Math 104-006

Chapter 8.7: Approximate Integration

Outline For Today

- Left/Right End Point Approximation
- Midpoint Approximation
- Trapazoidal Rule Approximation
- Simpson's Rule Approximation

- Error Bounds

Left End Point Approximation

The Left End Point Approximation is given by

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

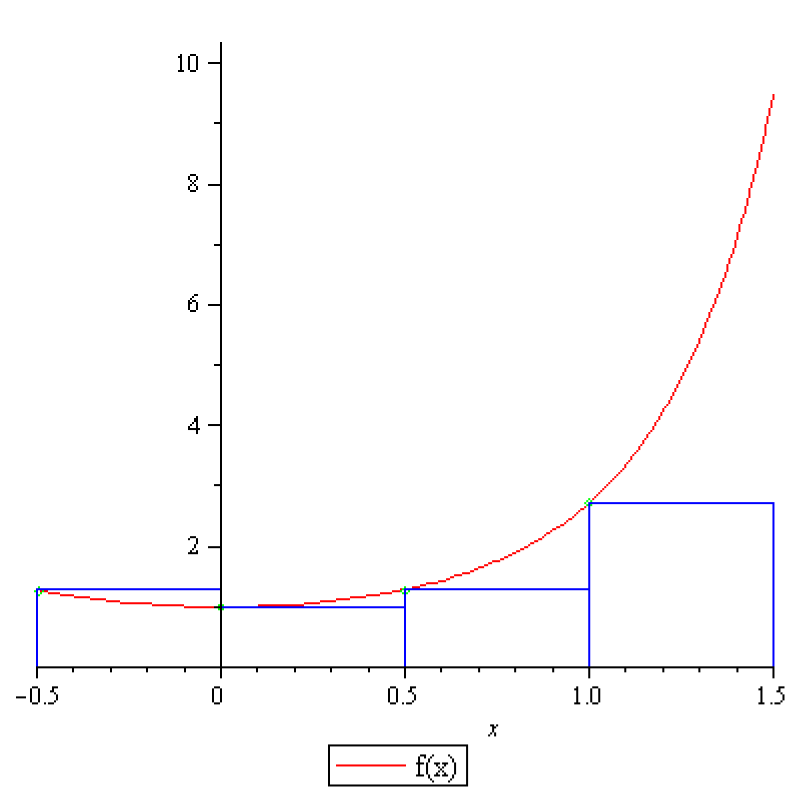
and

$$\int_a^b f(x)dx = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

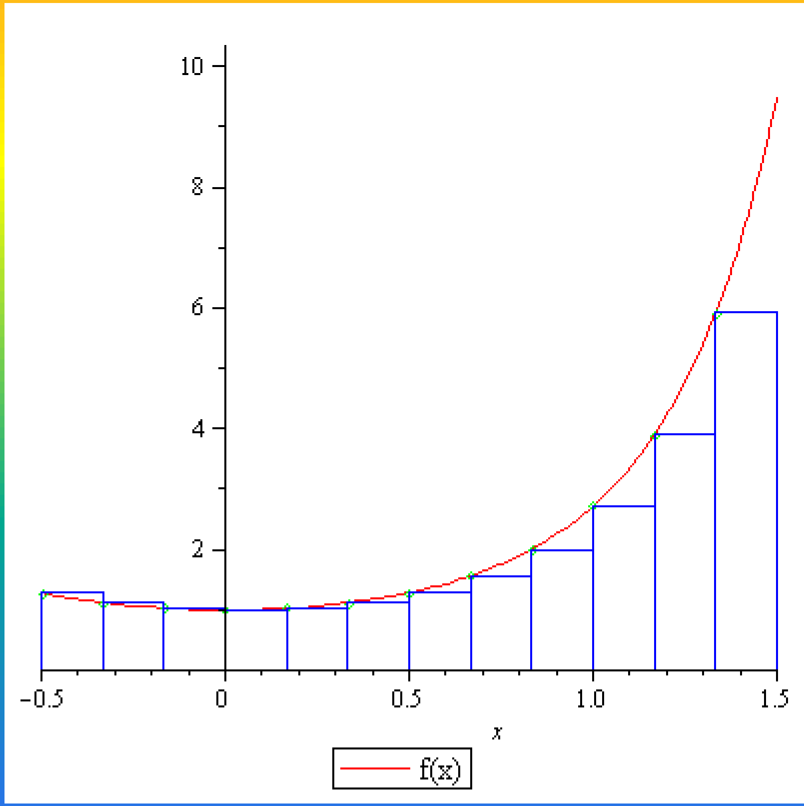
Left End Point Example

- We now consider the function $e^{(x*x)}$.

n = 4



n = 12



Right End Point Approximation

The Right End Point Approximation is given by

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

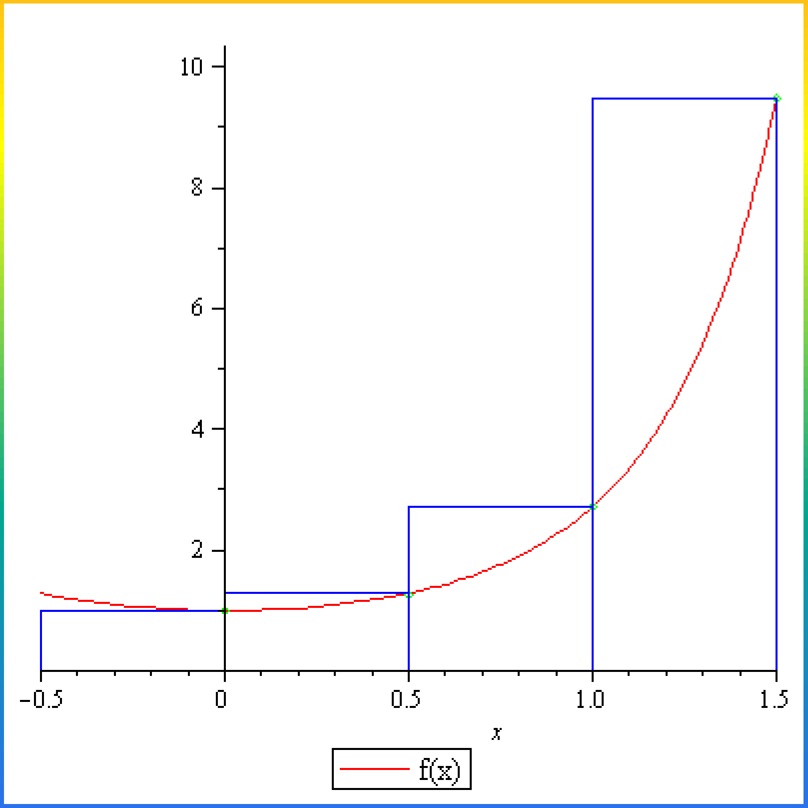
and

$$\int_a^b f(x)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

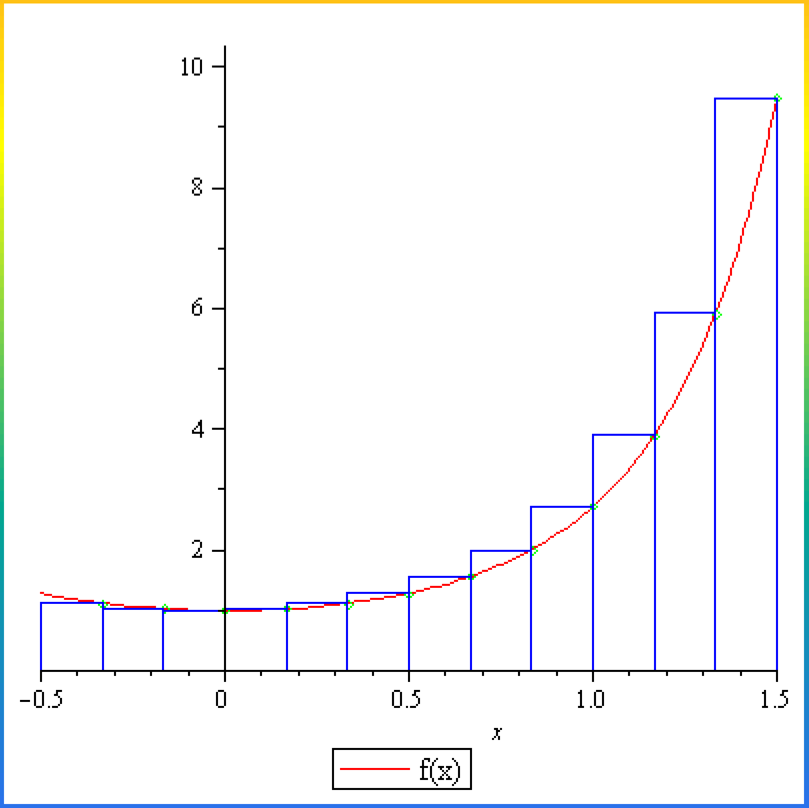
Right End Point Example

- We now consider the function $e^{(x*x)}$.

n = 4



n = 12



Midpoint Approximation

The Midpoint Approximation is given by

$$M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

where

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

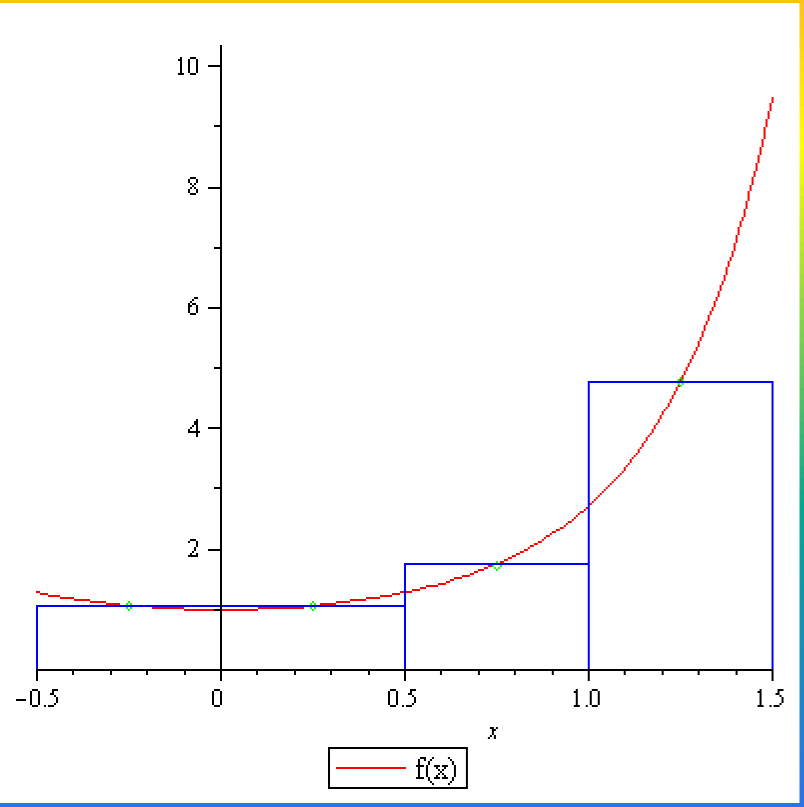
and the midpoint of the interval is $\dfrac{x_{i-1}+x_i}{2}$

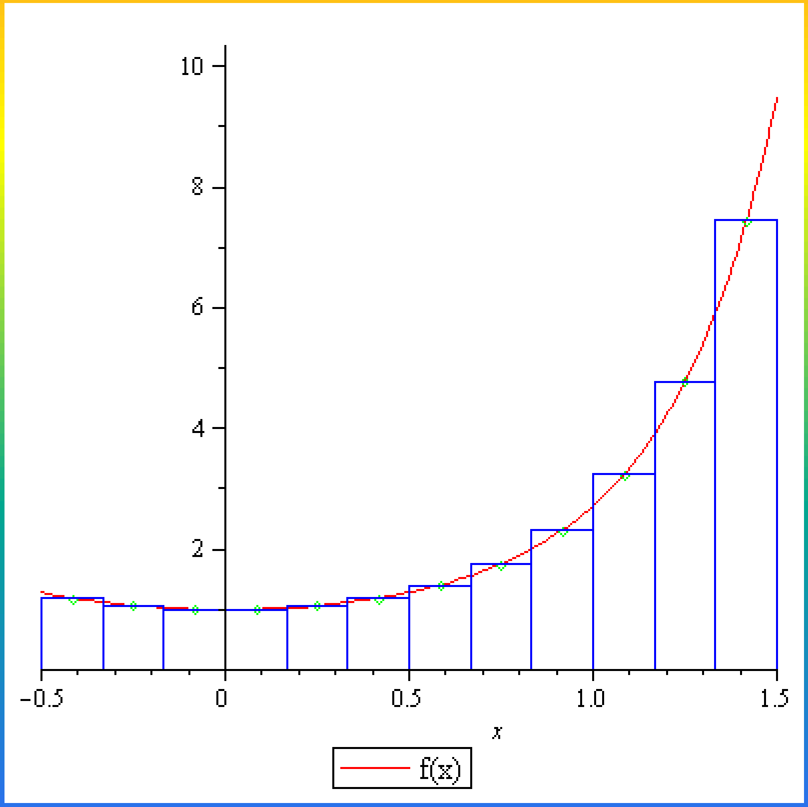
$$\int_a^b f(x)dx = \lim_{n\to\infty} M_n = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

Midpoint Approximation

- We now consider the function $e^{(x*x)}$.

$n = 4$ $n = 12$





Trapezoidal Rule

The Trapezoidal Rule is given by

$$T_n = \sum_{i=1}^{n} \Delta x \left(\frac{f(x_{i-1}) + f(x_i)}{2} \right)$$

$$= \left(\frac{\Delta x}{2} \right) \left[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n) \right]$$

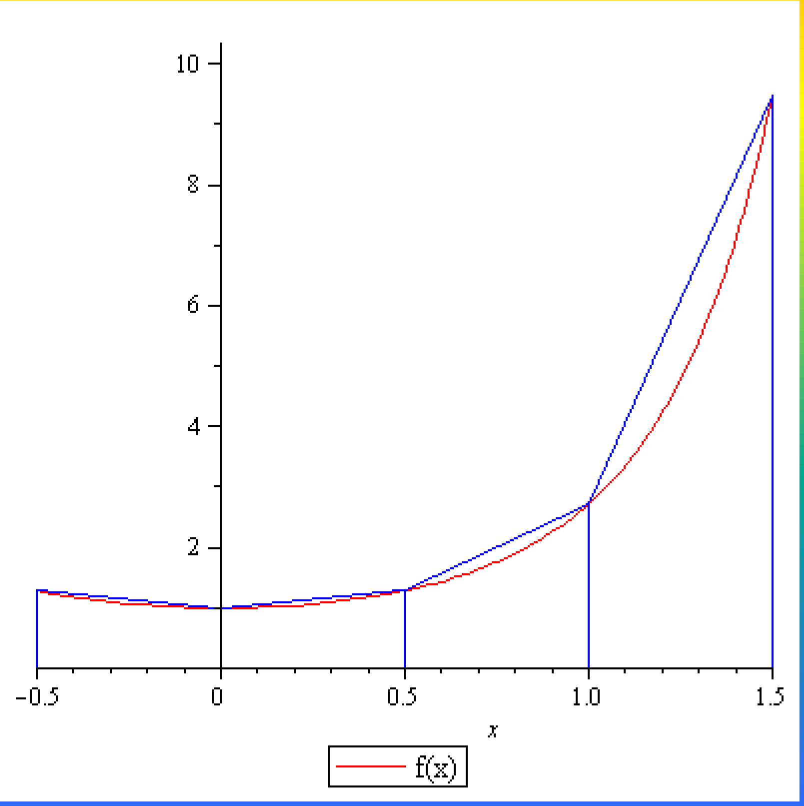
where $\Delta x = \dfrac{b-a}{n}$

$$x_i = a + i\Delta x$$

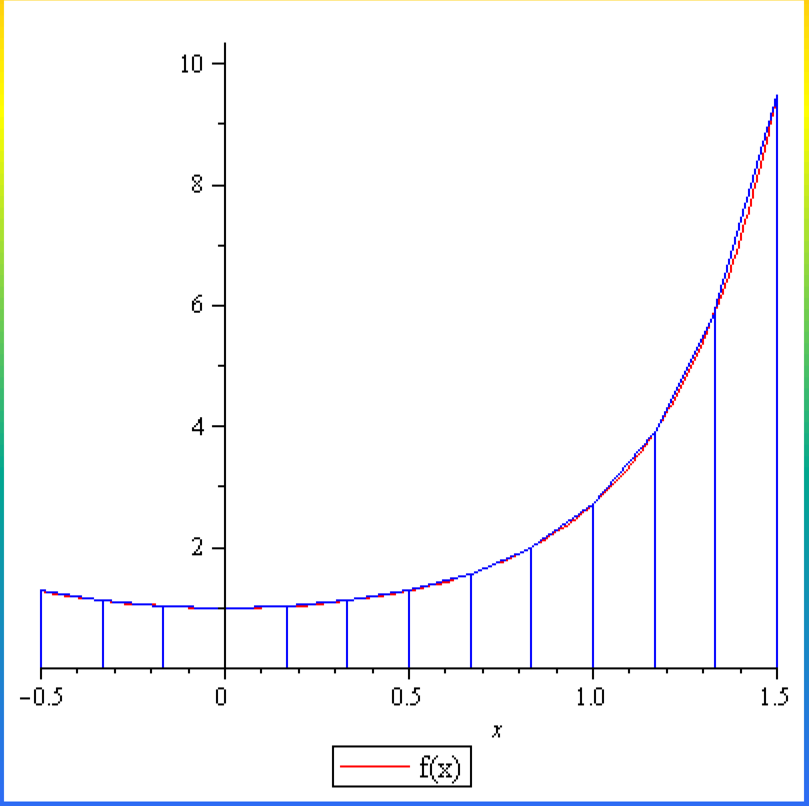
Trapezoidal Rule

- We now consider the function $e^{(x*x)}$.

n = 4

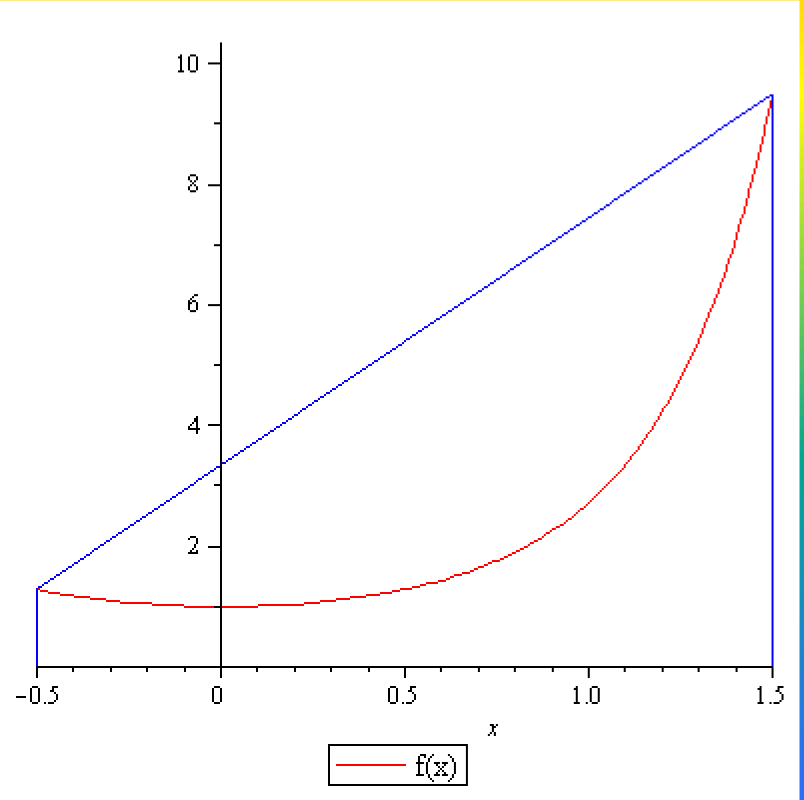


n = 12



Motivation

- This rule comes from the fact that the area of a trapezoid is



$$\Delta x\left(\frac{f(x_{i-1})+f(x_i)}{2}\right)$$

Simpsons Rule

The Simpsons Rule is given by

$$S_n = \left(\frac{\Delta x}{3}\right)\Big[f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]$$

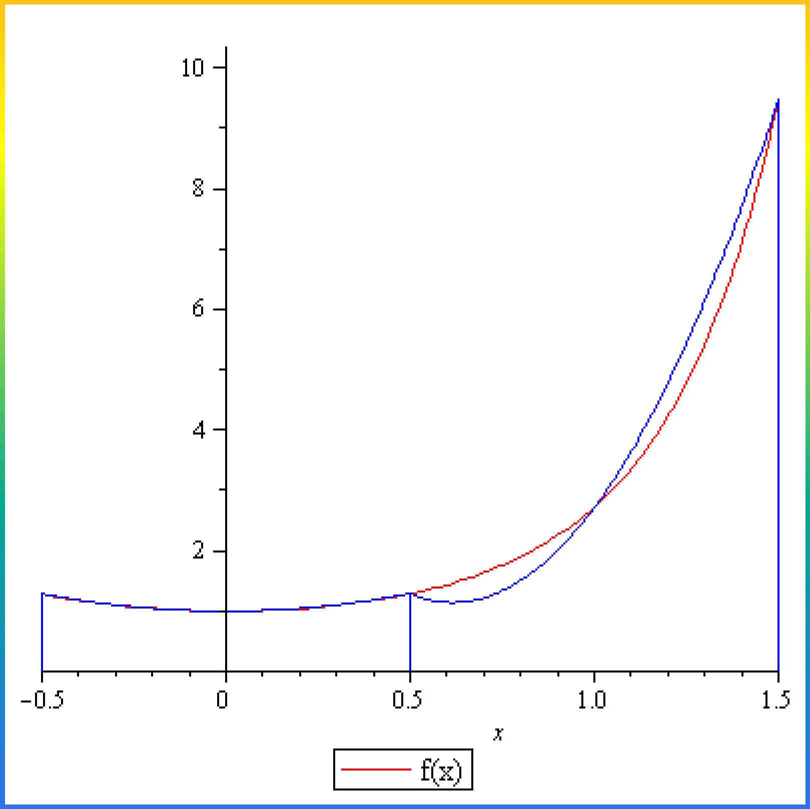
Where n is even and $\Delta x = \dfrac{b-a}{n}$

$$x_i = a + i\Delta x$$

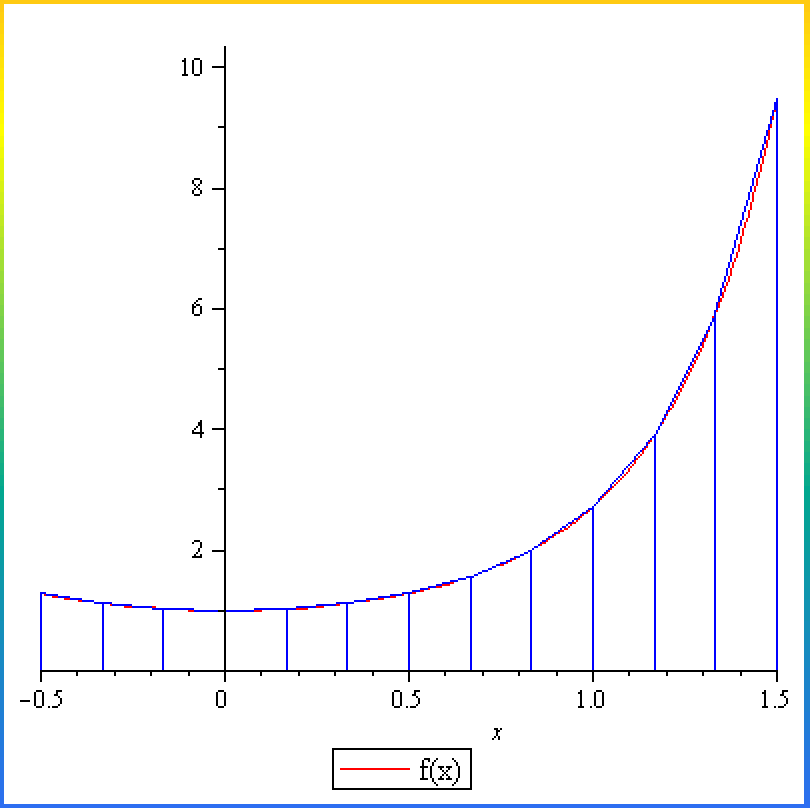
Trapezoidal Rule

- We now consider the function $e^{(x*x)}$.

n = 4



n = 12



Try An Example

Use the Trapezoidal method to approximate

$$\int_0^4 x^2 dx$$

With n = 4

A) 20

B) 22

C) 23

D) 19

E) 15

F) None of the Above

Try An Example

Use the Trapezoidal method to approximate

$$\int_0^4 x^2 dx$$

With n = 4

A) 20

B) 22

C) 23

D) 19

E) 15

F) None of the Above

Approximations to the Integral of x^2 from 0 to 4

RULE	N = 5	N = 10	N = 20
Left Endpoint	15.36	18.24	19.76
Right Endpoint	28.16	24.64	22.96
Midpoint	21.12	21.28	21.32
Trapezoidal	21.76	21.44	21.36
Simpson	N/A	21.33	21.33

Error (Integral – Approximation)

RULE	N = 5	N = 10	N = 20
Left Endpoint	5.9733	3.0933	1.5733
Right Endpoint	-6.8267	-3.3067	-1.6267
Midpoint	0.2133	0.0533	0.0133
Trapezoidal	-0.4267	-0.1067	-0.2667
Simpson	N/A	0	0

Observations

- All methods get more accurate as the number of partitions increases
- The errors in the Left and Right approximations are opposite in sign and decrease by about a factor of 2 when n doubles
- The Trapezoidal and Midpoint rules are much more accurate than the endpoint rules

Observations (continued)

- The errors in the Trapezoidal and Midpoint Rules are opposite in sign and decrease by a factor of about 4 when n doubles.
- The size of the error in the Midpoint Rule is about half the size of the error in the Trapezoidal rule

Error Bounds

- Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the error in the Trapezoidal and Midpoint Rules then

$$E_T \leq \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad E_M \leq \frac{K(b-a)^3}{24n^2}$$

Try An Example

How large would n have to be to guarantee the midpoint rule approximations for $\int_1^2 x^3 dx$ is accurate within 0.01?

A) 1

B) 2

C) 4

D) 6

E) 9

F) None of the Above

Try An Example

How large would n have to be to guarantee the midpoint rule approximations for $\int_1^2 x^3\,dx$ is accurate within 0.01?

A) 1

B) 2

B) 4

D) 6

E) 9

F) None of the Above

Error Bounds (Simpson's Rule)

- Suppose $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If E_S is the error in Simpsons Rule then

$$E_S \leq \frac{K(b-a)^5}{180n^4}$$