# Math 104-006

## Chapter 8.8: Improper Integerals

# Outline For Today

- Improper Integrals of Type 1
- Improper Integrals of Type 2
- Comparison Test

# Improper Integral: Type 1

- We say $\int_a^{\infty} f(x)dx = \lim_{t\to\infty}\int_a^{t} f(x)dx$ if the limit

exists and $\int_a^{t} f(x)dx$ is defined for all t > a.

Otherwise $\int_a^{\infty} f(x)dx$ is undefined otherwise

# Improper Integral: Type 1

- We say $\int_{-\infty}^{b} f(x)dx = \lim_{s\to-\infty}\int_{s}^{b} f(x)dx$ if the limit

exists and $\int_{s}^{b} f(x)dx$ is defined for all s < b.

Otherwise $\int_{-\infty}^{b} f(x)dx$ is undefined otherwise

# Example

- Lets find $\int_{1}^{\infty} \frac{1}{x} dx$
- So we need to find

$$\lim_{t\to\infty} \int_{1}^{t} \frac{1}{x} dx = \lim_{t\to\infty} \ln|x| \Big|_{1}^{t} = \lim_{t\to\infty} \ln|t| - 0 = \infty$$

- Hence $\int_{1}^{\infty} \frac{1}{x} dx$ is undefined

# Try An Example

For which values of p does $\int_1^{\infty} \frac{1}{x^p} dx$ converge?

A) $p < 1$

B) $p > 1$

C) $p \geq 1$

D) $p > 0$

E) $p < 0$

F) $p \leq 1$

# Try An Example

For which values of p does $\int_1^{\infty} \frac{1}{x^p} dx$ converge?

A) $p < 1$

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# Improper Integral

- We say

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{s\to-\infty}\int_{s}^{a} f(x)dx + \lim_{t\to\infty}\int_{a}^{t} f(x)dx$$

- if both limits $\lim_{s\to-\infty}\int_{s}^{a} f(x)dx$ and $\lim_{t\to\infty}\int_{a}^{t} f(x)dx$

exist.

# Convergent Versus Divergent

- If $\int_{a}^{\infty} f(x)dx$ , $\int_{-\infty}^{b} f(x)dx$ or $\int_{-\infty}^{\infty} f(x)dx$

exists we say the improper integral converges.

- Otherwise we say the improper integral diverges

# Example

- Lets find $\int_{-\infty}^{\infty} x\,dx$

- We will solve it the wrong way (and get the wrong answer) and then solve it the right way.

# Example: Wrong Way

- The wrong way to do this is to go to positive infinity and negative infinity at the same time. i.e.

$$\int_{-\infty}^{\infty} x dx = \lim_{t \to \infty} \int_{-t}^{t} x dx = \lim_{t \to \infty} 0 = 0$$

# Example: Right Way

- We have $\int_{-\infty}^{\infty} x dx = \lim_{s\to-\infty}\int_{s}^{0} x dx + \lim_{t\to\infty}\int_{0}^{t} x dx$
- But $\lim_{t\to\infty}\int_{0}^{t} x dx = \lim_{t\to\infty}\left.\frac{x^2}{2}\right|_0^t = \infty$
- So $\int_{-\infty}^{\infty} x dx$ is undefined because one of the

limits is undefined.

# Improper Integral: Type 2

- If f is continuous on [a, b) and discontinuous

at b then $$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

if the limit exists.

Otherwise $\int_a^b f(x)dx$ is undefined.

# Improper Integral: Type 2

- If f is continuous on (a, b] and discontinuous

at a then $\int_a^b f(x)dx = \lim_{s \to a^+} \int_s^b f(x)dx$

if the limit exists.

Otherwise $\int_a^b f(x)dx$ is undefined.

# Improper Integral: Type 2

- If f is defined everywhere on (a, b) except for a single discontinuity at x = c   (a < c < b) then

$$\int_a^b f(x)dx = \lim_{s \to c^+} \int_a^c f(x)dx + \lim_{t \to c^-} \int_t^b f(x)dx$$

if both limit exists and is undefined otherwise.

# Try An Example

For which values of p does $\int_0^1 \frac{1}{x^p} dx$ converge?

A) $p < 1$

B) $p > 1$

C) $p \geq 1$

D) $p > 0$

E) $p < 0$

F) $p \leq 1$

# Try An Example

For which values of p does $\int_0^1 \frac{1}{x^p}dx$ converge?

A) $p < 1$

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D) $p > 0$

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F) $p \leq 1$

# Limit Comparison Test

- Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$
- (a) If $\int_a^\infty f(x)dx$ is convergent then so is $\int_a^\infty g(x)dx$
- (b) If $\int_a^\infty g(x)dx$ is divergent then so is $\int_a^\infty f(x)dx$

# Example

- Does $\int_1^{\infty} e^{-x^2} dx$ converge?
- We know that $e^{-x^2} \le e^{-x}$ if x > 1
- So if $\int_1^{\infty} e^{-x} dx$ converges so does $\int_1^{\infty} e^{-x^2} dx$

# Example Continued

$$\begin{aligned}\int_1^{\infty} e^{-x}\,dx &= \lim_{t\to\infty}\int_1^{t} e^{-x}\,dx \\ &= \lim_{t\to\infty} -e^{-x}\Big|_1^{t} \\ &= \lim_{t\to\infty} -(e^{-t}-e^{0}) \\ &= 1\end{aligned}$$

So $\int_1^{\infty} e^{-x^2}\,dx$ converges.