Math 104-006

Chapter 10.1: Modeling with Differential Equations

Outline For Today

- Models of Population Growth
- Motion of a Spring
- General Differential Equations
- Initial Values

Models of Population Growth

- t = time (independent variable)
- P = the number of individuals in the population (the dependent variable).

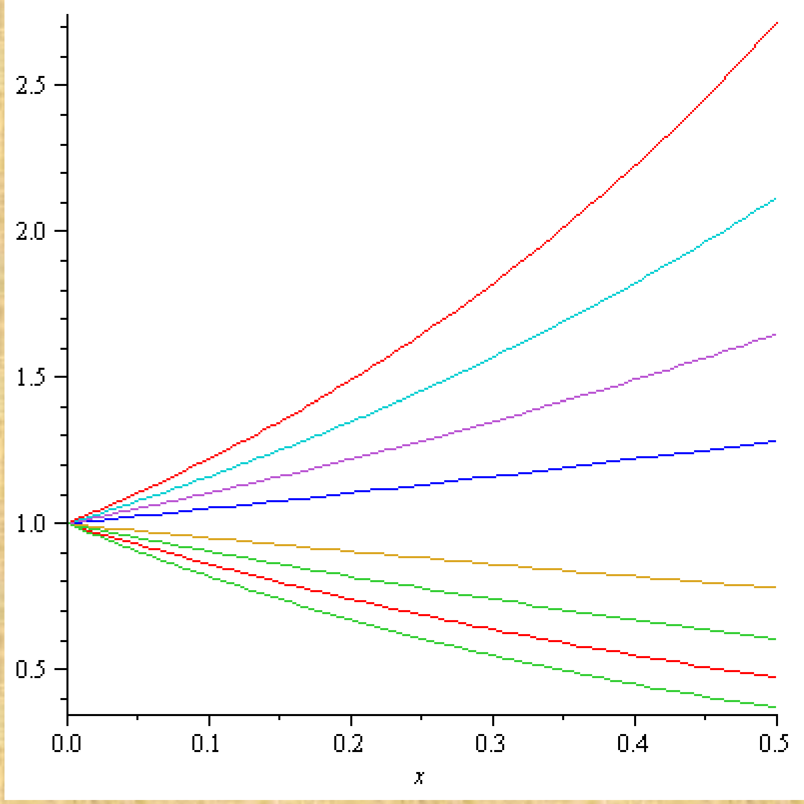
- **Assumptions**
- The rate of population growth is proportional to the size of the population.
- Population is always positive ($P(t) > 0$)
- Time is non-negative ($t \geq 0$)

Models of Population Growth Continued

- So we have $\frac{dP}{dt} = kP$

- If k > 0 then the population is always increasing

- If k < 0 then the population is always decreasing



Models of Population Growth Continued

- We know from our experience that if

$$P = Ce^{kt} \quad \text{then} \quad \frac{dP}{dt} = Cke^{kt} = kP$$

This then gives us a collection of solutions to the differential equation.

Models of Population Growth: Initial Values

- Often though we aren't interested in a family of solutions but a specific one.
- In this situation we often know the value at a given point in time (like say t = 0)
- Then we know that $P(0) = Ce^{k0} = C$
- So in general we have $P(t) = P(0)e^{kt}$

Logistic Model of Population Growth

- The previous model was only an approximation.

- Many populations though increase exponentially for a while until it approaches a **carrying capacity K** (or decreases towards **K** if it ever exceeds it).

Logistic Model of Population Growth

- We make two assumptions for this model.
- $\frac{dP}{dt} = kP$ if P is small (Initially the growth rate is proportional to P)
- $\frac{dP}{dt} < 0$ if P > K (P decreases if it ever exceeds K)

Logistic Model of Population Growth

- A simple expression which incorporates both of these is

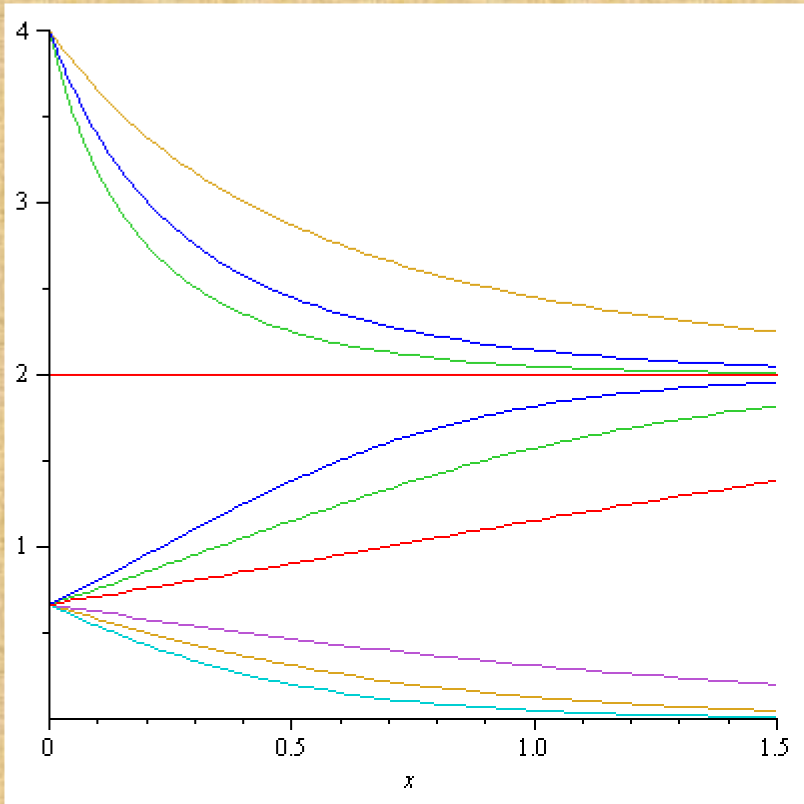
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

- This is called the **logistic differential equation**.

Logistic Model of Population Growth Continued

- This model was first proposed by the Dutch mathematical biologist Pierre-Francois Verhulst in the 1840s

- Graphing possible solutions we get



Motion of a Spring

- The restoring force = -k x.
  - k is called the spring constant

- By Newton's second law we know that
  - Force = Acceleration * Mass

- So we have $m\frac{d^2x}{dt^2} = -kx$

Motion of a Spring Continued

We also have $\frac{d^2x}{dt^2} = -\frac{k}{m}x$

We know two functions which have this property: sin(t) and cos(t)

The general solution is of the form

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right)$$

Try An Example

What is a solution to the differential equation

$$\frac{d^2x}{dt^2} = -x$$ where x(0) = 1, x'(0) = -1

A) $\sin(x) + \cos(x)$

B) $\sin(x) - \cos(x)$

C) $-\sin(x) + \cos(x)$

D) $-\sin(x) - \cos(x)$

E) $2\sin(x) + 2\cos(x)$

F) None of the above

Try An Example

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B) $\sin(x)-\cos(x)$

C) $-\sin(x)+\cos(x)$

D) $-\sin(x)-\cos(x)$

E) $2\sin(x)+2\cos(x)$

F) None of the above

General Differential Equation

In general a **differential equation** is an equation relating a function with its differentials.

The **order** of such an equation is the largest derivative which occurs in it.

Any function which satisfies a differential equation is called a solution.

General Differential Equation Continued

Usually there is a family of solutions (called the **general solution**)

Sometimes though we are interested in a specific solution which satisfies a given condition (i.e. $y(t_0) = y_0$)

This is called the **initial value problem** and $y(t_0) = y_0$ is an **initial condition**.

Example

Lets show that every member of the family

$$y = \frac{1+ce^t}{1-ce^t}$$

is a solution of the differential equation

$$y' = \frac{1}{2}\left(y^2 - 1\right)$$

Try An Example

What value of c gives a solution with y(0)=2?

A) c = 1/3

B) c = 1/2

C) c = 1

D) c = 2

E) c = 3

F) None of the above

Try An Example

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