Math 104-006

Chapter 8.4: Rational Functions

Outline For Today

- Method of Partial Fractions
- Rational Substitutions

Rational Functions

A rational function is one of the form:

$$\frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \ldots + b_m x^m} = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i}$$

First Step

If the degree of the numerator is greater than the degree of the denominator use long division to get

$$\frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i} = \sum_{i=0}^{r} c_i x^i + \frac{\sum_{i=0}^{s} d_i x^i}{\sum_{i=0}^{m} b_i x^i}$$

with s < n.

First Step

Or equivalently

$$\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

$$\deg(R) < \deg(D)$$

Example

Lets find $\int \frac{x^4 - x + 1}{x - 2} dx$

First we need to do long division with remainder.

$$\begin{array}{r} x^3 + 2x^2 + 4x + 7 \quad r\ 15 \\ \hline x - 2\,\big|\, x^4 + 0x^3 + 0x^2 - x + 1 \end{array}$$

$$- \quad x^4 - 2x^3 \quad +$$

$$2x^3 + 0x^2$$

$$- \quad 2x^3 - 4x^2$$

$$4x^2 - x$$

$$- \quad 4x^2 - 8x$$

$$7x + 1$$

$$- \quad 7x - 14$$

$$15$$

Example Continued

So

$$\frac{x^4 - x + 1}{x-2}dx = \int x^3 + 2x^2 + 4x + 7 + \frac{15}{x-2}dx$$

$$= \frac{1}{4}x^4 + \frac{2}{3}x^3 + 2x^2 + 7x + 15\ln|x-2| + C$$

Now You Try One

What is $\int \frac{2x^4 + x^3 - 4x^2 + 4}{2x+1}\,dx$?

A) $\frac{1}{2}x^4 - 2x^2 + 2x + \frac{1}{2}\ln|x+\frac{1}{2}| + C$

B) $\frac{1}{4}x^4 - \frac{1}{2}x^2 + 2x + \frac{1}{2}\ln|x+\frac{1}{2}| + C$

C) $\frac{1}{4}x^4 - x^2 + x + \frac{3}{2}\ln|2x+1| + C$

D) $x^4 - \frac{1}{2}x^2 + x + \frac{3}{2}\ln|2x+1| + C$

E) $\frac{1}{4}x^4 + x^2 - x + \frac{3}{2}\ln|2x+1| + C$

F) None of the Above

Now You Try One

What is $\int \frac{2x^4 + x^3 - 4x^2 + 4}{2x+1}\,dx$?

A) $\frac{1}{2}x^4 - 2x^2 + 2x + \frac{1}{2}\ln|x + \frac{1}{2}| + C$

B) $\frac{1}{4}x^4 - \frac{1}{2}x^2 + 2x + \frac{1}{2}\ln|x + \frac{1}{2}| + C$

C) $\frac{1}{4}x^4 - x^2 + x + \frac{3}{2}\ln|2x+1| + C$

D) $x^4 - \frac{1}{2}x^2 + x + \frac{3}{2}\ln|2x+1| + C$

E) $\frac{1}{4}x^4 + x^2 - x + \frac{3}{2}\ln|2x+1| + C$

F) None of the Above

Factorization

- It can be shown that any polynomial can be factored into linear factors and quadratic factors.
- That is factors of the form
- $ax + b$ and
- $ax^2 + bx + c$ where $b^2 - 4ac < 0$

Second Step

We now want to write our ration as the sum of terms of the form

$$\frac{A_i}{(a_i x + b_i)^n} \quad \text{or} \quad \frac{A_i x + B_i}{(a_i x^2 + b_i x + c_i)^n}$$

Case 1

- $D(x) = (a_1x + b_1)\,(a_2x + b_2)\ldots\,(a_nx + b_n)$

- Then

$$\frac{R(x)}{D(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \ldots + \frac{A_n}{a_nx+b_n}$$

Example

- Lets find $\int \frac{4dx}{x^3 - 4x}$

- First we see x^3-4x = x(x-2)(x+2)

- So...

Example Continued

$$\int \frac{4dx}{x^3 - 4x} = \int \frac{4dx}{x(x-2)(x+2)}$$

$$= \int \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} dx$$

- So…

Example Continued

$$\frac{4}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

- And hence

$$\begin{aligned} 4 &= A(x^2-4) + B(x^2+2x) + C(x^2-2x) \\ &= (A+B+C)x^2 + (2B-2C)x - 4A \end{aligned}$$

- which means...

Example Continued

$$\begin{aligned} 4 &= -4A \\ 0x &= (2B-2C)x \\ 0x^2 &= (A+B+C)x^2 \end{aligned}$$

And hence

$$\begin{aligned} A &= -1 \\ B &= C \\ 1+2B &= 0 \end{aligned}$$

Example Continued

So

$$\begin{aligned} A &= -1 \\ B &= \frac{1}{2} \\ C &= \frac{1}{2} \end{aligned}$$

And hence

$$\begin{aligned} \int \frac{4dx}{x^3 - 4x} &= \int -\frac{1}{x} + \frac{1}{2(x-2)} + \frac{1}{2(x+2)} dx \\ &= -\ln|x| + \frac{1}{2}\ln|x-2| + \frac{1}{2}\ln|x+2| + C \end{aligned}$$

Now you try one

What is $\int \frac{x+3}{x^2+3x+2}\,dx$?

A) $-\ln|x+2|+2\ln|x+1|+C$

B) $2\ln|x+2|+3\ln|x+1|+C$

C) $-2\ln|x+2|+\ln|x+1|+C$

D) $-\ln|x+2|+2\ln|x+3|+C$

E) $\ln|x+2|+\ln|x+1|+C$

F) None of the Above

Now you try one

What is $\int \frac{x+3}{x^2+3x+2}\,dx$?

A) $-\ln|x+2|+2\ln|x+1|+C$

B) $-\ln|x+2|+2\ln|x+1|+C$

C) $-2\ln|x+2|+\ln|x+1|+C$

D) $-\ln|x+2|+2\ln|x+3|+C$

E) $\ln|x+2|+\ln|x+1|+C$

F) None of the Above

Case 2

- $D(x) = (a_1x + b_1)^{r1} (a_2x + b_2)^{r2} \ldots (a_nx + b_n)^{rn}$
- Then

$$\frac{R(x)}{D(x)} = \left[\frac{A_{1,1}}{a_1x+b_1}+\frac{A_{2,1}}{(a_1x+b_1)^2}+\ldots+\frac{A_{r1,1}}{(a_1x+b_1)^{r1}}\right]$$

$$+\left[\frac{A_{1,2}}{a_2x+b_2}+\frac{A_{2,2}}{(a_2x+b_2)^2}+\ldots+\frac{A_{r2,2}}{(a_2x+b_2)^{r2}}\right]$$

$$+\ldots+\left[\frac{A_{1,n}}{a_nx+b_n}+\frac{A_{2,n}}{(a_nx+b_n)^2}+\ldots+\frac{A_{rn,n}}{(a_nx+b_n)^{rn}}\right]$$

Example

- Lets find $\int \frac{16dx}{(x-2)^2(x+2)}$
- We want

$$\int \frac{16dx}{(x-2)^2(x+2)} = \int \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2} dx$$

- So...

Example Continued

$$\frac{16}{(x-2)^2(x+2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2}$$

- And hence

$$16 = A(x^2-4) + B(x+2) + C(x-2)^2$$

- which means...

Example Continued

If x = 2 then

$$16 = 0A + B(2+2) + 0C$$

so B = 4

If x= -2 then

$$16 = 0A + 0B + 16C$$

so C=1

Example Continued

If x = 0 then

$$\begin{aligned} 16 &= -4A + 2B + 4C \\ &= -4A + 8 + 4 \end{aligned}$$

So A = -1

Example Continued

Hence

$$\int \frac{16dx}{(x-2)^2(x+2)} = \int -\frac{1}{x-2} + \frac{4}{(x-2)^2} + \frac{1}{(x+2)}\,dx$$

$$= -\ln|x-2| + \ln|x+2| - \frac{4}{x-2} + C$$

Case 3

- $D(x) = (a_0x^2 + b_0x + c_0)(a_1x^2 + b_1x + c_1)\ldots(a_nx^2 + b_nx + c_n)$

- then

$$\frac{R(x)}{D(x)} = \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{a_2x^2 + b_2x + c_2} + \ldots + \frac{A_nx + B_n}{a_nx^2 + b_nx + c_n}$$

Example

- Lets find $\int \frac{dx}{(x^2-2x+2)(x-1)}$

- We need to first break the integral up.

Now you try one

$$\frac{1}{(x^2-2x+2)(x-1)} = \frac{Ax+B}{x^2-2x+2} + \frac{C}{x-1}$$

A) $A=-1, B=1, C=2$

B) $A=1, B=-1, C=1$

C) $A=-2, B=2, C=1$

D) $A=-1, B=1, C=1$

E) $A=1, B=1, C=2$

F) None of the Above

Now you try one

$$\frac{1}{(x^2-2x+2)(x-1)}=\frac{Ax+B}{x^2-2x+2}+\frac{C}{x-1}$$

A) $A=-1, B=1, C=2$

B) $A=1, B=-1, C=1$

C) $A=-2, B=2, C=1$

D) $A=-1, B=1, C=1$

E) $A=1, B=1, C=2$

F) None of the Above

Example Continued

So

$$\int \frac{dx}{(x^2-2x+2)(x-1)} = \int \frac{-x+1}{x^2-2x+2} + \frac{1}{x-1}\,dx$$

$$= -\frac{1}{2}\ln|x^2-2x+2| + \ln|x-1| + C$$

Completing The Square

- $x^2 + ax + b = (x + c)^2 + d$
- $x^2 + ax + b = x^2 + 2cx + c^2 + d$

- So $a=2c$ and $a/2 = c$

and $c^2 + d = (a/2)^2 + d = b$

Now you try one

What is $\int \frac{1}{x^2+4x+5}\,dx$?

A) $\arcsin(x+2)+C$

B) $\arctan(x+2)+C$

C) $\arctan(x+1)+C$

D) $\arctan(x-2)+C$

E) $\arcsin(x-2)+C$

F) None of the Above

Now you try one

What is $\displaystyle\int \frac{1}{x^2+4x+5}\,dx$?

A) $\arcsin(x+2)+C$

B) $\arctan(x+2)+C$

C) $\arctan(x+1)+C$

D) $\arctan(x-2)+C$

E) $\arcsin(x-2)+C$

F) None of the Above

Rational Substitution

- If $\sqrt[n]{g(x)}$ occurs in the integral where g(x) is a polynomial then sometimes using

$$u = \sqrt[n]{g(x)}$$

$$u^n = g(x)$$

we get $nu^{n-1}du = g'(x)dx$

Example

- Lets find $\int \frac{\sqrt{x-2}}{x+1}\,dx$

$$u = \sqrt{x-2}$$

$$u^2 = x-2$$

$$x = u^2 + 2$$

$$dx = 2u\,du$$

- So…

Example Continued

$$
\begin{aligned}
\int \frac{\sqrt{x-2}}{x+1}\,dx &= \int \frac{u}{u^2+2-1}(2u\,du) \\
&= \int \frac{2u^2}{u^2+1}\,du \\
&= \int 2-\frac{2}{u^2+1}\,du \\
&= 2u-2\arctan(u)+C \\
&= 2\sqrt{x-2}-2\arctan(\sqrt{x-2})+C
\end{aligned}
$$