Math 104-006

Chapter 10.3: Separable Equations

Outline For Today

- Separable Equations
- Orthogonal Trajectories
- Mixing Problems

Separable Equation

- A separable equation is a first order differential equation of the form

- $y' = g(x)f(y)$

- or equivalently if $f(y) \neq 0$ and $h(y) = 1/f(y)$

- $y' = g(x)/h(y)$

Separable Equation Continued

- If dy/dx = g(x)/f(y)
- Then we have g(x) dx = f(y) dy
- So $\int g(x)dx = \int f(y)dy$

Example

- Lets find a solution to $y' = x^3 y$
- We have $\int \frac{dy}{y} = \int x^3 dx$
- So $\ln |y| = \frac{x^4}{4} + C$
- and $y = Ce^{\frac{x^4}{4}}$

Try An Example

What is a solution to the differential equation

$$\frac{dy}{dx} = x \cdot \sec(y)$$

A) $\sin(y) = x + C$

B) $\cos(y) = \ln|x| + C$

C) $\tan(y) = \frac{x^2}{2} + C$

D) $\sin(y) = \frac{x^2}{2} + C$

E) $\ln|\sec(y) + \tan(y)| = \frac{x^2}{2} + C$

F) None of the above

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F) None of the above

Orthogonal Trajectory

- An **orthogonal trajectory** of a family of curves is a curve which is orthogonal to every curve in the family.

- For example the family y=mx of strait lines through the origin is an orthogonal trajectory of the family of curves $x^2 + y^2 = r^2$

Orthogonal Trajectory Example

- Lets find the orthogonal trajectories of the of the family of curves $x = ky^2$

- If we differentiate both sides we get
- $1 = 2k\frac{dy}{dx}y$ so $\frac{dy}{dx} = \frac{1}{2ky}$

Orthogonal Trajectory Example Continued

- But this equation depends on k and we want a curve which is orthogonal for all values of k

- So notice that $k = \frac{y}{2x}$ so

$$\frac{dy}{dx} = \frac{1}{2(x / y^2)y} = \frac{y}{2x}$$

Orthogonal Trajectory Example Continued

- Hence our curve must satisfy the equation

$$\frac{dy}{dx} = -\frac{2x}{y}$$

- So $\int ydy = \int -2xdx$

- and $\frac{y^2}{2} = -x^2 + C$ or $\frac{y^2}{2} + x^2 = C$

Mixing Problems

- A typical mixing problem involves a tank of fixed capacity filled with a thoroughly mixed solution of some substance (such as a salt). A solution of a given concentration enters at a given rate (and is thoroughly stirred) while the mixture leaves at a fixed (possibly different) rate

Mixing Example

- A tank contains 20kg of salt dissolved in 5000L of water. Brine that contains 0.03kg of salt per liter of water enters the tank at a rate of 25L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate.

- How much salt remains after 30 minute

Mixing Example

- Let y(t) be the amount of salt in the tank after t minutes
- dy/dt = (rate in) - (rate out)

- rate in = (0.03 kg/L)(25 L/min) = 0.75 kg/min
- rate out = (y(t)/5000 kg/L)(25 L/min) = y(t)/200 kg/min

Mixing Example Continued

- So we have

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

- And hence

$$\int \frac{dy}{150 - y(t)} = \int \frac{dt}{200}$$

- So $-\ln|150 - y(t)| = \frac{t}{200} + C$

Mixing Example Continued

- Further we know that y(0) = 20 so

$$-\ln|150-20| = \frac{0}{200} + C$$

- And hence $-\ln 130 = C$ so

$$-\ln|150-y(t)| = \frac{t}{200} - \ln 130$$

- And $|150-y(t)| = 130e^{-\frac{t}{200}}$

Mixing Example Continued

- But as y(t) is continuous and $130e^{-\frac{t}{200}}$
- Is never 0 we know that

$$|150-y(t)|=150-y(t)$$

- Hence $y(t)=150-130e^{-\frac{t}{200}}$

- And $y(30)=150-130e^{-\frac{30}{200}}\approx 38.1kg$