Math 104-006

Chapter 8.2: Trigonometric Integrals

Outline For Today

- Integration of $\sin^m(x)\cos^n(x)$
- Integration of $\sec^m(x)\tan^n(x)$
- Integration of $\sin(mx)\sin(nx)$
- Integration of $\sin(mx)\cos(nx)$
- Integration of $\cos(mx)\cos(nx)$

Integrate $\sin^{2k+1}(x)\cos^{m}(x)$

Use $\sin^2(x) = 1-\cos^2(x)$ to get

$$\int \sin^{2k+1}(x)\cos^{m}(x)\,dx = \int \cos^{m}(x)(1-\cos^{2}(x))^{k}\sin(x)\,dx$$

Integrating $\sin^{2k+1}(x)\cos^{m}(x)$ Continued

Then use $u = \cos(x)$

$du = -\sin(x)\, dx$

To get

$$\int \cos^{m}(x)(1-\cos^{2}(x))^{k}\sin(x)dx = -\int u^{m}(1-u^{2})^{k}du$$

Lets Do An Example

Lets find $\int \sin^3(x)\cos^2(x)dx$

$$\int \cos^2(x)\sin^3(x)dx = \int \cos^2(x)(1-\cos^2(x))\sin(x)dx$$

so if we let u = cos(x)

du=-sin(x)dx

$$\int \cos^2(x)(1-\cos^2(x))\sin(x)dx = -\int u^2(1-u^2)du$$

$$= -u^3/3+u^5/5+C$$

$$= -\cos(x)^3/3+\cos(x)^5/5+C$$

Integrate $\sin^m(x)\cos^{2k+1}(x)$

Use $\cos^2(x) = 1-\sin^2(x)$ to get

$$\int \sin^m(x)\cos^{2k+1}(x)dx = \int \sin^m(x)(1-\sin^2(x))^k \cos(x)dx$$

Integrating $\sin^m(x)\cos^{2k+1}(x)$ Continued

Then use $u = \sin(x)$

$du = \cos(x)\,dx$

To get

$$\int \sin^m(x)(1-\sin^2(x))^k \cos(x)dx = \int u^m(1-u^2)^k du$$

Now You Try One

What is $\int \sin^5(x)dx$?

A) $-\cos(x)+\frac{2}{3}\cos^3(x)-\cos^5(x)+C$

B) $\sin(x)+\frac{2}{3}\sin^3(x)-\sin^5(x)+C$

C) $\cos(x)-\cos^3(x)+\cos^5(x)+C$

D) $\sin(x)+\frac{1}{3}\sin^3(x)-\sin^5(x)+C$

E) $\sin(x)+\frac{1}{3}\sin^3(x)-\frac{1}{5}\sin^5(x)+C$

F) $\cos(x)+\frac{2}{3}\cos^3(x)-\frac{4}{5}\cos^5(x)+C$

Now You Try One

What is $\int \sin^5(x)dx$?

A) $-\cos(x)+\frac{2}{3}\cos^3(x)-\cos^5(x)+C$

B) $\sin(x)+\frac{2}{3}\sin^3(x)-\sin^5(x)+C$

C) $\cos(x)-\cos^3(x)+\cos^5(x)+C$

D) $\sin(x)+\frac{1}{3}\sin^3(x)-\sin^5(x)+C$

E) $\sin(x)+\frac{1}{3}\sin^3(x)-\frac{1}{5}\sin^5(x)+C$

F) $\cos(x)+\frac{2}{3}\cos^3(x)-\frac{4}{5}\cos^5(x)+C$

Integrate $\sin^{2m}(x)\cos^{2n}(x)$

Use the identities

$$\cos^2(x) = 1/2(1+\cos(2x))$$
$$\sin^2(x) = 1/2(1-\cos(2x))$$
$$\sin(x)\cos(x) = 1/2(\sin(2x))$$

Another Example

What is $\int \sin^2(x)\cos^2(x)dx$?

A) $\frac{1}{8}[2x - \sin(2x)] + C$

D) $\frac{1}{8}[2x - \cos(2x)] + C$

B) $\frac{1}{32}[4x - \sin(4x)] + C$

E) $\frac{3}{4}[x + \sin(2x)] + C$

C) $\frac{1}{32}[4x + \cos(4x)] + C$

F) $\frac{1}{8}[2x + \cos(2x)] + C$

Another Example

What is $\int \sin^2(x)\cos^2(x)dx$?

A) $\frac{1}{8}[2x - \sin(2x)] + C$

B) $\frac{1}{32}[4x - \sin(4x)] + C$

C) $\frac{1}{32}[4x + \cos(4x)] + C$

D) $\frac{1}{8}[2x - \cos(2x)] + C$

E) $\frac{3}{4}[x + \sin(2x)] + C$

F) $\frac{1}{8}[2x + \cos(2x)] + C$

Integrate sec^{2k+2}(x)tan^{m}(x)

Use sec^{2}(x) = 1+tan^{2}(x) to get

$$\int \sec^{2k+2}(x)\tan^{m}(x)dx = \int \tan^{m}(x)(1-\tan^{2}(x))^{k}\sec^{2}(x)dx$$

Integrating $\sec^{2k+2}(x)\tan^{m}(x)$ Continued

Then use $u = \tan(x)$

$du = \sec^2(x)\, dx$

To get

$$\int \tan^m(x)(1-\tan^2(x))^k \sec^2(x)dx = \int u^m(1-u^2)^k du$$

Integrate $\sec^{m+1}(x)\tan^{2k+1}(x)$

Use $\tan^2(x) = \sec^2(x) - 1$ to get

$$\int \sec^{m+1}(x)\tan^{2k+1}(x)dx = \int \sec^{m}(x)(\sec^2(x)-1)^k \sec(x)\tan(x)dx$$

Integrating $\sec^{m+1}(x)\tan^{2k+1}(x)$ Continued

Then use $u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

To get

$$\int \sec^m(x)(\sec^2(x)-1)^k \sec(x)\tan(x)\,dx = \int u^m (u^2-1)^k\,du$$

Another Example

What is $\int \tan^2(x)\sec^4(x)dx$?

A) $\frac{1}{3}\sec^3(x)+\frac{1}{5}\sec^5(x)+C$

B) $\frac{1}{2}\sec^2(x)+\frac{1}{4}\sec^2(x)+C$

C) $\frac{1}{3}\tan^3(x)-\frac{1}{5}\tan^5(x)+C$

D) $\tan^3(x)\sec^5(x)+C$

E) $\frac{1}{3}\sec^3(x)-\frac{1}{5}\tan^5(x)+C$

F) $\frac{1}{3}\tan^3(x)+\frac{1}{5}\tan^5(x)+C$

Another Example

What is $\int \tan^2(x)\sec^4(x)dx$?

A) $\frac{1}{3}\sec^3(x)+\frac{1}{5}\sec^5(x)+C$

B) $\frac{1}{2}\sec^2(x)+\frac{1}{4}\sec^2(x)+C$

C) $\frac{1}{3}\tan^3(x)-\frac{1}{5}\tan^5(x)+C$

D) $\tan^3(x)\sec^5(x)+C$

E) $\frac{1}{3}\sec^3(x)-\frac{1}{5}\tan^5(x)+C$

F) $\frac{1}{3}\tan^3(x)+\frac{1}{5}\tan^5(x)+C$

The Other Cases Are Less Clear

To find $\int \tan(x)dx$ Use u = cos(x)

du = -sin(x)dx

To get

$$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx$$

$$= \int \frac{-1}{u}du$$

$$= -\ln|u| + C$$

$$= \ln\left|\frac{1}{u}\right| + C$$

$$= \ln|\sec(x)| + C$$

Integral of sec(x)

This requires a trick:

$$\int \sec(x)\,dx = \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$$

$$= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$$

Integral of sec(x) Continued

We then use u = sec(x) + tan(x)

du = $\sec^2(x)+\tan(x)dx$

to get :

$$\int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}dx = \int \frac{1}{u}du$$

$$= \ln|u|+C$$

$$= \ln|\sec(x)+\tan(x)|+C$$

Integral of $\sec^3(x)$

First we integrate by parts with:

$u = \sec(x)$ $dv = \sec^2(x)dx$

$du = \sec(x)\tan(x)$ $v = \tan(x)$

$$\begin{aligned} \int \sec^3(x)dx &= \sec(x)\tan(x) - \int \tan^2(x)\sec(x)dx \\ &= \sec(x)\tan(x) - \int (\sec^2(x) - 1)\sec(x)dx \\ &= \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx \end{aligned}$$

Integral of $\sec^3(x)$ Continued

So

$$2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \int \sec(x)\,dx$$
$$= \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| + C$$

and we therefore get:

$$\int \sec^3(x)\,dx = \frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|) + C$$

Other Trig Identities

$$\sin(A)\cdot\cos(B)=\frac{1}{2}[\sin(A-B)+\sin(A+B)]$$

$$\sin(A)\cdot\sin(B)=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\cos(A)\cdot\cos(B)=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

Other Trig Identities Continued

We can use these identities to get:

$$\int \sin(mx)\cdot\cos(nx)dx = \int \frac{1}{2}[\sin(mx-nx)+\sin(mx+nx)]dx$$

$$\int \sin(mx)\cdot\sin(nx)dx = \int \frac{1}{2}[\cos(mx-nx)-\cos(mx+nx)]dx$$

$$\int \cos(mx)\cdot\cos(nx)dx = \int \frac{1}{2}[\cos(mx-nx)+\cos(mx+nx)]dx$$

Yet Another Example

What is $\int \sin(2x)\cos(x)dx$?

A) $\frac{1}{3}\cos(3x)-\cos(x)+C$

B) $-\frac{1}{6}\sin(3x)-\frac{1}{2}\sin(x)+C$

C) $-\frac{1}{6}\cos(3x)-\frac{1}{2}\cos(x)+C$

D) $\frac{1}{6}\cos(3x)+\frac{1}{2}\cos(x)+C$

E) $\frac{1}{6}\cos(3x)-\frac{1}{2}\cos(x)+C$

F) $-\cos(3x)-\sin(x)+C$

Yet Another Example

What is $\int \sin(2x)\cos(x)dx$?

A) $\frac{1}{3}\cos(3x)-\cos(x)+C$

D) $\frac{1}{6}\cos(3x)+\frac{1}{2}\cos(x)+C$

B) $-\frac{1}{6}\sin(3x)-\frac{1}{2}\sin(x)+C$

E) $\frac{1}{6}\cos(3x)-\frac{1}{2}\cos(x)+C$

C) $-\frac{1}{6}\cos(3x)-\frac{1}{2}\cos(x)+C$

F) $-\cos(3x)-\sin(x)+C$