(Math 360) Homework 10:

Due April 16, 2009

All numbered exercises are from Rudin's Principles of Mathematical Analysis.

Exercise 1: Chapter 6, Exercise 19.

Exercise 2: (a) Suppose $A \subseteq A_1 \cup A_2 \cup \cdots \cup A_n \subseteq \mathbb{R}$ and each of A_i and A all have length. Show that $\mu(A) \leq \sum_{i=1}^{n} \mu(A_i)$.

(b) If A is compact show that A has measure zero if and only if it has length zero.

Exercise 3: Let $R([a,b]) = \{f : [a,b] \to \mathbb{R} | f \in \mathcal{R}\}$. Define

$$d(f,g) = \int_a^b |f(x) - g(x)| dx$$

Is d a metric on $R([a,b])$.

Exercise 4: Suppose $f(x), g(x)$ are continuous functions and $\int_b^a f(x)dx = \int_b^a g(x)dx$. Show there is an $x \in [a,b]$ such that $f(x) = g(x)$.

Exercise 5: Find an example of a continuous function $f : [0, \infty) \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in [0, \infty)$, $\int_0^\infty f(x)dx$ is defined, but f is not bounded.

Exercise 6: Suppose f is continuously differentiable on $[0,1]$ such that $\sup_{x \in [0,1]} |f'(x)| = M < \infty$. Prove that

$$\left| \int_0^1 f(x)\ dx - \sum_{i=1}^{n} \frac{f(i/n)}{n} \right| \leq \frac{M}{n}$$

Exercise 7: Show that if $\varphi(x) \geq 0$ on a connected compact set $A \subseteq \mathbb{R}$, φ is continuous and increasing in x, and f is positive and integrable, then $f\varphi$ is integrable and $\int_A f\varphi = \varphi(c) \int_A f$ for some point $c \in A$ (here $\int_A g(x)dx = \int_a^b \xi_A(x) g(x)dx$ where $A \subseteq [a, b]$.

Exercise 8: (a) Show that if $f : [0, \infty)$ is a non-negative, integrable and uniformly continuous then $\lim_{x\to\infty} f(x) = 0$

(b) Show (by example) that if you remove the word non-negative, the word integrable, or replace uniformly continuous with continuous then part (a) doesn't necessarily hold.