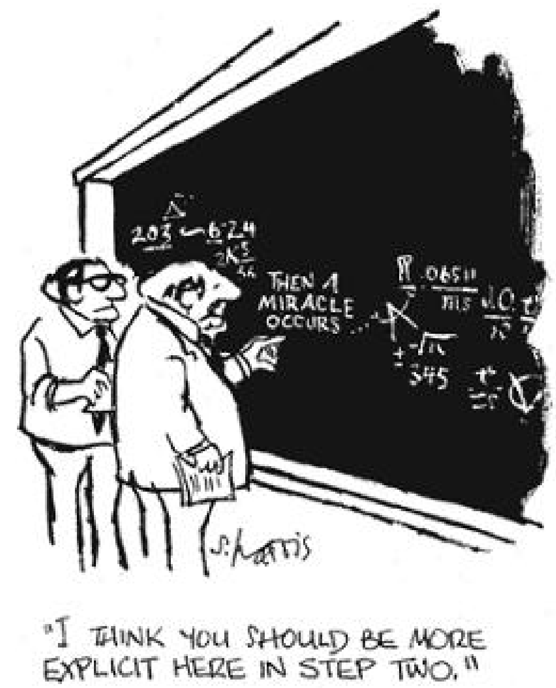
(Math 360) Free Response Final:

April 28, 2009



Rules

Preparation:

In preparing for this part of the final exam you are allowed to use any books as well as any source you find on the internet. You are also allowed to talk with anyone involved with this class, including Dr. Ackerman and Mr. DeVito (as such it is STRONGLY encouraged that you work together to try and solve these problems as well that you ask questions during recitation and office hours).

However, the people involved in this class (i.e. the teachers and students) are the ONLY people you are allowed to discuss the test with (i.e. you are not allowed to get help from people on the internet or from other teachers).

Exam:

The problems on this exam are broken into two categories: Those worth 1 point and those worth 2 points. During class on April 28 you will need to answer 6 points worth of questions in any combination of 1 point and 2 point questions that you choose (notice that, in our opinion, the 2 point questions are significantly harder than the 1 point questions).

In answering the questions you will not be allowed any notes or calculators. You will however be allowed to mention without proof any theorem you choose (up to Chapter 7) from Rudin. You will also be able to mention without proof any part of any problem in any other problem.

For example, suppose you choose to answer questions 1, 2, 3 and 4, 6. But, in your proof of Question 4 you need to reference question 8 (b). It is fine to do this even though you have not proved any part of question 8.

Useful Definitions

Here are some important definitions.

Definition 0.0.0.1. A *nice measure* on $\mathbb{R}$ is a partial function μ from $PowerSet(\mathbb{R})$ to $\mathbb{R} \cup \{\infty\}$ such that

- If $\mu(E)$ is defined then $\mu(E) \geq 0$
- $\mu(\emptyset) = 0$
- $\mu([a, b]) = b - a$ for any $a \leq b$
- If μ is defined on E , $r \in \mathbb{R}$, and $E_r = \{x + r : x \in E\}$ then μ is defined on E_r and $\mu(E) = \mu(E_r)$.
- If μ is defined on D and E then μ is defined on $E - D = \{x \in E : x \notin D\}$.
- If $\{E_i\}_{i \in \mathbb{N}}$ is a collection of sets such that $\mu(E_i)$ is defined for each $i \in \mathbb{N}$ then
  - $\mu\left(\bigcap_{i \in \mathbb{N}} E_i\right)$ is defined.
  - $\mu\left(\bigcup_{i \in \mathbb{N}} E_i\right)$ is defined.
  - If the $\{E_i\}$ are pairwise disjoint then
$$\mu\left(\bigcup_{i \in \mathbb{N}} E_i\right) = \sum_{i \in \mathbb{N}} \mu(E_i)$$

Definition 0.0.0.2. Suppose μ is a nice measure and f is a function from the $\mathbb{R}$ to $\mathbb{R}$. Define $\int f d\mu$ as follows

- If χ_S is the characteristic function of S then

$$\int \chi_S d\mu = \mu(S)$$

- We call functions $f = \sum_{k=1}^{n} a_k \chi_{S_k}$ where $\mu(S_k) < \infty$ *simple* and define

$$\int f d\mu = \sum_{k=1}^{n} a_k \mu(S_k)$$

  If B is a measurable set then we also define

$$\int_B f d\mu = \int \chi_B \cdot f d\mu = \sum_{k=1}^{n} a_k \mu(S_k \cap B)$$

- If f is non-negative function on a measurable set B then we define

$$\int_B f d\mu = \sup \left\{ \int_B s d\mu : 0 \leq s \leq f : s \text{ is simple} \right\}$$

  (here $s \leq f$ if and only if $(\forall x \in B) s(x) \leq f(x)$)

- If f is any function on a measurable set B define
  - $f^+(x) = f(x)$ if $f(x) \geq 0$ and $f^+(x) = 0$ otherwise.
  - $f^-(x) = -f(x)$ if $f(x) \leq 0$ and $f^-(x) = 0$ otherwise.

  If $\int_B f^+ d\mu < \infty$ and $\int_B f^- d\mu < \infty$ then $\int_B f d\mu = \int_B f^+ d\mu - \int_B f^- d\mu$

Definition 0.0.0.3. Suppose that μ is a nice measure. We say a function is μ-measurable (or $f \in \mathcal{M}(\mu)$) if $(\forall a \in \mathbb{R})\mu$ is defined on $f^{-1}(\{x \in \mathbb{R} \cup \{\infty\} : x > a\})$.

If $f \in \mathcal{M}(\mu)$ and $\int f d\mu$ exists then we say $f \in \mathcal{L}(\mu)$.

1 Point Problems

(1) Let f be a real-valued function defined on the closed interval $[a, b]$.

2/10 (a) Suppose there is a positive α such that

$$|f(x) - f(y)| \leq |x - y|^{\alpha}$$

for all $x, y \in [a, b]$. Prove that f is uniformly continuous.

4/10 (b) Let $\gamma : [0, 1] \to \mathbb{R}^k$ define a continuously differentiable curve in the plane. Show that $\Lambda(\gamma) \geq \|\gamma(1) - \gamma(0)\|$ (i.e. that the shortest distance between two points is a straight line)

4/10 (c) Let $\gamma : \mathbb{R} \to \mathbb{R}^k$ define a continuously differentiable curve in the plane. If in the Euclidean norm $\|\gamma'(t)\| \leq c$ for some constant c, show that $\|\gamma(t) - \gamma(0)\| \leq c|t|$

(2)

5/10 (a) Prove that there exists a subset A of $\mathbb{R}$ such that for any nice measure μ, $\mu(A)$ is undefined.

5/10 (b) Suppose $A \subseteq [0, 1]$ has positive measure. Then show that there are $x, y \in A$ such that $x \neq y$ and $x - y \in \mathbb{Q}$.

(3)

4/10 (a) Prove that if μ is a measure, f is a function on $[a, b]$ and $f \in \mathcal{R}$ then $\int_{[a,b]} f d\mu$ exists and

$$\int_{[a,b]} f d\mu = \int_a^b f(x) dx$$

(where the integral on the right is the usual Riemann integral)

1/10 (b) Given an example of a function for which $\int_{[a,b]} f d\mu \neq \int_a^b f(x)dx$ (Hint: It suffices to find a function for which $\int_{[a,b]} f d\mu$ is defined but $\int_a^b f(x)dx$ is undefined).

3/10 (c) Show that if $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of non-negative functions on a measurable set B such that

* $f_k(x) \leq f_{k+1}(x)$ for all $k \in \mathbb{N}$ and $x \in B$
* $\int_B f_k d\mu$ is defined for all $k \in \mathbb{N}$

then

$$\lim_{k\in\mathbb{N}} \int_B f_k d\mu = \int_B \lim_{k\in\mathbb{N}} f_k d\mu$$

2/10 (d) Show that if $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of non-negative functions on a measurable set B and $f(x) = \sum_{i=1}^{\infty} f_n(x)$ for all $x \in B$. Then

$$\int_B f d\mu = \sum_{i=1}^{\infty} \int_B f_n d\mu$$

(4)

2/10 (a) Show that μ is a nice measure and $f, g \in \mathcal{M}(\mu)$ then for any real numbers $a, b \in \mathbb{R}$, $af(x) + bg(x) \in \mathcal{M}(\mu)$.

3/10 (b) Show that if $\{f_n\}$ is a sequence of measurable functions and

$$g(x) = \sup_{n\in\mathbb{N}} f_n(x)$$

$$h(x) = \lim_{m\to\infty} \{\sup_{n\geq m} f_n(x)\}$$

then both g and h are measurable.

2/10 (c) Show that if $\{f_n\}$ is a sequence of measurable functions which converge pointwise to f then f is measurable.

3/10 (d) Suppose E is a measurable set and $\{f_n\}$ is a sequence of non-negative measurable functions with

$$f(x) = \lim_{m\to\infty} \inf\{f_n(x) : n \geq m\}$$

Then

$$\int_E f d\mu \leq \lim_{m\to\infty} \inf\{\int_E f_n d\mu : n \geq m\}$$

(5)

8/10 (a) Suppose f, g are continuous with the domain of f and g containing a measurable set A. Further suppose that $S = \{x \in A : f(x) \neq g(x)\}$ and $\mu(S) = 0$. Then show that $\int_A f(x)d\mu = \int_A g(x)d\mu$.

2/10 (b) Suppose f, g are continuous and bounded and $\int_{[a,b]} |f(x)-g(x)|d\mu = 0$. Show if $S = \{x \in [a, b] : f(x) \neq g(x)\}$ then $\mu(S) = 0$

2 Point Problems

(6)

5/10 (a) Let $f(x)$ be a function on $[0, \infty)$ and assume that the improper integral

$$\int_0^\infty e^{-sx} f(x)dx$$

converges for all real numbers $s \geq s_0$ where $s_0 \in \mathbb{R}$. Prove that this integral tends towards 0 as $s \to \infty$

5/10 (b) Compute

$$\lim_{n\to\infty} \int_0^1 (n+1)x^n \cdot \left(x^{(x+1)^{(x+2)^{(x+3)}}} + x^{(x+1)^{(x+2)}} + x^{(x+1)} + x + 1 \right) dx$$

(7)

3/10 (a) Say $f_n : [0,1] \to \mathbb{R}$ is a sequence of continuous functions such that $f_{n+1}(x) \leq f_n(x)$ for all n and all $x \in [0,1]$. Suppose $f : [0,1] \to \mathbb{R}$ and that the sequence $f_n(x) \to f(x)$ for all x. Show that if f is continuous then $\{f_n(x)\}$ converges to $f(x)$ uniformly. If we don't assume f is continuous do must $\{f_n(x)\}$ converge to f uniformly? Either show it must or give a counterexample.

3/10 (b) Let f be a non-negative monotone decreasing differentiable function and let

$$\int_0^\infty f(x)dx$$

converge. Prove $\lim_{x\to\infty} xf(x) = 0$. If $\lim_{x\to\infty} xf(x) = 0$ must $\int_0^\infty f(x)dx$ converge? If so prove it, if not give a counter example.

4/10 (c) Let $\{\varphi_n(x)\}$ be a sequence of continuous, real valued functions defined on $\mathbb{R}$ such that

* $\varphi_n(x) = 0$ for all $|x| \geq \frac{1}{n}$
* $\varphi_n(x) \geq 0$ for all $x \in \mathbb{R}$
* $\int_{-1}^{1} \varphi(x)dx = 1$

For each continuous $f : \mathbb{R} \to \mathbb{R}$ define the sequence $\{f_n\}$ by

$$f_n(x) = \int_{-\infty}^{\infty} \varphi_n(x-y)f(y)dy$$

Prove that the sequence $\{f_n(x)\}$ converges to $f(x)$ for every $x \in \mathbb{R}$. Further prove that if there is a real number c such that $f(x) = 0$ for all $|x| \geq c$ then $\{f_n(x)\}$ converges uniformly to $f(x)$

(8)

2/10 (a) For μ-measurable functions f, g whose domain is $[0,1]$ define $f \sim_\mu g$ if $\mu(\{x \in [0,1] : f(x) \neq g(x)\}) = 0$. Show that $\sim_\mu$ is an equivalence relation on $\mathcal{M}(\mu)$, the collection of μ-measurable functions.

3/10 (b) Let $\mathcal{L}^*(\mu)$ be the collection of equivalence classes under $\sim_\mu$ of elements of $\mathcal{L}(\mu)$ whose domain is $[0,1]$. If $f, g \in \mathcal{L}(\mu)$ and $[f], [g] \in \mathcal{L}^*(\mu)$ are the corresponding equivalence classes define

$$d_\mu([f],[g]) = \int_{[0,1]} |f - g| d\mu$$

Show $(\mathcal{L}^*(\mu), d_\mu)$ is a metric space. (Notice that by a previous problem our choice of representatives for the equivalence classes doesn't matter).

5/10 (c) Show $(\mathcal{L}^*(\mu), d_\mu)$ is complete.