# (Math 360) Homework 1:

Due January 22, 2007

Exercise 1: If $r$ is rational ($r \neq 0$) and $x$ is irrational, prove $r + x$ and $rx$ are irrational.

Exercise 2: Prove that there is no rational number whose square if 12

Exercise 3: Prove the axioms of multiplication for a field imply the following statements:

(a) If $x \neq 0$ and $xy = xz$ then $y = z$

(b) If $x \neq 0$ and $xy = x$ then $y = 1$

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$

(d) If $x \neq 0$ then $1/(1/x) = x$

Exercise 4: Let $E$ be a nonempty subset of an ordered set. Suppose $\alpha$ is a lower bound of $E$ and $\beta$ is an upper bound of $E$. Prove $\alpha \leq \beta$.

Exercise 5: Let $A$ be a nonempty set of real numbers which is bounded below and let $-A = \{-x : x \in A\}$ (i.e. the set of all $-x$ with $x \in A$). Prove

$$\inf A = -\sup(-A)$$

Exercise 6: Suppose $b > 0$ a real number. If $m, n, p, q$ are integers, $n > 0, q > 0$ and $r = m/n = p/q$ prove

$$(b^m)^{1/n} = (b^p)^{1/q}$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$ (Note: You can assume that for all positive real numbers $c$ and integers $p$ there is a unique real number $d$ with $d^p = c$ (i.e. $d = c^{1/p}$ is well defined))

Exercise 7: Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$ and $r > 0$. Prove

(a) If $2r > d$ there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r$$

(b) If $2r = d$ there is exactly one such $\mathbf{z}$

(c) If $2r < d$ there is no such $\mathbf{z}$

How must these statements be modified if $k$ is 2 or 1?

Exercise 8: If $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$ prove that there exists $\mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = \mathbf{0}$