Math 104-006

Chapter 9.1: Arc Length

Outline For Today

- Arc Length

Approximate Arc Length

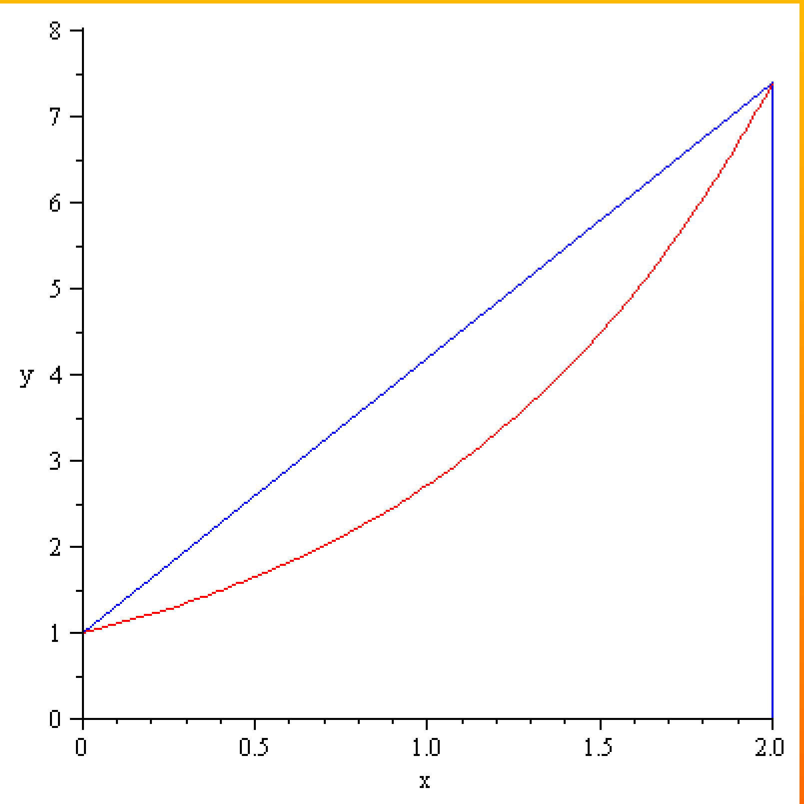
Suppose we have a curve given by $y = f(x)$ and we want to approximate its length on [a,b].

We do this by breaking up [a,b] into n intervals approximating the curve by a line on the interval and then adding up the lengths.

Length of a Segment

- Let $P_i = (x_i, y_i)$. The length of a segment is

$$
\begin{aligned}
|P_i P_{i-1}| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\
&= \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\
&= \sqrt{\Delta x^2 + \Delta y^2}
\end{aligned}
$$



Length of a Segment Continued

- By the mean value theorem there is an x^*_i such that $\Delta y = f(x_i)-f(x_{i-1}) = f'(x^*_i)\ \Delta x$

- Hence $|P_iP_{i-1}| = \sqrt{\Delta x^2 + (\Delta x f'(x^*_i))^2}$

$$= \sqrt{1+ f'(x^*_i)^2}\,\Delta x$$

Arc Length

- So the arc length is

$$\lim_{n\to\infty}\sum_{i=1}^{n}|P_{i-1}P_i| = \lim_{n\to\infty}\sqrt{1+f'(x^*{}_i)^2}\,\Delta x$$

$$= \int_a^b \sqrt{1+f'(x)^2}\,dx$$

Try An Example

Find the arc length from (0,0) to a point (2, 27) of

$$y = (2x + 1)^{3/2}$$

A) $\frac{46}{27}\sqrt{46} - \frac{10}{27}\sqrt{10}$

B) $\frac{15}{7}\sqrt{15} - \frac{10}{7}\sqrt{10}$

C) $\frac{40}{7}\sqrt{40} + \frac{10}{7}\sqrt{10}$

D) $\frac{46}{27}\sqrt{46} + \frac{10}{27}\sqrt{10}$

E) $\frac{15}{11}\sqrt{15} - \frac{7}{11}\sqrt{7}$

F) None of the Above

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F) None of the Above

Arc Length (for x = f(y))

- If x = g(y), c≤x≤d and g'(y) is continuous then the arc length is given by

$$L = \int_c^d \sqrt{1+(g'(y))^2}\,dy$$

$$= \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

General Arc Length

- If y = f(x) is a curve we call the function which calculates the arc length of a curve the arc length function.

$$s(x) = \int_a^x \sqrt{1+\left(f'(t)\right)^2}\,dt$$

- So

$$\frac{ds}{dx} = \sqrt{1+\left(f'(x)\right)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

General Arc Length Continued

- We can rewrite this as

$$
\begin{aligned}
ds &= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= \sqrt{(dx)^2+(dy)^2}
\end{aligned}
$$

- Or as $(ds)^2 = (dx)^2+(dy)^2$

General Arc Length Continued

- We then get the arc length can be expressed as

$$\int ds = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Example 2 (trick)

- Lets find the arc length from the point (1,1) to the point (t,f(t)) where
- $f(x)=x^2 - (1/8)\ln(x)$ and $x \geq 1$

Example 2 Continued (trick)

- Then $\frac{d}{dx} f(x) = 2x - \frac{1}{8x}$ so

- $$L = \int_1^t \sqrt{1+(2x-(\tfrac{1}{8x}))^2}\,dx$$
$$= \int_1^t \sqrt{1+(4x^2-2(\tfrac{2x}{8x})+\tfrac{1}{36x^2})}\,dx$$
$$= \int_1^t \sqrt{1+(4x^2-\tfrac{1}{2}+\tfrac{1}{36x^2})}\,dx$$
$$= \int_1^t \sqrt{4x^2+\tfrac{1}{2}+\tfrac{1}{36x^2}}\,dx$$

Example 2 Continued (trick)

$$\begin{aligned} L &= \int_1^t \sqrt{(2x + (1/8x))^2}\,dx \\ &= \int_1^t 2x + (1/8x)\,dx \\ &= x^2 + 1/8 \ln(x)\Big|_1^t \\ &= t^2 + 1/8 \ln(t) - 1 \end{aligned}$$