Math 104-006

Chapter 9.2: Arc Length

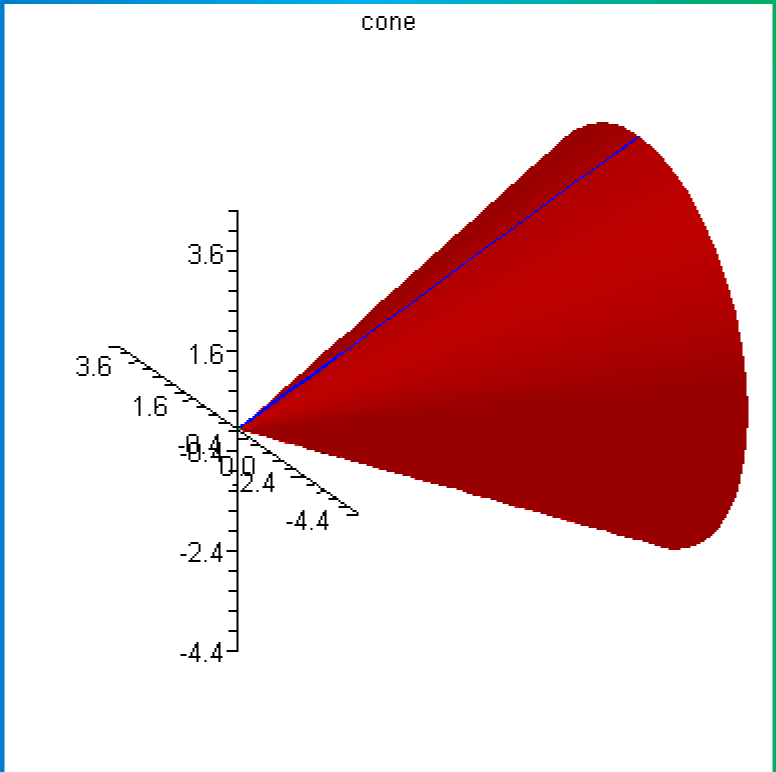
Outline For Today

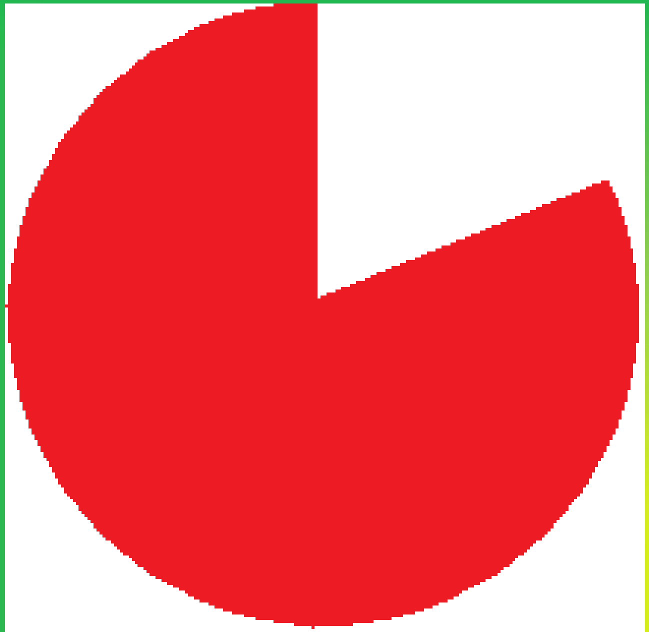
- Surface Area

Surface Area of a Cone

If the length of the slant line is L and the base radius is r we get a section of a circle of some angle θ. The are of such a section is

$$A = \frac{1}{2} L^2\theta$$





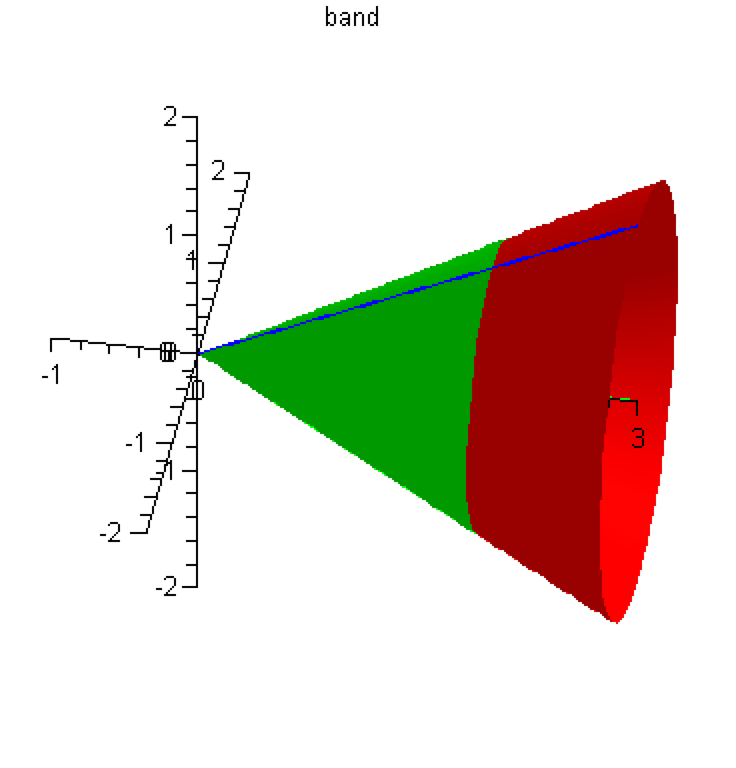
Surface Area of a Cone Continued

We also see that that $\theta = 2\pi r / L$

So $A = \frac{1}{2} L^2(2\pi r / L) = \pi r L$

Surface Area a Band

We want to find the surface area of a band with



Large Radius = r_2

Small Radius = r_1

Length of the Full Cone = L_1+L

Length of Cone Removed= L_1

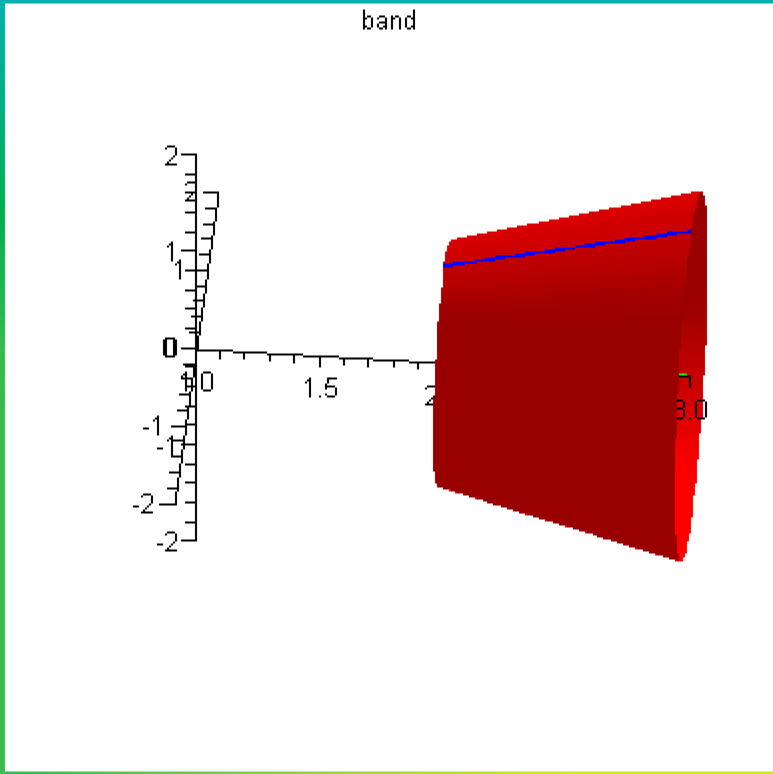
Length of Band= L

Surface Area a Band Continued

The Surface Area is then

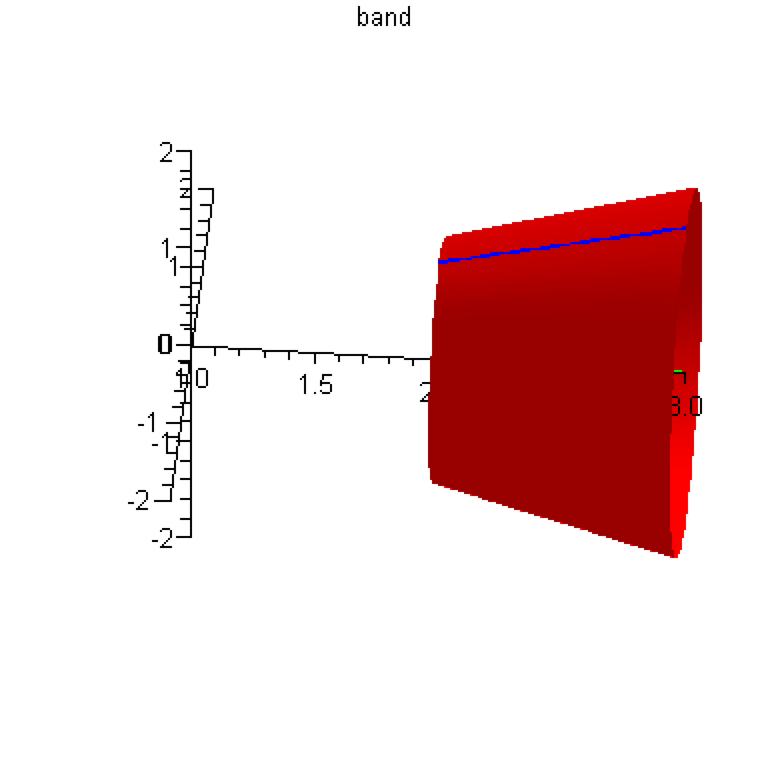
$$A = \pi(L_1 + L)r_2 - \pi L_1 r_1$$

$$= \pi\big((r_2 - r_1)L_1 + r_2 L\big)$$



Surface Area a Band Continued

And by similar triangles



$$\frac{L_1}{r_1} = \frac{L + L_1}{r_2}$$ so

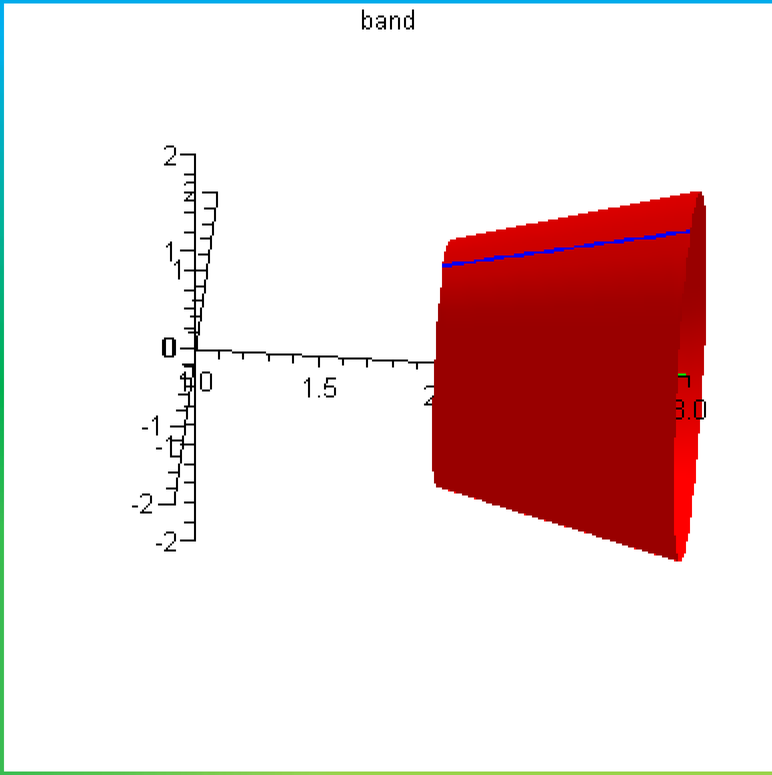
$$L_1 r_2 = L_1 r_1 + L r_1$$ or $$L_1(r_2 - r_1) = L r_1$$

Surface Area a Band Continued

So $A = \pi(r_1 L + r_2 L)$

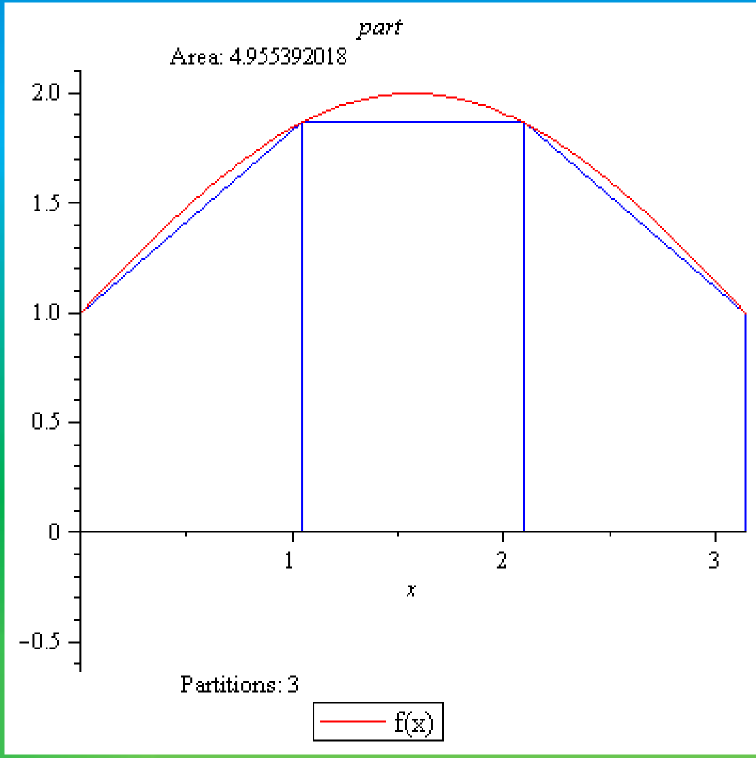
And if we let $r = \dfrac{r_1 + r_2}{2}$ then

$$A = 2\pi L r$$



Approximation to Surface Area

- Let $P_i = (x_i, y_i)$
- Then the surface area of the line connecting P_{i-1} and P_i rotated about the x-axis is a band with



radius $r=1/2(y_i+y_{i-1})$ and slant height = $|P_{i-1}P_i|$

Approximation to Surface Area Continued

- So the surface area of the the line connecting P_{i-1} and P_i rotated about the x-axis is

$$SA_{Band} = 2\pi \tfrac{1}{2}(y_i + y_{i-1})|P_iP_{i-1}|$$

Now $|P_iP_{i-1}| = \pi\sqrt{1+\left[f'(x_i^*)\right]^2}\Delta x$ for some $x_i^* \in [x_{i-1}, x_i]$

and when Δx is small we have

$y_i=f(x_i)\approx f(x_i^*)$ and $y_{i-1}=f(x_{i-1})\approx f(x_i^*)$

Surface Area

So $SA_{Band} \quad = \quad 2\pi \dfrac{(y_i + y_{i-1})}{2} | P_i P_{i-1} |$

$$\approx \quad 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\,\Delta x$$

And the surface area of a curve y = f(x) rotated about the x-axis is

$$SA \quad = \quad \lim_{n\to\infty} \sum_{i=0}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\,\Delta x$$

$$= \quad \int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

Surface Area (about x-axis)

The surface area of a curve y = f(x) from a ≤ x ≤ b rotated about the x-axis is

$$SA = \int_a^b 2\pi y \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

$$= 2\pi \int_a^b y\, ds$$

where $(ds)^2 = (dx)^2 + (dy)^2$

Surface Area (about y-axis)

The surface area of a curve x = g(y) from c ≤y≤ d rotated about the y-axis is

$$SA = \int_c^d 2\pi x \sqrt{1+\left[\frac{dx}{dy}\right]^2}\, dy$$

$$= 2\pi \int_c^d x\, ds$$

Try An Example

Find the surface area obtained by rotating the curve $y = x^2$ from $x = 1$ to 2 about the y-axis.

A) $\frac{\pi}{2}\left(17^{3/2} + 5^{3/2}\right)$

B) $\frac{\pi}{3}\left(12^{3/2} - 5^{3/2}\right)$

C) $\frac{\pi}{3}\left(7^{3/2} + 5^{3/2}\right)$

D) $\pi\left(7^{3/2} - 2^{3/2}\right)$

E) $\frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$

F) None of the above

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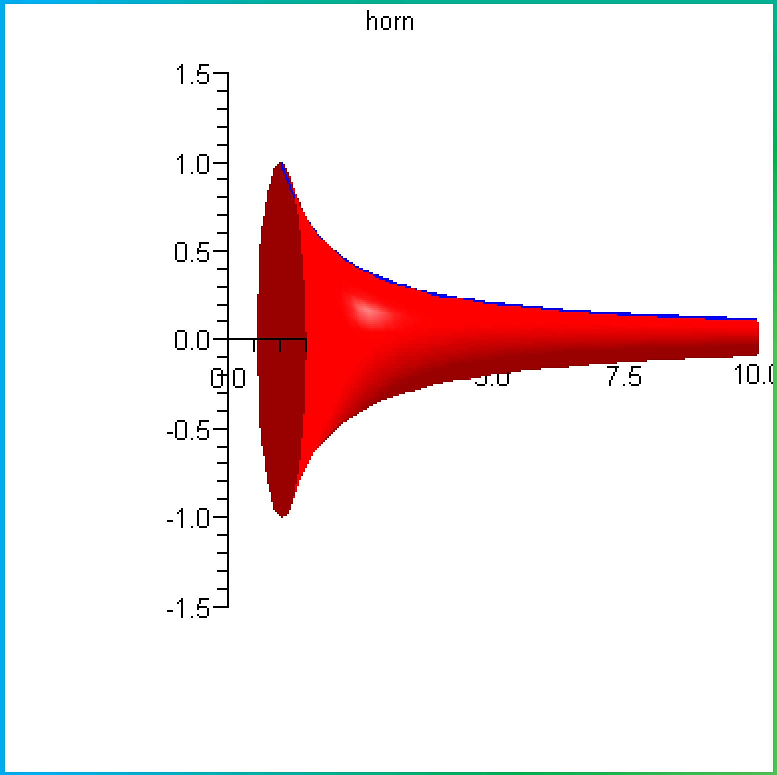
F) None of the above

Gabriel's Horn

- Lets find the surface area obtained by rotating the curve y = 1/x from 1 to ∞ about the x-axis



Gabriel's Horn Surface Area

y=1/x

dy/dx = -1/x^2

$$
\begin{aligned}
SurfaceArea &= 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\left[-\frac{1}{x^2}\right]^2}\,dx \\
&= 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\left[\frac{1}{x^4}\right]}dx \\
&\geq 2\pi\int_1^\infty \frac{1}{x}dx \\
&= \infty
\end{aligned}
$$

Gabriel's Horn Volume

$$\begin{aligned}
Volume &= \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx \\
&= \pi \int_1^\infty \frac{1}{x^2} dx \\
&= \lim_{t \to \infty} \int_1^t \frac{1}{x^2} dx \\
&= \lim_{t \to \infty} -\frac{1}{x}\bigg|_1^t \\
&= \lim_{t \to \infty} -\tfrac{1}{t} + 1 \\
&= 1
\end{aligned}$$