(Math 360) Midterm:

Due March 19, 2009

Allowed Materials: While working on this exam you are allowed to make use of either one of the class text books as well as your notes. However, these are the **only** materials you are allowed to you. Specifically you are not allowed to use other text books, the internet, or to discuss these problems with other people.

All page numbers are from Rudin's Principles of Mathematical Analysis.

Let $\boldsymbol{F}$ be the union of two distinct copies of $\mathbb{R}$. Let us put primes on the second copy to distinguish them. We then define

- $x + y$, $x \cdot y$ and $x \leq y$ as usual.
- $x + y' = y' + x = (x + y)'$, $x \cdot y' = y' \cdot x = (x \cdot y)'$ and $x \leq y'$ for all x, y'
- $x' + y' = y' + x' = x + y$, $x' \cdot y' = y' \cdot x' = x \cdot y$ and $x' \leq y'$ if and only if $x \leq y$

(20 points) <u>Problem 1:</u>

(12 points) (a) Show that $\boldsymbol{F}$ satisfies:

(A3) $(\forall x, y, z \in \boldsymbol{F})(x + y) + z = x + (y + z)$

(A4) $\boldsymbol{F}$ contains an element 0 such that

$$0 + x = x$$

for every $x \in \boldsymbol{F}$

(A5) To every $x \in \boldsymbol{F}$ there is an element $-x \in \boldsymbol{F}$ such that

$$x + (-x) = 0$$

(M3) $(\forall x, y, z \in \boldsymbol{F})(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(M4) $\boldsymbol{F}$ contains an element $1 \neq 0$ such that $1 \cdot x = x$ for every $x \in \boldsymbol{F}$

(These labels are from Definition 1.12).

(4 points) (b) We can associate to each natural number $n \in N$ an element $\mathbf{n} = 1 + 1 + \cdots + 1 \in \boldsymbol{F}$ (where there are n 1's added together). Show that there are $x, y \in \boldsymbol{F}$ with $x > 0$ such that for all $n \in N$, $nx \leq y$.

In other words show that $\boldsymbol{F}$ does not have the *archimedean* property.

(4 points) (c) Show that $\boldsymbol{F}$ does not have the least upper bound property (Definition 1.10)

(30 points) Problem 2:

For this problem it may be useful to keep in mind the example of the reals as the completion of the rationals. I.e. we can think of a real number as an equivalence class of Cauchy sequences of rational numbers.

Let (X, d) be a metric space

(6 points) (a) Call two Cauchy sequences $\{p_n\}, \{q_n\}$ in X *equivalent* if

$$\lim_{n\to\infty} d(p_n, q_n) = 0$$

Prove this is an equivalence relation (see page 25 for a definition of an equivalence relation)

(8 points) (b) Let X^* be the set of all equivalence classes so obtained. If $P, Q \in X^*$ and $\{p_n\}, \{p'_n\} \in P, \{q_n\}, \{q'_n\} \in Q$. Show

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n)$$

We know that each limit exists by a Chapter 3 Exercise 23.

We then define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n)$$

which, by part (b) is well defined.

(8 points) (c) Prove that the metric space (X^*, Δ) is complete

(2 points) (d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; Let P_p be the element of X^* which contains this sequence. Prove

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words the map $\varphi(p) = P_p$ is an isometry (i.e. a distance preserving mapping) of X into X^*.

(6 points) (e) Prove that $\varphi(X)$ is dense in X^* and that $\varphi(X) = X^*$ if X is complete.

By part (d) we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the *completion* of X

(10 points) Problem 3:

Show that if $f : [0,1] \to [0,1]$ is continuous then there is an element $x \in [0,1]$ such that $f(x) = x$. Such a point is called a *fixed point* of f. (Hint: Consider the function $g(x) = f(x) - x$).

(12 points) Problem 4:

Let $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Given $p, q \in S^1$ define $d_S(p,q)$ to the shortest distance along S^1 from p to q. For example, if $p = (1,0)$ and $q = (0,1)$ then $d_S(p,q) = \pi/2$ (Notice $d_S(p,q)$ is also the angle between the line connecting $(0,0)$ and p and the line connecting $(0,0)$ and q.)

(4 points) (a) Show (S^1, d_S) is a metric space.

(3 points) (b) Prove S^1 is compact.

(2 points) (c) Prove every continuous function $f : S^1 \to \mathbb{R}$ has a maximum value.

(3 points) (d) Prove S^1 is connected

(4 points) (e) Prove there is no continuous injective (i.e. one-to-one) map $f : S^1 \to \mathbb{R}$. (Hint: If $f(x)$ is the maximum value of f, what happens to $f(y)$ for y near x?)

For parts (b), (c), (d), and (e) you may either use the metric d_S from part (a) or the metric induced by the fact that S^1 is a subset of $\mathbb{R}^2$ (as (S^1, d_S) and $(S^1, d_{\mathbb{R}^2})$ are homeomorphic). Some of the parts of this problem may be easier to prove on $(S^1, d_{\mathbb{R}^2})$ than on (S^1, d_S).

(12 points) Exercise 5:

Suppose (X, d) is a metric space. Define a function

$$D(x, y) = \min\{d(x, y), 1\}$$

(3 points) (a) Show that (X, D) is a metric space. We call it the *bounded metric* associated to (X, d)

(9 points) (b) Show $Y \subseteq X$ is open in (X, d) if and only if it is open in (X, D)

(16 points) Exercise 6:

Let $\langle (X_i, d_i) : i \in \mathbb{N} \rangle$ be a countable collection of metric spaces. Let $\prod_{i \in \mathbb{N}} X_i = \{\langle x_i : i \in \mathbb{N} \rangle$ such that $x_i \in X_i\}$. For example, if $X_{2i} = \mathbb{R}$ and $X_{2i+1} = \mathbb{C}$ then $\prod_{i \in \mathbb{N}} X_i$ consists of all tuples $\langle x_0, x_1, \ldots \rangle$ such that $x_0, x_2, x_4, \ldots$ are real numbers and $x_1, x_3, x_5, \ldots$ are complex numbers.

We then define a distance function by letting

$$D_\Pi(\langle x_i : i \in \mathbb{N} \rangle, \langle y_i : i \in \mathbb{N} \rangle) = \sup \left\{ \frac{D_i(x_i, y_i)}{i} : i \in \mathbb{N} \right\}$$

Where (X_i, D_i) is the bounded metric associated to (X_i, d_i) (from Problem 5(a) on this exam)

(10 points) (a) Show that $(\prod_{i \in \mathbb{N}} X_i, D_\Pi)$ is a metric space.

(6 points) (b) Show that for all $j \in \mathbb{N}$ the map $\pi_j : (\prod_{i \in \mathbb{N}} X_i, D_\Pi) \to (X_j, D_j)$ given by $\pi_j(\langle x_i : i \in \mathbb{N} \rangle) = x_j$ is continuous.

(Notice by Problem 5(b) this is equivalent to showing that the map $\pi'_j : (\prod_{i \in \mathbb{N}} X_i, D_\Pi) \to (X_j, d_j)$ is continuous)

Exam Questions From Hell

Worth: 0 Points

The following is extra credit. It is worth no points. However if you can solve any of the following you will greatly impress both your professor as well as your TA.

Instructions: Read each question carefully. Answer all questions.

Time Limit - 4 hours. Begin immediately.

1. History

   Describe the history of the papacy from its origins to the present day, concentrating especially, but not exclusively, on its social, political, economic, religious, and philosophical impact on Europe, Asia, America, and Africa. Be brief, concise, and specific.

2. Medicine

   You have been provided with a razor blade, a piece of gauze, and a bottle of Scotch. Remove your appendix. Do not suture until your work has been inspected. You have fifteen minutes.

3. Public Speaking

   2,500 riot-crazed aborigines are storming the classroom. Calm them. You may use any ancient language except Latin or Greek.

4. Biology

   Create life. Estimate the differences in subsequent human culture if this form of life had developed 500 million years earlier, with special attention to its probable effect on the English parliamentary system. Prove your thesis.

5. Music

   Write a piano concerto. Orchestrate and perform it with flute and drum. You will find a piano under your seat.

6. Psychology

   Based on your degree of knowledge of their works, evaluate the emotional stability, degree of adjustment, and repressed frustrations of each of the following: Alexander of Aphrodisias, Rameses II, Gregory of Nicea, Hammurabi. Support your evaluations with quotations from each man's work, making appropriate references. It is not necessary to translate.

7. Sociology

   Estimate the sociological problems which might accompany the end of the world. Construct an experiment to test your theory.

8. Management Science

   Define Management. Define Science. How do they relate? Why? Create a generalized algorithm to optimize all managerial decisions. Assuming an 1130 CPU supporting 50 terminals, each terminal to activate your algorithm; design the communications interface and all necessary control programs.

9. Engineering

   The disassembled parts of a high-powered rifle have been placed in a box on your desk. You will also find an instruction manual, printed in Swahili. In ten minutes a hungry Bengal tiger will be admitted to the room. Take whatever action you feel is appropriate. Be prepared to justify your decision.

10. Economics

    Develop a realistic plan for refinancing the national debt. Trace the possible effects of your plan in the following areas: Cubism, the Donatist controversy, the wave theory of light. Outline a method for preventing these effects. Criticize this method from all possible points of view. Point out the deficiencies in your point of view, as demonstrated in your answer to the last question.

11. Political Science

    There is a red telephone on the desk beside you. Start World War III. Report at length on its socio-political effects, if any.

12. Epistemology

    Take a position for or against truth. Prove the validity of your position.

13. Physics

    Explain the nature of matter. Include in your answer an evaluation of the impact of the development of mathematics on science.

14. Philosophy

    Sketch the development of human thought; estimate its significance. Compare with the development of any other kind of thought.

15. General Knowledge

    Describe in detail. Be objective and specific.

(Extra Credit)

Define the Universe; give three examples.

————————————additions————————————

16. Religion

    (Take Home Essay)

    Define God, and prove the existence thereof. Personal interviews may be used as a source for this essay. Please use quotes when possible, and footnote any data which is not derived from personal revelations. This paper is due by the end of Reading Period. There is a word limit but no time limit other than the aforementioned deadline.

17. English

    Based on your knowledge of poets writing or translated into the English language, please describe their main themes. Further, discuss their relevance to contemporary life. Defend your positions and use quotes whenever necessary. Do not overgeneralize; be specific and concise in your answer. Self criticize your thesis, and be sure to pay attention to the use of writing in the English language.

18. Linguistics

    Linguistically analyze the following statement:

    > "I am lying to you now."

    Pay attention to usage of words in order to render a logical meaning. Bring to bear your etymological knowledge of the morphemes. Use this example to illuminate your discussion upon the use and history of language.

19. Astronomy

    Please catalog the stars, including data such as relative and absolute magnitude, stellar luminosity, distance from earth, and stellar class and size.

20. Music History

    Please describe the importance of pitch, rhythm, and timbre to the composer. Bring your knowledge of composers from the medieval to post-modern periods to bear.

21. Computer Science

    Prove that all problems can be solved in polynomial time. Extra credit: provide a solution to traveling salesperson, or prime factorization that executes in linear time. You have 10 minutes.

22. South Asian Studies

    Translate and explicate the unexpurgated Bhagvad Gita. You have one hour.

23. French

    Construisez cinq phrases, et utilisez toutes les formes du verbe avoir. Aussi, il faut que les phrase soient une petite histoire de votre vie. Apres avoir termine votre petit composition, traduisez ces trois pages de "La Peste" par Camus. Expliquez comment en se exprimant ses sentiments du monde, Camus est influencer par les Francais. Vous aurez quinze minutes. Bonne chance!

24. Human Emotion

    Define the concept of Love. Define the concept of Truth. Explain the ramifications of these definitions on socio-economic personal interaction in the Lodz Ghetto in the period 1942-1945 as they relate to the uprising. Do not over-generalize or include personal values or opinions.

25. Spanish

    Escriba todas las palabras de la lengua espanola. Use solamente una hoja de papel. Se pueden usar los dos lados.

26. Molecular Physics

    Rearrange the atoms in the pencil you are currently holding so as to create a functional nuclear accelerator. Use this apparatuses to discover at least four new elements (more for extra credit). Combine these new elements so that they form a matter-antimatter bridge, and then travel this bridge to a parallel universe (any one will do). Collect several examples of local flora and fauna and turn them in with this test. You have 30 minutes.

27. Advanced Philosophy

    Why?

28. German

    Ubersetzen Sie alles, was Sie zu diesen Zeit gelesen haben, ins deutsche. Beschreiben Sie die Wichtigkeit von der Studium der deutschen Sprache. Vergleichen Sie sie mit der Studium anderer Sprachen. Endlich, erzahlen Sie bitte "Steppenwolf", von Hermann Hesse. Beschreiben Sie die soziologische Aspekt des Schreibers. Sie haben zwanzig Minuten. Viel Gluck!

29. Logic and Number Theory

    Explain, based on your understanding of Godel-numbering, why it is impossible that artificial intelligence could be created. Include in your answer a description of proof-pairs, arithmoquining, undecidable propositions, and a concise but complete overview of typographical number theory. Now prove the possibility of the creation of artificial intelligence by means of an analogous argument to the human mind. Be brief, concise, and specific!