Math 104-006

Chapter 12.1: Sequences

Outline For Today

- Sequences
- Limits of Sequences
- Bounded Sequences

Sequence

- A sequence is a list of numbers:

- $\{a_1, a_2, a_3, \ldots\}$ which can also be writen as

- $\{a_n\}$ or $\{a_n\}_{n=0}^{\infty}$

Sequence Example

- The sequence $\left\{\frac{n+2}{5^n}\right\}_{n=0}^{\infty} = \left\{2, \frac{3}{5}, \frac{4}{25}, \frac{5}{125}, \ldots\right\}$

Limit of a Sequence

- A sequence $\{a_n\}_{n=0}^{\infty}$ has a limit L and we write
- $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$

if for every ε > 0 there is an integer N such that

if n > N then $|a_n - L| < \varepsilon$

Limit of a Sequence Continued

- If a sequence $\{a_n\}_{n=0}^{\infty}$ has a finite limit then we say
- $\{a_n\}_{n=0}^{\infty}$ is convergent
- Otherwise we say $\{a_n\}_{n=0}^{\infty}$ is divergent

Limit of a Sequence and a Function

If $f(n) = a_n$ for all natural numbers n

and

$$\lim_{x \to \infty} f(x) = L$$

then

$$\lim_{n \to \infty} a_n = L$$

Limit of a Sequence

We say $\lim_{n \to \infty} a_n = \infty$

if for every positive number M there is an integer N such that

whenever $n > N$ then $a_n > M$

Properties of Limits

If $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are convergent sequences and c is a constant then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \cdot \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

Properties of Limits Continued

$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} {a_n}/{b_n} = {\lim_{n\to\infty} a_n}/{\lim_{n\to\infty} b_n} \quad if \ \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p \ if \quad p > 0 \quad a_n > 0$$

Example

$$\lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}$$

$$= \frac{\lim_{n\to\infty}1}{\lim_{n\to\infty}1+\lim_{n\to\infty}\frac{1}{n}}$$

$$= 1$$

Example

- Is {-1,1,-1,1,...} divergent or convergent?

Squeeze Theorem

If $a_n \leq b_n \leq c_n$ for all natural numbers $n > n_0$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then

$$\lim_{x\to\infty} b_n = L$$

Continuous Function Theorem

If f(x) is continuous at L and $\lim_{n \to \infty} a_n = L$ then

$$\lim_{n \to \infty} f(a_n) = f(L)$$

Exponents

$\lim_{n\to\infty} r^n = 1$ if r = 1

$\lim_{n\to\infty} r^n = 0$ if -1 < r < 1

otherwise $\{r^n\}_{n=0}^{\infty}$ is divergent

Increasing and Decreasing Sequences

A sequence $\{a_n\}$ is **increasing** if

$a_0 < a_1 < a_2 < \ldots$

A sequence $\{a_n\}$ is **decreasing** if

$a_0 > a_1 > a_2 > \ldots$

A sequence $\{a_n\}$ is **monotonic** if it is either increasing or decreasing

Bounded Sequences

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that for all $n > 0$, $a_n \leq M$

A sequence $\{a_n\}$ is **bounded below** if there is a number M such that for all $n > 0$, $a_n \geq M$

A sequence $\{a_n\}$ is a **bounded sequence** if it is either bounded above or bounded below

Completeness Axiom

- Every increasing sequence which is bounded from above has a least upper bound
- Every decreasing sequence which is bounded from below has a greatest lower bound

- Every bounded, monotonic sequence is convergent.