(Math 360) Homework 2:

Due January 29, 2009

Exercise 1: Prove that no order can be defined in the complex field that turns it into an ordered field. (Hint: -1 is a square.)

Exercise 2: If z is a complex number, prove that there exists an $r \geq 0$ and a complex number $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

Exercise 3: If $z_1, \ldots, z_n$ are complex prove

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

Exercise 4: If x, y are complex then

$$||x| - |y|| \leq |x - y|$$

Exercise 5: Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same size. I.e. that there is a bijection (a one to one pairing such that every element of $\mathbb{N}$ is paired with exactly one element of $\mathbb{Z}$ and vice versa)

Exercise 6: Suppose Y and $\{X_i : i \in I\}$ are sets. Prove that $Y \cap (\bigcup_{i \in I} X_i) = \bigcup_{i \in I}(Y \cap X_i)$

Exercise 7: Construct a bounded set of real numbers with exactly three limit points.

Exercise 8: Let $A_1, A_2, \ldots$ be subsets of a metric space.

(a) Suppose $B_n = \bigcup_{i=1}^n A_i$. Prove $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$

(b) If $B = \bigcup_{i=1}^{\infty} A_i$ prove $\overline{B} \supset \bigcup_{i=1}^{\infty} \overline{A_i}$

(c) Given an example where $\overline{B} \neq \bigcup_{i=1}^{\infty} \overline{A_i}$

<u>Exercise 9:</u> Let X be an infinite set. For $p, q \in X$ let $d(p, q) = 1$ if $p \neq q$ and $d(p, q) = 0$ if $p = q$.

Show that (X, d) is a metric space. What are the open sets?

<u>Exercise 10:</u> For each of the following determine whether or not they define a metric on $\mathbb{R}^1$

(1) $d_1(x, y) = (x - y)^2$

(2) $d_2(x, y) = \sqrt{|x - y|}$

(3) $d_3(x, y) = |x^2 - y^2|$

(4) $d_4(x, y) = |x - 2y|$

(5) $d_5(x, y) = \frac{|x-y|}{1+|x-y|}$