \) factors as \( s = \sigma \circ \varpi \).

Proof. As noted in Remark 2.14, \( \mathcal{H} \) has the structure of a cyclic central extension of a finite-index subgroup \( \overline{\mathcal{H}} \leq \pi_1(\Sigma_p) \). Viewed as a subgroup of \( I(\Sigma_{p,1}) \), the center of \( \mathcal{H} \) consists of elements of the form \( T^k_b \). As already observed, \( \sigma(T^k_b) = T^k_b \), where \( c \) is the boundary of the subsurface \( \Sigma_{p,1,*} \leq \Sigma_{g,*} \).

The map \( p_{1,*} : \mathrm{PB}_{1,1}(\Sigma_p) \to \pi_1(\Sigma_p) \) is induced from the boundary-capping map \( \Sigma_{p,1,*} \to \Sigma_{p,*} \). The result follows. \( \Box \)

The construction of \( s \) allows us to continue the analysis of \( \sigma \) begun in Lemma 2.9, giving a complete description of \( \sigma \) on (powers of) bounding-pair maps.

Lemma 3.2. Let \( a, b \) form a bounding pair on \( \Sigma_g \). Then there exists a bounding pair \( \overline{a}, \overline{b} \) on \( \Sigma_{g,*} \) such that \( \sigma(T^k_bT^{-k}_b) = T^k_aT^{-k}_a \) for any \( k \) such that \( T^k_bT^{-k}_b \in \Gamma \).

Proof. Let \( c \) be a separating curve on \( \Sigma_g \) dividing \( \Sigma_g \) into components \( \Sigma_{p,1}, \Sigma_{q,1} \), each of genus \( p,q \geq 2 \). Let \( a, b \) be a bounding pair on \( \Sigma_g \) such that \( c \) forms a pair of pants; observe that for any bounding pair \( a, b \), there exists a curve \( c \) as above. For instance, in Figure 1, the curves \( \{ \gamma_L, \gamma_R \} \) form such a bounding pair relative to the \( c \) as shown. As \( T^k_bT^{-k}_b \) commutes with \( T^f_c \), the same is true for the lifts \( \sigma(T^k_bT^{-k}_b) \) and \( \sigma(T^f_c) = T^f_c \). In particular, \( \sigma(T^k_bT^{-k}_b) \) is supported on exactly one component of the surface \( \Sigma_{g,*} \setminus \{ \gamma \} \). There are thus two possibilities to consider, depending on whether this component also contains \( * \).

According to Lemma 2.9, there are simple closed curves \( \overline{a}, \overline{a}' \subset \Sigma_{g,*} \) and an integer \( m \) such that
\[ \sigma(T^k_bT^{-k}_b) = T^k_aT^{-k}_a \]
(10)
The curves \( \overline{a} \) and \( \overline{a}' \) are isotopic on \( \Sigma_g \), but may not be isotopic on \( \Sigma_{g,*} \), i.e. \( \overline{a} \cup \overline{a}' \) can bound an annulus \( A \) containing the marked point \( * \). If this is not the case, then \( \overline{a}, \overline{a}' \) determine the same isotopy class on \( \Sigma_{g,*} \), and the result follows. Note that in the case where \( A \) and \( * \) are contained in distinct components of \( \Sigma_{g,*} \setminus \{ \gamma \} \), this must necessarily hold.

We therefore assume that \( A \) and \( * \) are contained in the same component \( \Sigma_{p,1,*} \subset \Sigma_{g,*} \). Since \( a, b, c \) form a pair of pants on \( \Sigma_g \), it follows that \( T^k_aT^{-k}_a \in \mathcal{H} \), the handle-pushing subgroup. In fact, there is a one-to-one correspondence between elements of \( \overline{\mathcal{H}} \) represented by (a power of) a simple closed curve on \( \Sigma_{p} \), and the set of bounding pairs \( a, b \) under consideration. We write \( \alpha(a,b) \in \pi_1(\Sigma_p) \) for the element of \( \overline{\mathcal{H}} \) corresponding to the bounding pair \( T^k_aT^{-k}_a \). Our proof now proceeds by analyzing \( s \) on such elements of \( \overline{\mathcal{H}} \).

As observed above, \( * \) may or may not be contained in the annulus \( A \). If \( * \) is not, we can reformulate the above argument by observing that \( s(\alpha(a,b))^k = 1 \). In the remaining case, we aim to show that either \( m = 0 \) or \( m = k \) in (10). As (without loss of generality) \( \overline{a}' \) becomes isotopic to \( \overline{b} \) upon capping \( c \) by a disk, it follows that
\[ s(\alpha(a,b))^k = p_{1,*}(T^k_aT^{-k}_a) = T^k_aT^{-k}_a = \alpha(a,b)^{m-k}. \] (11)
To summarize, we have shown that for all bounding pairs \( a, b \) under consideration, there is an integer \( m(a, b, k) \) such that
\[
s(\alpha(a, b)^k) = \alpha(a, b)^{m(a, b, k)}.
\]
The desired assertion \( m = 0 \) or \( m = k \) now follows from Lemma 3.3 below.

**Lemma 3.3.** Let \( G \leq \pi_1(\Sigma_p) \) be a subgroup of finite index, and let \( f : G \to \pi_1(\Sigma_p) \) be an arbitrary homomorphism. Suppose that for all simple elements \( \alpha \in \pi_1(\Sigma_p) \), there is an integer \( m(\alpha, k) \) such that
\[
f(\alpha^k) = \alpha^{m(\alpha, k)}.
\]
Then either \( m(\alpha, k) = 0 \) or else \( m(\alpha, k) = k \), independent of \( \alpha \).

**Proof.** Suppose \( \alpha, \beta \) are simple elements. Then for any \( \ell \), the conjugate \( \beta^\ell \alpha \beta^{-\ell} \) is also simple. Choose \( k, \ell \) such that \( \alpha^k \) and \( \beta^\ell \) are both elements of \( G \). Then definitionally,
\[
f(\beta^\ell \alpha^k \beta^{-\ell}) = (\beta^\ell \alpha \beta^{-\ell})^{m(\beta^\ell \alpha \beta^{-\ell}, k)}.
\]
On the other hand, it is clear that \( m(\beta, -\ell) = -m(\beta, \ell) \), and so
\[
f(\beta^\ell \alpha^k \beta^{-\ell}) = f(\beta^\ell) f(\alpha^k) f(\beta^{-\ell}) = \beta^{m(\beta, \ell)} \alpha^{m(\alpha, k)} \beta^{-m(\beta, \ell)}.
\]
For an arbitrary nontrivial element \( \gamma \in \pi_1(\Sigma_p) \) and integers \( m, n \), the elements \( \gamma^m \) and \( \gamma^n \) are conjugate if and only if \( m = n \). It follows that \( m(\alpha, k) = m(\beta^\ell \alpha \beta^{-\ell}, k) \). Thus,
\[
(\beta^\ell \alpha \beta^{-\ell})^{m(\alpha, k)} = \beta^{m(\beta, \ell)} \alpha^{m(\alpha, k)} \beta^{-m(\beta, \ell)},
\]
and so
\[
\beta^{\ell-m(\beta, \ell)} \alpha^{m(\alpha, k)} \beta^{m(\beta, \ell)} = \alpha^{m(\alpha, k)}.
\]
Nontrivial elements \( x, y \in \pi_1(\Sigma_p) \) commute if and only if there are nonzero integers \( c, d \) such that \( x^c = y^d \). As \( \alpha, \beta \) were assumed to be simple, we conclude that one of three conditions must hold: (1) \( \alpha = \beta^{\pm 1} \), or (2) \( \ell = m(\beta, \ell) \) or else (3) \( m(\alpha, k) = 0 \).

Case (1) provides no further information; we henceforth assume that \( \alpha \neq \beta^{\pm 1} \). To finish the argument, we must show that if \( m(\alpha, k) = 0 \), then \( m(\beta, \ell) = 0 \) for all \( \beta, \ell \). Suppose to the contrary that there is some \( \beta \) such that \( m(\beta, \ell) \neq 0 \). Reversing the roles of \( \alpha \) and \( \beta \) in the above argument, we see that (2) must hold and so \( k = m(\alpha, k) \), but this contradicts the assumption \( m(\alpha, k) = 0 \).

Translated into the setting of the homomorphism \( s : \mathcal{H} \to \pi_1(\Sigma_p) \), Lemmas 3.2 and 3.3 combine to give the following immediate but crucial corollary.

**Corollary 3.4.** The homomorphism \( s : \mathcal{H} \to \pi_1(\Sigma_p) \) has one of the following properties:

(A) \( s(\alpha^k) = \alpha^k \) for all elements \( \alpha^k \in \mathcal{H} \) such that \( \alpha \in \pi_1(\Sigma_p) \) is simple.

(B) \( s(\alpha^k) = 1 \) for all elements \( \alpha^k \in \mathcal{H} \) such that \( \alpha \in \pi_1(\Sigma_p) \) is simple.

The next step of the argument considers cases (A) and (B) separately. In both cases, we will see that the formula defining \( s \) on simple elements extends to all of \( \mathcal{H} \).
3.2. Case (A).

Lemma 3.5. Suppose $s$ has property (A) of Corollary 3.4. Then $s : H \to \pi_1(\Sigma_p)$ is given by the inclusion map.

Proof. This follows easily from the method of proof of Lemma 3.3. Let $\beta \in H$ be an arbitrary element, let $\alpha \in \pi_1(\Sigma_p)$ be simple, and let $\alpha^k \in H$. Then $\beta\alpha\beta^{-1}$ is also simple, and $\beta\alpha^k\beta^{-1} \in H$. As $\beta\alpha\beta^{-1}$ is simple,

$$f(\beta\alpha^k\beta^{-1}) = \beta\alpha^k\beta^{-1};$$
onumber

on the other hand,

$$f(\beta\alpha^k\beta^{-1}) = f(\beta)\alpha^k f(\beta)^{-1}.$$  

Arguing as in Lemma 3.3, this implies $f(\beta) = \beta$ as desired. \qed

3.3. Case (B).

Lemma 3.6. Suppose $s$ has property (B) of Corollary 3.4. Then $s : H \to \pi_1(\Sigma_p)$ is the trivial homomorphism.

The proof of Lemma 3.6 will require a further analysis of $s$. This will require some preliminary work to describe. By passing to a further finite-index subgroup $\Gamma' \leq \Gamma$ if necessary, we can assume that $H \leq \pi_1(UT\Sigma_p)$ is characteristic and hence the conjugation action of $I(\Sigma_p,1)$ on $\pi_1(UT\Sigma_p)$ preserves $H$. This descends to an action of $I(\Sigma_p,*)$ on $H$. Thus there is a homomorphism

$$\lambda : I(\Sigma_p,*) \to \text{Aut}(H).$$

Consider now the images $\Gamma \leq I(\Sigma_p,*)$ and $\Gamma \leq I(\Sigma_p)$. By construction, $\Gamma \cap \pi_1(\Sigma_p) = H$. As conjugation by $H$ is an inner automorphism, $\lambda$ descends to a homomorphism

$$\lambda : \Gamma \to \text{Out}(H).$$

Lemma 3.7. The homomorphism $s$ is $\Gamma$-equivariant. That is, for any outer automorphism $[\alpha] \in \Gamma$ and any $x \in H$, the conjugacy classes of $s(\alpha \cdot x)$ and $\alpha \cdot s(x)$ in $\pi_1(\Sigma_g)$ coincide.

Proof. Let $a \in \Gamma$ be given. Choose an element $\alpha \in \Gamma$ descending to the outer automorphism class $a$. By construction, for $x \in H$, the image $s(x)$ is given by $(p_{1,*} \circ \sigma)(\tilde{x})$, where $\tilde{x} \in H$ is any lift. On $H$, the action of $\Gamma$ is induced by the conjugation action $\tilde{x} \mapsto \alpha\tilde{x}\alpha^{-1}$. Thus

$$s(a \cdot x) = p_{1,*}(\sigma(\alpha\tilde{x}\alpha^{-1})) = p_{1,*}(\sigma(\alpha)) s(x) p_{1,*}(\sigma(\alpha))^{-1}.$$  

Here we exploit the fact that $p_{1,*} : \text{PB}_{1,1}(\Sigma_p) \to \pi_1(\Sigma_p)$ is the restriction of the forgetful homomorphism

$$p_{1,*} : I(\Sigma_{p,1,*}) \to I(\Sigma_{p,*}).$$

To finish the argument, it suffices to show that $[p_{1,*}((\sigma(\alpha)))] = a$ as elements of $I(\Sigma_p)$. This follows from the fact that $\sigma : \Gamma \to I(\Sigma_{p,1,*})$ is a section of the map $p_{2,*} : I(\Sigma_{p,1,*}) \to I(\Sigma_{p,1})$ in combination
with the commutativity of the diagram

$$I(\Sigma_{p,1}) \xrightarrow{p_1 \cdot *} I(\Sigma_{p,1}) \xrightarrow{p_2 \cdot *} I(\Sigma_{p,1}) \xrightarrow{} I(\Sigma_p).$$

Proof. (of Lemma 3.6) Let \( x \in \overline{\mathcal{H}} \) be an arbitrary element, and let \( d \) be an arbitrary separating curve on \( \Sigma_{p,1} \). Taking \( k \) such that \( T_d^k \in \Gamma \) and applying Lemma 3.7, there is an equality

$$s(T_d^k(x)) = T_d^k(s(x))$$

of conjugacy classes in \( \pi_1(\Sigma_p) \). To proceed, we will analyze the conjugacy class of \( T_d^k(x) \) in \( \overline{\mathcal{H}} \). This is complicated by the fact that in this expression, \( T_d^k \) acts on \( x \) not as a separating twist on \( \Sigma_p \), but rather as the lift of such a twist to the finite-sheeted cover \( \Sigma_r \to \Sigma_p \) corresponding to the finite-index subgroup \( \overline{\mathcal{H}} \).

Lemma 3.8. Let \( T_d \) be a Dehn twist on \( \Sigma_{p,1} \), and let \( x \in \overline{\mathcal{H}} \) be an arbitrary element. Then there exists some \( k \geq 1 \), simple elements \( \gamma_1, \ldots, \gamma_N \) of \( \pi_1(\Sigma_p) \) and integers \( f_1, \ldots, f_N \), such that \( \gamma_i^{f_i} \in \overline{\mathcal{H}} \) for all \( i \), and there is an expression

$$T_d^k(x) = \gamma_1^{f_1} \cdots \gamma_N^{f_N} x$$

of elements of \( \overline{\mathcal{H}} \).

Proof. Let \( \pi : \Sigma_r \to \Sigma_p \) be the covering map associated to the containment \( \overline{\mathcal{H}} \leq \pi_1(\Sigma_p) \). For \( k \) sufficiently large, \( T_d^k \) lifts to a mapping class on \( \Sigma_r \). This lift is not unique, but there is a unique lift up to the action of the deck group of \( \pi \). Since \( T_d \) is a Dehn twist on \( \Sigma_p \), there is a distinguished lift

$$\widetilde{T}_d^k = \prod T_{d_i}^{k_i}$$

of \( T_d^k \) as a multitwist on \( \Sigma_r \), for certain integers \( k_i \). Here, the set \( \{ \widetilde{d}_i \} \) consists of all components of the preimage \( \pi^{-1}(d) \). Observe that each curve \( \widetilde{d}_i \) is contained in the \( \pi_1(\Sigma_p) \)-conjugacy class of \( d^{e_i} \) for some \( e_i \), and that also the conjugacy class of \( d^{e_i} \) is contained in \( \overline{\mathcal{H}} \). As the deck group is finite, we can assume that \( T_d^k \) acts on \( \overline{\mathcal{H}} \) as a genuine multitwist, possibly after further increasing \( k \).

Choose representative curves for each \( \widetilde{d}_i \), and represent \( x \in \overline{\mathcal{H}} \) as a map \( x(t) : [0,1] \to \Sigma_r \), chosen so as to intersect the set \( \{ d_i \} \) in minimal position. This determines a sequence of arcs \( \alpha_1, \ldots, \alpha_{N+1} \) as follows. The points of intersection between \( x \) and \( \{ d_i \} \) can be enumerated via \( 0 < t_1 < \cdots < t_N < t_{N+1} = 1 \) such that \( x(t) \) intersects the multicurve \( \{ d_i \} \) if and only if \( t = t_m \) for some \( 1 \leq m \leq N \). The arc \( \alpha_m \) is then defined as the image of \( x \) restricted to the interval \( [0,t_m] \) (so in particular, \( \alpha_{N+1} = x \)).

Each arc \( \alpha_m \) connects \( * \) to one of the curves \( \widetilde{d}_i \), and thus determines an element \( \gamma_m' \) of \( \overline{\mathcal{H}} \) in the conjugacy class of the appropriate \( \widetilde{d}_i \). The geometric description of \( T_d^k \) as a multitwist allows one to obtain an expression for \( T_d^k(x) \) of the desired form. The curve \( T_d^k(x) \) can be described as follows: first \( T_d^k(x) \) follows \( \alpha_1 \) to the first point of intersection with \( \{ d_i \} \); this is the curve corresponding to \( \gamma_1'. \)
Then $T_d^k(x)$ winds around $\gamma'_1$ a number of times $f'_1$ as specified by (14). Then $T_d^k(x)$ continues along the portion of $\alpha_2$ running from $t = t_1$ to $t = t_2$, and continues, winding around each $\gamma'_i$ some number of times $f'_i$ in succession.

By construction, after each crossing of $\gamma'_m$, the curve $T_d^k(x)$ traverses the portion of $\alpha_{m+1}$ from $t_m = t_1$ to $t_{m+1}$. This can be replaced by first backtracking along $\alpha_m$, and then traversing the entirety of $\alpha_{m+1}$. Written as an element of $\pi_1(\Sigma_r) = \mathcal{H}$, this analysis produces an expression

$$T_d^k(x) = \gamma'_1f'_1 \cdots \gamma'_Nf'_N x.$$ 

The claim now follows from the observation that each $\gamma'_m$ is a based loop on $\Sigma_r$ corresponding to a curve $\tilde{d}_i$. Each $\tilde{d}_i$ is a component of the preimage of $d$. As an element of $\pi_1(\Sigma_p)$, each $\gamma'_m$ is thus of the form $\gamma'_m = \gamma^e_m$ for some simple curve $\gamma_m \in \pi_1(\Sigma_p)$ in the conjugacy class of $d$. Taking $f_m = e_m f'_m$, the result follows. □

Applying Lemma 3.8, there is an equality

$$s(T_c^k(x)) = s(\gamma_1f_1 \cdots \gamma_Nf_N x) = s(\gamma_1f_1) \cdots s(\gamma_Nf_N)s(x) = s(x),$$

(15)

with the last equality holding by Corollary 3.4(B) since all the $\gamma_i$ are simple. We conclude that there is an equality of $\pi_1(\Sigma_g)$-conjugacy classes

$$T_c^k(s(x)) = s(x).$$

By Lemma 2.8 this implies that $s(x)$ is disjoint from $c$ as curves on $\Sigma_p$. As this argument applies for every separating curve on $\Sigma_p$, we conclude that $s(x)$ must be disjoint from every separating curve $c$ on $\Sigma_p$. Since $p \geq 2$, an easy argument with the change-of-coordinates principle implies that any nontrivial element $y \in \pi_1(\Sigma_p)$ must intersect some separating curve $c$. This shows that $s(x)$ must be trivial as claimed. □

3.4. **Finishing the argument.** The final stage of the argument exploits the fact that the existence of a section $\sigma : \mathcal{H} \to \text{PB}_1(\Sigma_p)$ places strong homological constraints on the map $s$. Throughout this section, our cohomology groups will implicitly have rational coefficients. To simplify matters further, we forget the (inessential) tangential data encoded in the space $\text{PConf}_{1,1}(\Sigma_p)$, and consider instead the induced section

$$\sigma : \mathcal{H} \to \text{PB}_2(\Sigma_p);$$

here $\text{PB}_2(\Sigma_p) = \pi_1(\text{PConf}_2(\Sigma_p))$ is the fundamental group of the configuration space of two ordered points on $\Sigma_p$. The space $\text{PConf}_2(\Sigma_p)$ is, by definition, given as

$$\text{PConf}_2(\Sigma_p) := \Sigma_p \times \Sigma_p \setminus \Delta,$$

where $\Delta$ is the diagonal locus. In this setting, there is a factorization

$$s = p_{2,*} \circ \sigma.$$
A crucial consequence of this is that \( s^* : H^*(\Sigma_p) \to H^*(\overline{H}) \) factors through \( H^*(\text{PB}_2(\Sigma_p)) \). The following lemma is proved by a standard argument using the formulation of Poincaré duality via Thom spaces.

**Lemma 3.9.** Let \( [\Delta] \in H^2(\Sigma_p \times \Sigma_p) \) denote the Poincaré dual class of \( \Delta \), and let
\[
i : \text{PConf}_2(\Sigma_p) \to \Sigma_p \times \Sigma_p
\]
de note the inclusion map. Then \( i^*([\Delta]) = 0 \in H^2(\text{PConf}_2(\Sigma_p)) \).

**Concluding the proof.** Let \( i : \overline{H} \to \pi_1(\Sigma_p) \) denote the inclusion. Consider the product homomorphism
\[
i \times s : \overline{H} \to \pi_1(\Sigma_p) \times \pi_1(\Sigma_p) \cong \pi_1(\Sigma_p \times \Sigma_p).
\]
Observe that this coincides with the section map \( \sigma : \overline{H} \to \text{PB}_2(\Sigma_p) \), so that there is a factorization
\[
i \times s = i \circ \sigma.
\]
By Lemma 3.9 it follows that \( (i \times s)^*([\Delta]) = (i \circ \sigma)^*([\Delta]) = 0 \in H^2(\overline{H}) \).

Let \( x_1, y_1, \ldots, x_p, y_p \in H^1(\Sigma_p) \) denote a symplectic basis with respect to the cup product form; let also \( [\Sigma_p] \) denote the fundamental class. Then
\[
[\Delta] = 1 \otimes [\Sigma_p] + [\Sigma_p] \otimes 1 + \sum_{i=1}^p x_i \otimes y_i - y_i \otimes x_i
\]
as a class in
\[
H^2(\Sigma_p \times \Sigma_p) \cong (H^0(\Sigma_p) \otimes H^2(\Sigma_p)) \oplus (H^2(\Sigma_p) \otimes H^0(\Sigma_p)) \oplus (H^1(\Sigma_p) \otimes H^1(\Sigma_p)).
\]
Thus
\[
0 = (i \times s)^*([\Delta]) = i^*(1)s^*([\Sigma_p]) + i^*([\Sigma_p])s^*(1) + \sum_{j=1}^p i^*(x_j)s^*(y_j) - i^*(y_j)s^*(x_j).
\]
We will see that in both cases (A) and (B), this is a contradiction. Lemma 3.5 asserts that in Case (A), \( s^* = i^* \) in degree 1. Since \( H^*(\Sigma_p) \) is generated as an algebra in degree 1, this implies that \( s^* = i^* \) in degree 2 as well. Then a basic calculation shows that in this case,
\[
(i \times s)^*([\Sigma_p]) = (i \times i)^*([\Sigma_p]) = \chi(\Sigma_p)[\overline{H}],
\]
where \( \chi(\Sigma_p) \) denotes the Euler characteristic and \( [\overline{H}] \) denotes the fundamental class of the surface group \( \overline{H} \). As this is nonzero, we have arrived at a contradiction. Similarly, Lemma 3.6 asserts that in Case (B), \( s^* = 0 \) in positive degrees. Then \( (i \times s)^*([\Sigma_p]) = i^*([\Sigma_p]) = \chi(\Sigma_p)[\overline{H}] \neq 0 \), again a contradiction. □
THE BIRMAN EXACT SEQUENCE DOES NOT VIRTUALLY SPLIT

REFERENCES


