Problem 1

Proof. a)

In general, we need only check the axioms when primes are involved, as we know $\mathbb{R}$ is a field.

(A3) Additive associativity - (a+b) + c' = ((a+b) + c)' = (a+ (b+c))' = a + (b+c)' = a + (b+c'), and likewise for 2 or 3 primes

(A4)Additive identity: 0 + a' = (0+a)' = (a)' = a'.

(A5)Additive inverse: -(a') = (-a)' since a'+(-a)' = a -a = 0.

(M3)Multiplicative associativity - (ab)c' = ((ab)c)' = (a(bc))' = a(bc)' = a(bc'), and likewise for 2 or 3 primes.

(M4)Multiplicative identity: 1a' = (1a)' = (a)' = a'.

b) Let $x = 1$, $y = 0'$. Then $nx = n \in \mathbb{R}$ and so by definition, $nx < 0'$ since $0' \in \mathbb{R}'$.

c) Consider the set $\mathbb{R} \subseteq F$. Then $\mathbb{R}$ is bounded above by, say, $0'$, again by definition, but it has no least upperbound. To see this second statement, notice that every upper bound is x' for some x. But for any given x', $(x-1)'$ is smaller and an upperbound for $\mathbb{R}$. Thus, there cannot be a least upperbound. □

Problem 2.

Proof. 2a. $\{p_n\}$ is equivalent to $\{p_n\}$ since $d(p_n, p_n) = 0$ so that $\lim_{n\to\infty} d(p_n, p_n) = 0$.

If $\{p_n\}$ is equivalent to $\{q_n\}$, then $\{q_n\}$ is equivalent to $\{p_n\}$ since $\lim_{n\to\infty} d(q_n, p_n) = \lim_{n\to\infty} d(p_n, q_n) = 0$.

If $\{p_n\}$ is equivalent to $\{q_n\}$ and $\{q_n\}$ is equivalent to $\{r_n\}$, then $\{p_n\}$ is equivalent to $\{r_n\}$ since $\lim_{n\to\infty} d(p_n, r_n) \leq \lim_{n\to\infty} d(p_n, q_n) + d(q_n, r_n) = \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, r_n) = 0 + 0 = 0$.

Thus, this does give an equivalence relation.

2b. We know $d(p_n, q_n) \leq d(p_n, p'_n) + d(p'_n, q_n) \leq d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n)$ and thus $\lim_{n\to\infty} d(p_n, q_n) \leq \lim_{n\to\infty} d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n) = \lim_{n\to\infty} d(p_n, p'_n) + \lim_{n\to\infty} d(p'_n, q'_n) + \lim_{n\to\infty} d(q'_n, q_n) = 0 + \lim_{n\to\infty} d(p'_n, q'_n) + 0 = \lim_{n\to\infty} d(p'_n, q'_n)$.

2c. Let P_i be a cauchy sequence in X^*. We have to show P_i converges. To that end, for each P_i, let p^i_j be a sequence in X with $\{p^i_j\} \in Q_i$. For each i, since p^i_j is cauchy, there is a number N_i such that for all $n, m \geq N_i$, $d(p^i_n, p^i_m) < 1/i$. We may assume wlog that $N_1 < N_2 < N_3, ...$

Define a sequence $q_i = p^i_{N_i}$. I claim that q_i is cauchy and that P_i converges to the equivalence class of q_i, denoted by $[q_i] = Q$.

To see this, let $\epsilon > 0$. Since P_i forms a cauchy sequence, there is an N such that for any $n, m \geq N$, $\Delta(P_n, P_m) < \epsilon/3$. But $\Delta(P_n, P_m) = \lim_{i\to\infty} d(p_i^n, p_i^m) < \epsilon/3$ so there is an M' suchtaht $\forall\ j \geq M'$, we have $d(p_j^n, p_j^m) < \epsilon/3$.

We also know there is an N such that $1/N < \epsilon/3$.

Let $N' = \max\{M, M', N\}$ and let $n, m \geq N'$. Then for any $j \geq \max\{N_n, N_m, M, M', N\}$, we'll have $d(q_n, q_m) = d(p_{N_n}^n, p_{N_m}^m) \leq d(p_{N_n}, p_j^n) + d(p_j^n, p_j^m) + d(p_j^m, p_{N_m}^m) < \epsilon/N' + \epsilon/3 + \epsilon/N' < \epsilon$.

Thus, $\{q_n\}$ forms a Cauchy sequence. Let Q be the equivalence class of $\{q_n\}$.

Now I claim that P_n converges to Q. To see this, let $\epsilon > 0$. Again, since P_n is Cauchy, there is an N such that for $n, m \geq N$, $\Delta(P_n, P_m) < \epsilon/2$. But $\Delta(P_n, P_m) = \lim_{i\to\infty} d(p_i^n, p_i^m) < \epsilon/2$, so there is an M' such that for $i \geq M'$, $d(p_i^n, p_i^m) < \epsilon/2$.

Then, for any $n \geq \max\{N, M'\}$, we have $\Delta(P_n, Q) = \lim_{i\to\infty} d(p_i^n, q_i)$. Then, for $i \geq M'$, we have $d(p_i^n, q_i) = d(p_i^n, p_{N_i}^i) \leq d(p_i^n, p_i^i) + d(p_i^i, p_{N_i}^i) \leq \epsilon/2 + \epsilon/i \leq \epsilon$. Thus, we have that $\lim_{i\to\infty} d(p_i^n, q_i) \leq \epsilon$. But this implies that for $n \geq \max\{N, M'\}$, $\Delta(P_n, Q) \leq \epsilon$ for any $\epsilon > 0$. It follows that $P_n \to Q$.

Thus, since every Cauchy sequence converges, it follows that X^* is complete.

2d. We've already showed that $\Delta(P, Q)$ is independent of the choice of Cauchy sequence in P and Q. Thus, to evaluate $\Delta(P_p, P_q)$, we may as well choose the two Cauchy sequences $(p, p, p, p, ...)$ and $(q, q, q, q, ...)$.

Then $\Delta(P_p, P_q) = \lim_{i\to\infty} d(p, q) = d(p, q)$.

2e. To see $\phi(X)$ is dense, let $P \in X^*$. We must show P is a limit point of $\phi(X)$. Recall that this is the same as saying P is a limit of a sequence all of whose terms lie in $\phi(X)$. So, now, choose any Cauchy sequence p_i in P. Consider the sequence P_{p_i}. Each element of this sequence is by definition in $\phi(X)$. I claim that P_{p_i} converges to P.

To see this, let $\epsilon > 0$. Then, since p_i is Cauchy, there is an N such that for any $n, m \geq N$ we have $d(p_n, p_m) < \epsilon$. Letting $i > N$, we have that $\Delta(P_{p_i}, P) = \lim_{n\to\infty} d(p_i, p_n)$. But since $i > N$ and since $n \to \infty$, it follows that $\lim_{n\to\infty} d(p_i, p_n) \leq \epsilon$. Thus, the sequence P_{p_i} converges to P. Thus, $\phi(X)$ is dense.

Now, if X was complete to begin with, I want to show that $X^* \subseteq \phi(X)$. Since $\phi(X) \subseteq X^*$ holds automatically, from this I'll conclude that $X^* = \phi(X)$. So, let $P \in X^*$. Let p_i be any sequence in P. Then, by definition, p_i is Cauchy. Since X is complete, p_i converges to p for some $p \in X$. But then $\lim_{i\to\infty} d(p_i, p) \to 0$, but this means that the sequence $\{p_i\}$ and the constanst sequence $\{p\}$ are in the same equivalence class. But the constant sequence is in the equivalence class given by P_p, so we must have $P = P_p$.

Thus, $P = \phi(p)$, so $P \in \phi(X)$.

□

Problem 3.

Proof. Consider $g(x) = f(x) - x$. Notice first that g is continuous since it's a difference of two continuous functions. Also note that $g(0) = f(0) - 0 \in [0, 1]$ and $g(1) = f(1) - 1 \in [-1, 0]$.

Now, either $g(0) = 0$ or not. If $g(0) = 0$, then by definition, $f(0) = 0$, so $x = 0$ is the desired fixed point.

In the case that $g(0) \neq 0$, it follows from the above that $g(0) > 0$. Now, we either have that $g(1) = 0$ or not. Again, if $g(1) = 0$, then $f(1) = 1$ so that $x = 1$ is the desired fixed point. If not, then $g(1) < 0$. But since g is continuous, $g(0) > 0$ and $g(1) < 0$, it follows from the Intermediate Value Theorem that there is some $x_0 \in (0, 1)$ such that $g(x_0) = 0$. Then $0 = g(x_0) = f(x_0) - x_0$ so $f(x_0) = x_0$. Thus, x_0 is the desired fixed point.

Thus, in every case there is a fixed point. □

Problem 4.

Proof. a)I'll check all the axioms.

1. $d(x, y) \geq 0$. This occurs since we're measuring the distance along a path, and distances (in $\mathbb{R}^2$) satisfy this. 2. $d(x, x) = 0$. This occurs because the shortest "'path"' from x to x is just the constant path, which clearly has length 0. 3. $d(x, y) = 0$ implies $x = y$. If $d(x, y) = 0$, then there is a path from x to y of length 0. But the only choice for such a path is clearly the constant path. Thus, $x = y$. 4. $d(x, y) \leq d(x, z) + d(z, y)$. Suppose the shortest path from x to z is p_1 and the shortest path from z to y is p_2. Then moving along p_1 and then p_2 is path from x to y. But $d(x, y)$ is the lenght of the SHORTEST path from x to y so that $d(x, y) \leq$ the length of the concatenated path $= d(x, z) + d(z, y)$.

b) For this part, I'll use the usual metric is gets as a subspace in $\mathbb{R}^2$. Then S^1 is compact because it's closed and bounded. It's clearly bounded (it's completely contained in $N_2(0)$). But why is it closed? Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2 - 1$. Then the zero set of f, $Z(f) = \{(x, y)|x^2 + y^2 = 1\}$. That is, the zero set of f is precisely S^1. But problem 3 in Chapter 4 shows that $Z(f)$ is closed. Thus, S^1 is closed and bounded, and hence compact.

c) This follows immediately from theorem 4.16, since S^1 is a compact metric space.

d) (In this problem, I'll use the usual metric on S^1, i.e., the one inherited from $\mathbb{R}^2$).

First, I claim that if U and V are two connected open sets such that $U \cap V \neq \emptyset$, then $U \cup V$ is also connected. To see this, assume $U \cup V = A \cup B$ where A and B are separated. Then $U \subseteq A \cup B$ and likewise $V \subseteq A \cup B$. But, U is connected, so it can't be separated. It follows that $U \subseteq A$ or $U \subseteq B$. Assume wlog $U \subseteq A$. Likewise V must be contained in just one of A and B. But since $\exists p \in V \cap U$, it follows that $p \in A$, so that V must also be entirely contained in A. But then, we have $A \cup B = U \cup V = A$, but this implies that $B \subseteq A$, so that $B \cap A \neq \emptyset$, contradicting the separatedness of A and B.

Thus, $U \cup V$ is connected.

Now to apply this.

Consider the two functions $f : [-1, 1] \to S^1$ and $g : [-1, 1] \to S^1$ given by $f(x) = (x, \sqrt{1 - x^2})$ and $g(x) = (x, -\sqrt{1 - x^2}$.

By theorem 4.10, it follows that these functions are continuous iff both slots are. Since $h(x) = x$ is continuous, we need only argue that $\pm\sqrt{1 - x^2}$ is continuous. Since $h(x) = x$ is continuous, it follows from theorem 4.9 that $1 - x^2$ is continuous. Since $j(x) = x^2$ (here, j has the restricted domain $[0, \infty)$) is also continuous (by theorem 4.9), it follows from theorem 4.17 that $j^{-1} = \sqrt{x}$ is also continuous. Finally, by theorem 4.7, $\sqrt{1 - x^2}$ is continuous (and likewise $-\sqrt{1 - x^2}$).

Thus, both f and g are continuous. But if you apply a continuous function to a connected set, then you get a connected output 4.22. Thus, $U = f([-1, 1])$ and $V = g([-1, 1])$ are connected. But then $U \cap V = \{(1, 0), (-1, 0)\} \neq \emptyset$, so it follows from the above lemma that $U \cup V$ is connected. But $S^1 = U \cup V$. To see this, note first that by definition, $U \cup V \subseteq S^1$. To see the other way, for any $(x, y) \in S^1$, we have $x^2 + y^2 = 1$, so that $y = \pm\sqrt{1 - x^2}$ which is in U or V depending on the sign of y.

Thus, $S^1 = U \cup V$ is connected.

2e. Let $h : S^1 \to \mathbb{R}$ be any continuous function. By part c), there is an x_0 in S^1 such that $h(x_0) = M$ is the maximum value of h. By rotating everything if neccesary, we may assume wlog that $x_0 = (1, 0)$. Consider the point $(-1, 0)$. Then $h(-1, 0)$ is either $h(x_0)$ or it's not. In the case that $h(-1, 0) = h(x_0)$, h is not 1-1, so we'd be done. Thus, we can assume $h(-1, 0) \neq h(x_0)$. Further, since x_0 is the point where the maximum occurs, we must have $h(-1, 0) < h(x_0)$.

Now, consider the two functions $h \circ f$ and $h \circ g$. These are boths functions from $[-1, 1]$ into $\mathbb{R}$.

We know that $h(f(-1)) = h(-1, 0) < h(x_0) = h(f(1))$ and likewise $h(g(-1)) < h(g(1))$ (infact, $h(g(-1)) = h(f(-1))$ and $h(g(1)) = h(f(1))$). But by the intermediate value theorem, for any real r with $h(f(-1)) < r <$

$h(f(1))$, there is some x_0 in $(-1,1)$ such that $h(f(x_0)) = r$ and there is some (most likely different) $y_0 \in (-1,1)$ such that $h(g(x_0)) = r$.

But notice then that since both x_0 and y_0 are in the interval $(-1,1)$, it follows that $f(x_0) \neq g(x_0)$. However, $h(f(x_0)) = r = h(g(x_0))$. Thus, h is not 1-1.

□

Problem 5.

Proof. a) Every axiom is obvious except the triangle inequality. So, pick any x, y, and z in X.

Assume first $d(x,y) \leq 1$. Then, we have $d(x,y) \leq d(x,z) + d(z,y)$ since (X,d) is a metric space. If neither of $d(x,z)$ nor $d(z,y)$ is bigger than 1, then the triangle inequality holds for D since it holds for d. If at least one of the two is bigger than 1, then D of that one will contribute at least one, so we'll have $d(x,y) \leq 1 \leq 1+$ some other term, which is true.

Assume second that $d(x,y) \leq 1$. Then we have $D(x,y) = 1 \leq d(x,y) \leq d(x,z)+d(z,y)$. Again, if neither of the second two distances are bigger than 1, we're done, and if at least one is bigger than 1, the nthe sum on the right is $1 +$ something, so we're done again.

b) Let $Y \subset X$ and assume Y is open with respect to D. Let $y \in Y$. Then, there, we know there is some neighborhood such that $N_r(y) \subseteq Y$ (here, the radius is measured with respect to D). We may assume wlog that $r < 1/2$ since $N_{1/2}(y) \subseteq N_r(y)$ if $r > 1/2$.

But $\{x \in X | D(x,y) < r\} = \{x \in X | d(x,y) < r\} = N_r(y)$ where we measure radius here with respect to d. Thus, $N_r(y) \subseteq Y$ regardless of how we measure r. Thus, Y is open with respect to d.

Conversely, assume Y is open with respect to d. Then, as above, there is a neighborhood $N_r(y) \subseteq Y$, with r measured relative to d. Again, as above, we may assume wlog that $r < 1/2$. Thus, by exactly the same argument, we find that Y is open relative to D. □

Problem 6.

Proof. a) Again, just check all the axioms.

1. D_π is a supremum of a set of numbers, all of which are ≥ 0 since each D_i is a metric. Thus, the supremum is also bigger than or equal to 0.

2. For $x = \langle x_1, x_2, x_3, ...\rangle$, $D_\pi(x,x) = \sup\{D_i(x_i,x_i)/i\} = \sup\{0\} = 0$.

3. If $x = \langle x_1, x_2, x_3, ...\rangle$ and $y = \langle y_1, y_2, y_3, ...\rangle$, if $D_\pi(x,y) = 0$, then we have that $D_i(x_i,y_i)/i = 0$ for all i (since if some $D_i(x_i,y_i)/i$ were nonzero, the supremum would have to be at least as large as it). Thus, $D_i(x_i,y_i) = 0$

for every i. But the D_i are metrics, so this implies $x_i = y_i$ fore very i, and hence that $x = y$.

4. Triangle inequality. For x, y, z where $x = \langle x_1, x_2, x_3, ...\rangle$ and likewise for y and z, then we have that $D_\pi(x, y) = \sup\{D_i(x_i, y_i)/i\} \leq \sup\{1/i(D_i(x_i, z_i)+ D_i(z_i, y_i))\} \leq \sup\{1/iD_i(x_i, z_i)\} + \sup\{1/iD_i(z_i, y_i)\} = D_\pi(x, z) + D_\pi(z, y)$.

b) Let $\epsilon > 0$. Let $\delta = \epsilon/j < 1$. Then, for x and y, if $D_\pi(x, y) < \delta$ then we'd have that $D_j(x_j, y_j)/j < \delta$ since otherwise, $D_\pi(x, y) \geq \delta$.

Thus, $D_\pi(x, y) < \delta$ implies that $D_j(x_j, y_j) < \delta j = \epsilon$. But $D_j(x_j, y_j) = D_j(\pi_j(x), \pi_j(y))$.

Thus, we've shown that given any ϵ, there is a δ such that if $D_\pi(x, y) < \delta$, then we must have that $D_j(\pi(x), \pi(y)) < \epsilon$. But this is precisely what is meant by (uniform) continuity. Hence, π_j is continuous. $\square$