# Math 104-006

## Chapter 8.3: Trigonometric Substitution

# Outline For Today

- Inverse Substitution
- Trigonometric Substitutions

# Inverse Substitution

If x = g(t)     and   dx=g'(t)dt   then

$$\int f(x)dx = \int f(g(t)) \cdot g'(t)dt$$

# sin(θ) substitution

Expression: $\sqrt{a^2 - x^2}$ $\qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Substitution: $x = a \cdot \sin(\theta)$

$dx = a \cdot \cos(\theta) d\theta$

Identity: $1 - \sin^2(\theta) = \cos^2(\theta)$

# Example

Lets find $\int \frac{x^2}{\sqrt{4-x^2}}dx$

We use x = 2 sin(θ) to get

dx= 2 cos(θ)dθ

$$\int \frac{x^2}{\sqrt{4-x^2}}dx = \int \frac{(2\sin(\theta))^2}{\sqrt{4-(2\sin(\theta))^2}} 2\cos(\theta)d\theta$$

# Example Continued

$$\int \frac{(2\sin(\theta))^2}{\sqrt{4-(2\sin(\theta))^2}} 2\cos(\theta)d\theta = \int \frac{4\sin^2(\theta)}{\sqrt{4-4\sin^2(\theta)}} 2\cos(\theta)d\theta$$

$$= \int \frac{4\sin^2(\theta)}{2\cos(\theta)} 2\cos(\theta)d\theta$$

$$= \int 4\sin^2(\theta)d\theta$$

$$= 4\int \frac{1}{2}(1-\cos(2\theta))d\theta$$

$$= 2\theta - \sin(2\theta) + C$$

$$= 2\theta - 2\sin(\theta)\cos(\theta) + C$$

# Example Continued

We then have

$$\begin{aligned} \frac{x}{2} &= \sin(\theta) \\ \sqrt{1-(\tfrac{x}{2})^2} &= \cos(\theta) \\ \sin^{-2}(\tfrac{x}{2}) &= \theta \end{aligned}$$

So

$$\begin{aligned} 2\theta - 2\sin(\theta)\cos(\theta) + C &= 2\sin^{-1}(\tfrac{x}{2}) - 2(\tfrac{x}{2})\sqrt{1-(\tfrac{x}{2})^2} + C \\ &= 2\sin^{-1}(\tfrac{x}{2}) - x\sqrt{1-(\tfrac{x}{2})^2} + C \end{aligned}$$

# tan(θ) substitution

Expression: $\sqrt{a^2 + x^2}$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Substitution: $x = a \cdot \tan(\theta)$

$$dx = a \cdot \sec^2(\theta)d\theta$$

Identity: $1 + \tan^2(\theta) = \sec^2(\theta)$

# sec(θ) substitution

Expression: $\sqrt{x^2 - a^2}$ $0 \le \theta < \dfrac{\pi}{2}$ or $\pi \le \theta < \dfrac{3\pi}{2}$

Substitution: $x = a \cdot \sec(\theta)$

$$dx = a \cdot \tan(\theta) \cdot \sec(\theta) d\theta$$

Identity: $\sec^2(\theta) - 1 = \tan^2(\theta)$

# Now You Try One

What is $$\int_0^{3\sqrt{3}/2} \frac{x^3}{\sqrt{(4x^2+9)^{3/2}}}\,dx$$?

A) $\frac{1}{32}$

B) $\frac{\sqrt{3}}{8}$

C) $\frac{3}{32}$

D) $\frac{3\sqrt{3}}{4}$

E) $\frac{3\sqrt{3}}{32}$

F) $\frac{9}{16}$

# Now You Try One

What is $$\int_0^{3\sqrt{3}/2} \frac{x^3}{\sqrt{(4x^2+9)^{3/2}}}\,dx$$

A) $\frac{1}{32}$

B) $\frac{\sqrt{3}}{8}$

C) $\frac{3}{32}$

D) $\frac{3\sqrt{3}}{4}$

E) $\frac{3\sqrt{3}}{32}$

F) $\frac{9}{16}$

# Lets Work Through One

What is $\displaystyle\int \frac{1}{\sqrt{\left(8-2\cdot x-x^{2}\right)^{1/2}}}\,dx$ ?

# Answer

First we need to turn the denominator into something we can handle.

$8-2x-x^2 = 8-(x^2+2x) = 8+1-(x^2+2x+1) = 9- (x+1)^2$

# Answer Continued

So we have

$$\int \frac{1}{\sqrt{(8-2\cdot x-x^2)^{1/2}}}\,dx \quad = \quad \int \frac{1}{\sqrt{(9-(x+1)^2)^{1/2}}}\,dx$$

$$= \quad \int \frac{1}{3\sqrt{(1-(\frac{x+1}{3})^2)^{1/2}}}\,dx$$

# Answer Continued

So if we let u = x+1/3

du=dx/3 and

$$\int \frac{1}{\sqrt{(8-2\cdot x-x^2)^{1/2}}}dx = \int \frac{1}{3\sqrt{(1-u^2)^{1/2}}}(3du)$$

$$= \int \frac{1}{\sqrt{(1-u^2)^{1/2}}}(du)$$

$$= \sin^{-1}(u)+C$$

$$= \sin^{-1}(\tfrac{x+1}{3})+C$$